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BULLETIN
OF THE
AMERICAN MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE

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THE OCTOBER MEETING IN NEW YORK

The two hundred fifty-seventh regular meeting of the American Mathematical Society was held at Columbia University, on Saturday, October 29, 1927, extending through the usual morning and afternoon sessions. Attendance included the following ninety-five members of the Society:

Alexander, Allison, R. C. Archibald, R. G. Archibald, J. W. Arnold, A. A. Bennett, Benton, Bratton, Brinkmann, R. W. Burgess, Carrié, E. T. Carroll, Coble, Cramlet, Edmondson, Eisenhart, Engstrom, Fenn, Fiske, Fite, D. A. Flanders, Forsyth, Fort, M. C. Foster, Frink, Garver, Gehman, Gill, Glenn, P. H. Graham, Gronwall, C. C. Grove, Guggenbühl, Hebbert, L. S. Hill, Hille, Hofmann, Hoyt, Huber, Dunham Jackson, Joffe, R. A. Johnson, Kasner, Kholodovsky, Kline, Koopman, Kormes, Lamson, Langman, Lefschetz, MacColl, S. P. Mead, Meder, H. H. Mitchell, Richard Morris, Mullins, Murnaghan, C. A. Nelson, Newman, K. E. O'Brien, Ore, Paradiso, Pfeiffer, Pierpont, Post, R. G. Putnam, Reddick, R. G. D. Richardson, D. E Richmond, Ritt, Robertson, Rutledge, Schoonmaker, Seely, Siceloff, Simons, Slotnick, Smail, P. A. Smith, Virgil Snyder, Stetson, Tamarkin, J. E. Thompson, Veblen, H. E. Webb, Wedderburn, Weida, Weisner, A. P. Wheeler, Whittemore, Widder, W. A. Wilson, Margaret M. Young, Zippin, Zobel.

The Secretary announced the election of the following persons to ordinary membership in the Society:

Mr. Abraham Adrian Albert, University of Chicago;
Dr. Ralph George Archibald, Columbia University;
Mr. Armond W. Bear, Marquette University;
Mr. Richard Stevens Burington, Case School of Applied Science;
Professor Harold Hardesty Downing, University of Kentucky;
Mr. Walter Henry Gage, Victoria College, Victoria, British Columbia;
Mr. Ellis Richard Heineman, Michigan Agricultural College;
Mr. Isaac Newton Heminger, San Jose, Calif;
Professor Edward Milton Little, University of Montana;
Professor Ernest Lloyd Mackie, University of North Carolina;
Professor Marion Lee MacQueen, Southwestern College;
Professor Max Morris, Case School of Applied Science;
Mr. Kenneth Ernest Moyle, High School, Somerville, N. J.;
Professor Oystein Ore, Yale University;
Mr. Rafael Sanchez-Diaz, University of Porto Rico;

Miss Mildred Ellen Taylor, Knox College;
Mr. Carleton Russell Worth, University of Arkansas;
Professor Frank Lynwood Wren, George Peabody College for Teachers;

Nominees of the Georgia Power Company as sustaining member:

Mr. C. E. Bennett, Georgia Power Company, Atlanta;
Mr. S. C. Bleckley, Georgia Power Company, Atlanta;
Mr. F. A. Brine, Georgia Power Company, Atlanta;
Mr. W. A. Hammel, Georgia Power Company, Atlanta;
Mr. W. P. Hammond, Alabama Power Company, Birmingham.

At the meeting of the Board of Trustees, the editors of the Bulletin and of the Transactions were authorized to sign printing contracts with the Banta Publishing Company.

The council fixed the dates for regular New York meetings of the Society in New York during 1928 as follows: February 25, October 27, and, in place of the meeting usually held on the last Saturday in April or the first Saturday in May, a two-day meeting on April 6–7, the Friday and Saturday before Easter. An invitation for the summer meeting of 1931 was received from the University of Minnesota.

On nomination of the Western group, a committee consisting of Professors E. W. Chittenden (chairman), L. M. Graves, and M. H. Ingraham was appointed to choose a Symposium speaker for the Easter, 1928, meeting at Chicago.

Nominations for officers and other members of the Council was adopted, and ordered printed on the official ballot.

President Snyder presided at the morning session. In the afternoon two sectional meetings were held; Dr. Gronwall presided over the section of Analysis and Point Sets, at which papers numbered 11–26, 46–48 were read, and Professor Pierpont over that of Algebra and Geometry, at which papers numbered 27–45 were read.

At the request of the Program Committee, Professor F. D. Murnaghan delivered an address at the end of the morning session, entitled *Modern hydrodynamical theory, with special reference to aeronautics*. This address will appear in full in an early number of this Bulletin. Titles and abstracts of the other papers read at this meeting follow below. The papers of Ayres, Bray, Copeland, Douglas, Frink (second and third papers), Garver (third paper), Hickey, Hollcroft, Michal, Roos, Seely, Stetson, Weatherburn, and Whyburn were read by title. Mr. Gourin and Mr. Wang were introduced by Professor Ritt, Miss Hickey by Professor Evans, and Professor Weatherburn by Professor Kasner.

1. Professor C. H. Forsyth: *The amount of a single sum of money at any number of rates of interest.*

In the issue of this Bulletin for June-July, 1921, the author presented certain generalizations of some of the most fundamental formulas of the mathematical theory of finance. Those generalizations were based upon the use of two and three rates of interest. The purpose of the present paper is to show the nature and form of the function which is obtained when the number of rates of interest is increased indefinitely.

2. Professor C. H. Forsyth: *The determination of monthly items from yearly items of the secular trend of a time series when that trend is assumed to be other than linear.*

In the case of a linear trend, a straight line is fitted to yearly items or totals of a time series, and the coefficients of the linear equation divided by 12 give the monthly items. Occasionally, the trend is clearly other than linear; then the yearly items are easily obtained when once the type of curve is selected, but the corresponding monthly items cannot be found by simple division. The author shows they may be found by interpolation.

3. Dr. H. P. Robertson (National Research Fellow): *An alternative to De Sitter's cosmology.*

The 4-dimensional manifold defined by the line element $ds^2 = -e^{2kct} \cdot (dx^2+dy^2+dz^2)+c^2dt^2$, $k>0$, is of constant Riemannian curvature, and is therefore mathematically equivalent to De Sitter's universe. Interpreting the variable t occurring in this form as the proper-time of an observer at a point x, y, z = constant, the "mass-horizon" and "arrest of time at the horizon" paradoxes of De Sitter are obviated. Because of the dynamical form of the line element, however, natural processes are not reversible; in particular, a shift of spectral lines in light from distant objects is predicted which is in accordance with existing data on extra-galactic nebulae and which leads to a value $k=5 \cdot 10^{-28}(cm)^{-1}$. The universe may be represented as a hypersphere in a flat 5-space, hence the relations existing between freely moving observers (the analogue of the Lorentz transformations of special relativity) are equivalent to the rotations of flat 5-space.

4. Dr. H. P. Robertson: *Conditions under which a certain type of solution of Schrödinger's wave equation can be found*

The conditions under which the fundamental equation of Schrödinger's "wave-mechanics" is solvable by separation of variables are found for the case of a conservative system whose kinetic energy is expressed in orthogonal coordinates. It is shown that in addition to the conditions given by P. Stäckel for the solution of the corresponding Hamilton-Jacobi equation a certain functional relation must be satisfied.

5. Dr. L. S. Hill: *Certain matrices in finite algebraic fields, and their relation to code telegraphy.*

Let a matrix in which no determinant of any order vanishes be called, for convenience, uniform; the present paper directs attention to the bearing of such matrices, constructed in various finite algebraic fields, upon certain important problems of accuracy assurance in the transmittal of electrical communications,—telegraph, cable, radio. A special type of uniform matrix, which appears to be directly amenable to practical application, is treated with particular care. It would be easy to construct a mechanical device, similar to the ordinary adding machine but somewhat less complex, which would automatically accomplish all checking operations based upon this special type. The paper leads to a further problem which is of purely algebraic interest, and which will be studied later.

6. Professor D. E. Richmond: *A complete set of relations between type numbers of extremals joining a pair of points.*

Let A and B be a pair of points in an extremal-convex region for a calculus of variations problem. An extremal arc AB may be assigned a type number n equal to the number of points on the arc conjugate to A. Let M_n represent the number of extremals of type n joining A to B. The author has proved the completeness of a certain set of number relations between the integers M_n, when no type number n exceeds 2. When extremals of type 2 exist joining A to B, the relations are $M_0=2$, $M_0-M_1+M_2=1$. It has been shown that there exists a calculus of variations problem and a pair of points A and B such that the number of extremals of type i ($i=1, 2, 3$) is M_i, where the M_i's are arbitrary positive integers satisfying the relations stated.

7. Dr. A. H. Copeland: *Admissible numbers in the theory of probability.*

The problem of establishing the consistency of the fundamental assumptions of the theory of probability is proved in this paper to be equivalent to that of showing the existence of certain sets of numbers, $A(p)$. These assumptions are consistent in the case of a given probability, p, if and only if the corresponding set, $A(p)$, is non-empty. Admissible numbers are the numbers that belong to the sets $A(p)$. Every admissible number must satisfy an infinite set of independent equations and hence the consistency of the fundamental assumptions seems doubtful. Not only is the consistency of the assumptions open to question, but also the customary definition of independence is so loosely stated that, if one takes the most obvious interpretation of this definition, the theorem of compound probability is false. However, it is shown that this definition of independence can be so stated that this theorem is true. It is also proved that the sets $A(p)$ are non-empty and hence that the fundamental assumptions are consistent. Moreover, it is proved that each point x of the interval $0<x<1$ is a point of condensation of each of the sets $A(p)$ where $0<p<1$.

8. Dr. A. H. Copeland: *Stability of the motion of a gyroscope in contact with a smooth horizontal plane.*

The problem considered in this paper is the stability of the motion of a gyroscope which is a smooth body of revolution and which is acted upon only by gravity and a constraint which keeps it in contact with a smooth horizontal table. The motion is described in terms of a curve Γ traced by the axis of the gyroscope on the unit sphere whose center is at the center of mass of the gyroscope and whose orientation is fixed in space. The north pole is taken at the top of the sphere. The curve Γ is bounded by two parallels of latitude. It is proved that for slight changes in the initial conditions large changes can occur in the positions of these parallels, that is, the motion can be unstable; but these unstable motions can be cut out by giving the gyroscope a sufficiently large spin. The method used is an extension of the method employed in the author's paper entitled *Types of motion of the gyroscope* (see abstract No. 10, below).

9. Dr. A. H. Copeland: *The motion of a gyroscope with variable spin.*

This paper is a discussion of the motion of a gyroscope which is acted upon by gravity, by a constraint which keeps one point on its axis fixed in space, by a torque, S, which tends to increase or decrease the spin of the gyroscope about its axis, and by no other forces. The case in which S is identically zero gives a classical problem. In this paper it is assumed that S is a function of the time, the position of the gyroscope, and the velocities of the coordinates determining the position. The effect of S is traced throughout the entire motion. The major portion of this paper is devoted to the case in which the spin is monotone and nowhere stationary. All the possible types of motions of gyroscopes with always increasing spins, or always decreasing spins, and all the possible changes in these types are determined. It is also shown that some types of motion are unstable and break down into other types when small changes are made in the spin.

10. Dr. A. H. Copeland: *Types of motion of the gyroscope.*

A method of graphical representation of gyroscopic motion is developed in this paper. A simple representation is given of the complete history of all motions of all gyroscopes subject to the following restrictions: each gyroscope is acted upon by gravity and by a constraint which keeps one point on its axis fixed in space, and by no other forces. With the aid of Osgood's intrinsic equations, this history is extended to include new intrinsic properties of the space cone traced by the axis of the gyroscope. By means of these methods it is possible to list the properties of the motion more fully than has hitherto been done, exhibit their dependence upon the initial conditions, and show how one type of motion changes into another as the initial conditions are varied continuously.

11. Professor Edward Kasner: *Non-monogenic (or polygenic) functions.*

The author proposes the term *polygenic* for arbitrary functions $w=\phi(x, y)+i\psi(x, y)$ of the complex variable $z=x+iy$ where the derivative depends on the slope m as well as the point z. To each point z we have then ∞^1 values of $dw/dz=\alpha+i\beta$, which when plotted as points in a new plane (α, β) form a circle called the *derivative circle.* Thus to the entire polygenic function corresponds a *congruence* of circles. This in special cases degenerates; e.g., if the given function is monogenic the circles all become points; for a function of $x-iy$, we obtain in the (α, β) plane ∞^1 circles with centers at the origin; if the centers all lie on the axis of α, the function must obey merely one of the Cauchy-Riemann equations, etc. The *mean derivative* of any polygenic function is defined as $H+iK$, where H, K are the coordinates of the center of the derivative circle. The mean derivative is monogenic when and only when ϕ and ψ obey the Laplace equation. The transformation is then termed general harmonic by the author; this includes conformal as a very special case. The point (α, β) on any derivative circle always moves, as m changes, with angular velocity twice that of m and in the opposite direction. For each circle a certain initial radius must be fixed; the complete graph is thus a congruence not of circles but of *clocks*, each clock with four coordinates. A general theory of such congruences is given.

12. Professor J. F. Ritt: *Certain points in the theory of Dirichlet series.*

The results of this paper, all of which deal with Dirichlet series, bear upon the following topics: Picard's theorem, integral functions, asymptotic convergence. The first part contains generalizations of the theorems of Landau and Schottky. The second part deals with integral functions defined by everywhere absolutely convergent Dirichlet series, and develops the relations between the growth of the modulus of the function and the coefficients of the series. The third part carries over to Dirichlet series the theory of asymptotic convergence of power series.

13. Dr. B. O. Koopman: *Some theorems on inversion.*

Consider the substitution (1). $y_i=f_i(x_1, \cdots, x_n)$, $i=1, 2, \cdots, n$, where the f's are analytic functions of $(x_1, \cdots, x_n)=(x)$ for $(x)=(0)$. This substitution has only been studied completely when the Jacobian J does not vanish for $(x)=(0)$, when the transformation has a unique analytic inverse. The object of our paper is to investigate the substitution in the case where $(x)=(0)$ is a non-specialized point of the complex locus $J=0$. By means of non-singular substitutions of the variables (x) and (y) we reduce the substitution (1) to a standard form, from which it appears that it may be inverted by means of algebroid functions. The results are then applied to the extension of the theorem of functional dependence. (See Osgood, *Lehrbuch der Funktionentheorie*, II, chap. 2, §§19, 20, 23.)

14. Mr. Eli Gourin: *On irreducible polynomials.*

Given an irreducible polynomial $Q(y_1, y_2, \cdots, y_p)$ of p variables $y_1, y_2, \cdots, y_p$, the object is to determine all positive integers $t_1, t_2, \cdots, t_p$ for which the polynomial $Q(y_1^{t_1}, y_2^{t_2}, \cdots, y_p^{t_p})$ is reducible.

15. Mr. Eli Gourin: *Periodic Weierstrass zeta functions.*

This paper presents a study of conditions for which Weierstrass's zeta function becomes a singly periodic function.

16. Mr. S. C. Wang: *On a linear addition theorem in three arguments.*

It is shown in this paper that all continuous solutions of the functional equation of real variables $f(x+y+z)=f(x+y)+f(x+z)+f(y+z)-f(x)-f(y)-f(z)$ are of the form $f(t)=At^2+Bt$, where A and B are two independent arbitrary constants.

17. Dr. T. C. Benton: *On continuous curves homogeneous except for certain points.*

This paper proves the following theorems: (1) A point set M is homogeneous except for n points $c_1, c_2, \cdots, c_n$ if for any pair of points a_1, a_2 distinct from all c_i a continuous (1-1) correspondence ψ exists such that $\psi(M)=M$ and $\psi(a_1)=a_2$; and for every i there exists a point x_i such that no correspondence π exists such that $\pi(M)=M$ and $\pi(x_i)=c_i$. (2) A bounded continuous curve homogeneous except for one point c_1 is a countable set ($\geqq 2$) of simple closed curves having only c_1 in common and such that only a finite number of them are of diameter greater than any given positive number. (3) A bounded continuous curve homogeneous except for c_1 and c_2 is a finite number ($\neq 2$) of arcs joining c_1 to c_2. (4) A bounded continuous curve homogeneous except for c_1, c_2, c_3 is either two sets homogeneous except for two points which have only one c_i in common and have the same number of arcs, or three such sets joined cyclically with the same number of arcs in each set. (5) An unbounded continuous curve homogeneous except for c_1 is a finite number of rays from c_1. (6) An unbounded continuous curve homogeneous except for c_1, c_2 does not exist.

18. Professor H. M. Gehman: *On extending a continuous (1-1) correspondence.* Second paper.

In continuation of results recently published (Transactions of this Society, vol. 28 (1926), pp 252–265), it is proved that if any two plane point sets which are closed and bounded and which have the properties that (1) every maximal connected subset is a continuous curve, and (2) not more than a finite number of maximal connected subsets are of diameter greater than any positive number, are in continuous (1-1) correspondence in such a way that the correspondence preserves sides *in the* *same sense*, then a continuous (1-1) correspondence of their planes can be defined which is identical with the given correspondence for points of the two sets.

19. Professor W. A. Wilson: *On bounded regular frontiers in the plane.*

This paper appears in full in the present number of this Bulletin.

20. Professor R. G. Putnam: *Note on countable sets of points of non-separable metric spaces.*

It is shown in this note that every countable set of points of a non-separable metric space is homeomorphic with a linear set.

21. Dr. W. L. Ayres: *An elementary property of bounded domains.*

This paper will appear in full in an early number of this Bulletin.

22. Dr. W. L. Ayres: *Concerning subsets of a continuous curve which can be connected through the complement of the continuous curve.*

Let H denote the sum of the boundaries of the complementary domains of a continuous curve M in a plane S. These theorems are proved: (1) if an M-domain is simply connected with respect to M, there exists a simply connected domain R such that $R \cdot M=D$; (2) if every two points x and y of H may be connected through $x+y$ and $S-M$, then H is a continuous curve and either (a) H is the boundary of one of the complementary domains of M, or (b) M consists of three arcs with common end points and no two having any other point in common, or (c) M consists of three rays with the same vertex and no two having any other only its common, or (d) M consists of an open curve plus an arc having point in end points in common with the open curve; (3) if every two points x and y of M may be connected through $x+y$ and $S-M$, then $M=H$; (4) if every three points of H lie together on some complementary domain of M, then H is the boundary of one of the complementary domains of M.

23. Dr. G. T. Whyburn and Dr. W. L. Ayres (National Research Fellow): *On continuous curves in n dimensions.*

This paper will appear in full in an early number of this Bulletin.

24. Professor A. D. Michal: *Invariant functionals of functional forms.*

In the theory of algebraic invariants of algebraic forms in n variables, the underlying group is the general group of linear homogeneous transformations with non-vanishing determinants. This paper develops the first essentials of a corresponding theory in the space of real continuous functions of a real variable x. The "algebraic" *functional* forms are restricted to normal types and the fundamental group is taken to be the group of all linear functional Fredholm transformations for which unity is not a characteristic value. "Tensor" invariant functionals of the coefficients of the functional forms are considered as well as scalar ones. The laws of transformation of the coefficients of the functional forms incidentally give rise to a theory of certain interesting integral equations which are not of the classic types.

25. Professor A. D. Michal: *Affinely connected function space manifolds.*

The first part of this paper develops a theory of functional tensors in the space of real continuous functions of a real variable x. The analogues of affine connection, infinitesimal parallelism, covariant differentiation, geodesics, normal coordinates, and tensor extension of n-dimensional geometry are then considered. The functional invariants of the functional affine connection arise as the integrability conditions of functional equations with funcional derivatives. The "geodesics" satisfy non-linear integro-differential equations of static type. The theory of the Fredholm integral equation underlies the theory of functional normal coordinates. The paper closes with a discussion of functional quadratic differential forms followed by a development of a theory of "Riemannian" function space manifolds. This paper combined with a part of the preceding paper will appear in the American Journal of Mathematics.

26. Miss Deborah M. Hickey: *The equilibrium point of Green's function for an annular region.*

This paper derives the necessary and sufficient condition for an equilibrium point in an annular region in terms of Fourier series and in terms of elliptic functions. Experimental investigations by W. S. Vaughn seemed to indicate that the equilibrium point remained fixed when the pole in the region was moved. A necessary and sufficient condition that this point be fixed is $2\zeta(\omega')-\omega'\wp(\omega')=0$. By means of Fourier series it is shown that this equation is satisfied for no ratio of the half periods ω, ω'. Nevertheless the experimental results are justified by obtaining theoretically the upper bound of the shift, which is very small.

27. Professor Oystein Ore: *Newton polygons in the theory of algebraic fields.*

The determination of the development of an algebraic function in power series corresponds to a certain extent to the determination of the prime-ideal decomposition of a prime p in an algebraic field. By generalizing the method of Newton polygons and by using the author's investigations on algebraic numbers (Mathematische Annalen, vols. 96 and 97) the complete decomposition of all primes p is obtained for every field given by an arbitrary equation.

28. Professor Oystein Ore: *Some relations between groups and ideals in Galois fields.*

As a generalization of the "Zerlegungsgruppe" of Hilbert, a group $G\{\mathfrak{A}\}$ is introduced corresponding to an arbitrary ideal $\mathfrak{A}$. The group $G\{\mathfrak{A}\}$ embraces all the substitutions for which the system of numbers in $\mathfrak{A}$ as ideal is invariant. The construction of $G\{\mathfrak{A}\}$ is determined, and some properties of the corresponding subfield studied.

29. Professor O. E. Glenn: *The invariancy of infinite series.*

The terms of a convergent infinite series $a, b\ c, \cdots, \infty$ may be considered to be the coefficients of a polynomial f of order n in an infinite number of variables. Let f, with $n=1$ for example, be transformed by the infinite cyclic transformation $x=\lambda x'+\mu y'+\nu z'+\cdots\infty$, $y=\lambda y'+\mu z'+\nu t'+\cdots\infty, z=\lambda z'+\mu t'+\nu u'+\cdots\infty, \cdots$, where the series $\lambda, \mu, \nu, \cdots, \infty$ is convergent. Then certain series in the form of infinite polynomials in the terms $a, b, c, \cdots$ are relative invariants. The sum of the series is an invariant, the invariantive relation being $a'+b'+c'+\cdots=(\lambda+\mu+\nu+\cdots) \cdot(a+b+c+\cdots)$. The property of convergence is an invariant property. A theorem on the product of two series is a theorem on the transformation of one series by the infinite cyclic transformation determined by the other.

30. Dr. Louis Weisner: *Quadratic fields in which cyclotomic polynomials are reducible.*

For any particular value of m, the quadratic fields in which the polynomial whose zeros are the primitive mth roots of unity is reducible can be determined by methods described in the second volume of Weber's *Algebra*; but it does not seem practicable to carry out the computations for a general m. The present writer avoids these computations by proving two lemmas from which it is inferred that the cyclotomic polynomial is reducible in certain quadratic fields. The number of these fields is found to be precisely the number of intransitive subgroups of index 2 of the group of the cyclotomic equation relative to $R(1)$. The quadratic fields determined are the only ones in which the cyclotomic polynomial is reducible.

31. Dr. R. G. Archibald: *Diophantine equations in division algebras.*

This paper obtains necessary and sufficient conditions for the solvability in integers of diophantine equations, of the type $U^2-aR^2=(A^2-abB^2) \cdot(F^2-bC^2)$, which arise in an attempt to satisfy the associativity conditions for an algebra Γ based on a quartic equation $x^4+px^2+n^2=0$, irreducible in the field of rational numbers. The irreducibility of this equation imposes the condition that each of $a=2n-p$, $b=-2n-p$, ab be different from a perfect square. The associativity conditions for such algebras Γ were obtained by Dickson (*New division algebras*, Transactions of this Society, April, 1926). In the present paper numerical examples are given which satisfy all the conditions obtained.

32. Dr. Orrin Frink (National Research Fellow): *An algebraic method of differentiation.*

The theory of analytic functions of a hypercomplex variable in the case of certain familiar commutative linear algebras containing nilpotent units is shown to lead to a method of obtaining the standard formulas of the differential calculus by purely algebraic means, without the use of limiting processes. The method can be used to find higher derivatives directly. It is shown that differentials may be treated as absolute infinitesimals when no higher derivatives are involved. In some cases the method can be used to find functions with prescribed addition theorems.

33. Dr. Orrin Frink: *A general type of algebras.*

The algebras studied here, which may be considered generalizations of Boolean algebras, are commutative and associative and obey the special laws $a+a=a$ and $a^2=a$ for all a's. Many examples are given, which are discussed with relation to supplementary postulates which some of them satisfy, such as $a+ab=a$, the dual of the distributive law, and the existence of identities of addition and multiplication. An important case is one in which the addition and multiplication operations are interpreted to mean least common multiple and greatest common divisor.

34. Dr. Orrin Frink: *Concerning the fundamental regions of elliptic functions.*

It is first shown that the only convex fundamental regions of elliptic functions are parallelograms and hexagons. Then the problem is solved of finding all possible sets of points in the plane such that a period parallelogram of fixed size, shape, and orientation will always contain exactly n of them no matter what its position in the plane.

35. Dr. Hazel E. Schoonmaker: *Non-monoidal involutions having a congruence of invariant conics.*

The purpose of this paper is to derive all birational involutorial point transformations of space which have the following properties: (a) each transforms every conic of a linear congruence into itself; (b) the transformations cannot be reduced birationally to the monoidal type.

36. Dr. Raymond Garver: *The binomial quartic as a normal form.*

This paper has appeared in the November-December number of this Bulletin.

37. Dr. Raymond Garver: *A rational normal form for certain quartics.*

This paper appears in full in the present number of this Bulletin.

38. Dr. Raymond Garver: *Tschirnhausen transformations on certain rational cubics.*

Normal forms for particular cubics are obtained, as well as a necessary and sufficient condition that a rational cubic may be transformed rationally into a binomial cubic.

39. Dr. Jesse Douglas (National Research Fellow): *The geometry of systems of K-spreads.*

This paper constructs for systems of K-spreads in N-space a theory analogous to the general geometry of paths ($K=1$) developed in a previous paper presented to the Society, May 7, 1927. If G is any group of coordinate transformations in the N-space, and H any group of parameter transformations on the K-spreads, we can have a G, H, geometry of K-spreads. This paper considers the affine, equivoluminar, and descriptive geometries of K-spreads, i.e., where G is the total group $\bar{x}^i=\bar{x}^i(x)$, and H is respectively (1) $u^\alpha=A^\alpha_\beta v^\beta$, (2) $u^\alpha=\phi^\alpha(v)$, where $|\partial u/\partial v|=\text{constant}$, (3) $u^\alpha=\phi^\alpha(v)$, ϕ^α arbitrary. For the affine geometry of K-spreads the fundamental partial differential equations are (4) $\partial^2 x^i/\partial u^\alpha \partial u^\beta=H^i_{\alpha\beta}\ (x, \partial x/\partial u)$, where $H^i_{\alpha\beta}$ must satisfy the following condition: if $q^j_\alpha=A^\beta_\alpha p^j_\beta$, then $H^i_{\alpha\beta}(x, q)=A^\rho_\alpha A^\sigma_\beta H^i_{\rho\sigma}(x,p)$. A double tensor analysis arises in which Greek indices run from 1 to K, Latin indices from 1 to N. The formula for the affine connection is $\Gamma^i_{jk}=(K(K+1))^{-1}\partial^2 H^i_{\alpha\beta}/\partial p^j_\alpha \partial p^k_\beta$. The integrability conditions for (4) are $B^i_{jkl}p^j_\alpha p^k_\beta p^l_\gamma=0$, where B^i_{jkl} is the curvature tensor based on the Γ's. A parameter transformation on the K-spreads converts (4) into $\partial^2 x^i/\partial v^\alpha \partial v^\beta =H^i_{\alpha\beta}(x, \partial x/\partial v)+(\partial x^i/\partial v^\lambda)G^\lambda_{\alpha\beta}(x, \partial x/\partial v)$. The equivoluminar geometry of K-spreads is characterized by the vanishing of the contracted G's: $G^\lambda_{\lambda\beta}=0$.

40. Dr. Jesse Douglas: *One-to-∞^1 surface transformations of space.*

The equations (1) $X=X(x, y, z, p, q, a)$, $Y=Y(x, y, z, p, q, a), \cdots$, $Q=Q(x, y, z, p, q, a)$ associate with each surface element x, y, z, p, q a series of ∞^1 surface elements X, Y, Z, P, Q. The ∞^2 elements of any surface Σ (or union) are thereby converted, in general, into ∞^3 elements so distributed that there is one and only one through each point of a certain region of the X, Y, Z space, thus defining a Pfaff equation in the X, Y, Z space. Under what circumstances will the Pfaff equation corresponding to an arbitrary Σ be exact, so that the transformation (1) may be regarded as converting any surface Σ into ∞^1 surfaces? The present paper proves that this occurs when and only when there exists a transformation of the parameter a, (2) $a=a(x, y, z, p, q, b)$, such that the equations which result by substituting (2) in (1): (1′) $X=\xi(x, y, z, p, q, b)$, $Y=\eta(x, y, z, p, q, b), \cdots$, $Q=\kappa(x, y, z, p, q, b)$, define a *contact* transformation for each fixed value of b. As far as the present author knows, this result is not contained in Goursat's monograph *Le Problème de Bäcklund*, to the order of ideas of which the present paper belongs.

41. Dr. M. M. Slotnick (National Research Fellow): *A contribution to the theory of fundamental transformations of surfaces.*

Graustein has introduced a projective invariant of transformations F which we call the invariant C. The dual of this invariant, that is, the cross ratio in which a pair of corresponding tangent planes is divided by the focal planes of the line of the harmonic congruence, is called the invariant H. These two invariants are in the relation $CH=e_1g/(g_1e)$, where e, g, e_1, g_1 are the coefficients of the second fundamental quadratic forms of the surfaces. A study of transformations F based on these two invariants yields dual fundamental theorems, and renders many facts easier to prove than by the classical methods. The perspective transformation for which $H=1$ enters as the dual of the radial transformation for which $C=1$; and in the same way the transformation $K(C=-1)$ and the transformation $\Omega(H=-1)$ are duals of one another. The theorems obtained are applied to transformations R in general, and also to those transformations R which are also K or Ω or both. Some attention has also been paid to transformations F of various special types when the nets are isothermal-conjugate.

42. Dr. M. M. Slotnick: *A method of applying tensor analysis to the study of rectilinear congruences.*

Some interesting results are obtained when the methods of tensor analysis are applied to deal with congruences of straight lines in a euclidean 3-space, the lines being represented by Study's "dual coordinates." The fundamental tensor is that of the linear element of the spherical representation of the congruence. We write $m_i=p_ia+q_i^la_l$, where m represents the coordinates of the middle point of the line of the congruence, a its direction cosines; and the subscripts on m and a represent differentiation with respect to the corresponding parameters. The coefficients of linear combination p_i and q_i^l are tensors which form the basis of the study. These are evaluated in terms of the fundamental tensor, and a second tensor whose components are the coefficients of the second fundamental quadratic form of the congruence.

43. Professor T. R. Hollcroft: *On nets of manifolds in i dimensions.*

The characteristics of a net in two dimensions have been fully determined, but little has been done in three dimensions and the author has been able to find nothing for dimensions greater than three. In this paper, the characteristics of a net of manifolds in any number of dimensions are obtained. This is done by establishing a (1, 1) correspondence between the lines of a plane and the manifolds of a net, and obtaining the characteristics of the branch-point-curve. The number of manifolds of the net that have two hypernodes, one hyperbinode, two simple contacts or a stationary contact with another member of the net, are obtained. A multiple basis point of a net reduces all the characteristic numbers except the order of the jacobian. When the basis point is simple, this reduction occurs only for a net of curves in two dimensions. The amount of this reduction is found for any number of basis points of any given multiplicity.

44. Dr. J. M. Stetson: *Periodic conjugate nets.*

A net which can be transformed into itself by a sequence of Levy transformations, p of them using derivatives with respect to u, and q of them using derivatives with respect to v, we call periodic (p, q). Such nets exist in spaces of $p+q-1$ or fewer dimensions. We find many types of transformations of nets periodic (p, q), all having theorems of permutability. Reciprocally derived nets and nets periodic under Laplace transformations seem to be the only cases hitherto considered. All the transformations known for these are simple special cases of the results of this paper.

45. Professor C. E. Weatherburn: *On curvilinear congruences.*

In an earlier paper (*On congruences of curves*, Tôhoku Mathematical Journal, 1927) the author has shown how the theory of curvilinear congruences in ordinary space of three dimensions may be extended along the lines followed for rectilinear congruences, making use of oblique curvilinear coordinates. The present paper contains a further extension of the theory.

46. Dr. C. F. Roos (National Research Fellow): *A dynamical theory of economics.*

In his recent papers in mathematical economics, the author has been primarily concerned with the mathematical aspects of the problems of the new dynamical economics, developed by G. C. Evans and himself, and references to existing economics papers have been avoided. It seems advisable, therefore, to give a history of these problems from the view point of the mathematical economist. In an attempt to do this, the present paper gives ample references to all existing related works known to the author, and shows how the work of Evans and the author is a generalization of existing theories of static equilibrium.

47. Professor H. E. Bray: *On uniform absolute continuity.*

Given a summable function $f(x)$ and a family of continuous functions $f(\mu, x)$ such that $\lim_{\mu\to 0} f(\mu, x) = f(x)$ almost everywhere; if $f(\mu, x)$ is non-negative, a necessary and sufficient condition that $\lim_{\mu\to 0}\int_a^b f(\mu, x)dx = \int_a^b f(x)dx$ is (a): that the absolute continuity of the indefinite integral $\int f(\mu, x)dx$ be uniform for all μ (de la Vallée Poussin). Another well known condition which is sufficient, being more restrictive than (a), is (b): that a summable function $g(x)$ exist such that $f(\mu, x) < g(x)$ for all μ. In this paper we find a necessary condition, bearing upon $f(x)$ itself, in order that a function $f(\mu, x)$ belonging to a certain class and satisfying (a) may also satisfy (b). The class of approximations considered includes the Poisson integral and also the Landau and Weierstrass integrals as special cases.

48. Dr. Caroline E. Seely: *Kernels of positive type.*

The author shows that if a kernel $K(s, t)$ is such that an infinite number of its iterated kernels are of positive type with respect to all linear combinations of its principal functions, then the resolvent kernel has all its poles real and simple.

R. G. D. RICHARDSON,
Secretary

THE FIFTY-THIRD REGULAR MEETING OF THE SAN FRANCISCO SECTION

The fifty-third regular meeting of the San Francisco Section of the Society was held at the University of California on Saturday, October 29, 1927. The total attendance was forty-four, including the following twenty-three members of the Society:

Alderton, Bernstein, Biggerstaff, Blichfeldt, Buck, Cajori, Corbin, Edwards, M. W. Haskell, E. R. Hedrick, Hotelling, Frank Irwin, R. L. Jackson, Vern James, Langford, D. N. Lehmer, S. H. Levy, McCarty, F. R. Morris, Noble, Pauline Sperry, Morgan Ward, A. R. Williams.

In the absence of the regular Chairman, Professor Allardice, Professor Blichfeldt acted as Chairman in the morning session and Professor Cajori in the afternoon session.

The following officers were elected for the coming year: Chairman, E. T. Bell; Secretary, B. A. Bernstein; Program Committee, F. L. Griffin, W. A. Manning, J. H. McDonald, B. A. Bernstein (ex officio).

The date of the next meeting at the University of California was fixed as October 20, 1928.

Titles and abstracts of papers read at the meeting follow. Professor Hedrick's address was delivered at the request of the Program Committee. Professor Cajori's second paper and Dr. Trjitzinski's paper were read by title. Professor Uspensky was introduced by Professor Blichfeldt, and Miss Weiss by Professor Manning.

1. Professor E. R. Hedrick: *Extensions of the theory of functions of a complex variable.*

The traditional theory of functions of a complex variable is limited by assumptions originally made by men often not conscious of the restrictions they imply. One extension of the traditional theory is made by acceptance of the Dirichlet definition of a function. While this creates a theory coextensive with the theory of point transformations of a plane, many ideas of the function theory are extensible: among those mentioned are Riemann surfaces, the Tissot indicatrix, pseudo-Stieltes integrals and integration theorems, extensions of the Morera and other tests for analytic functions. A well known extension of the traditional theory is the theory of functions of several complex variables, treated briefly by several writers, and very extensively by Osgood in the second volume of his *Funktionentheorie.* Certain facts not foreshadowed by the traditional theory are mentioned. Efforts toward extension in other directions have led to the development of hypercomplex numbers on the one hand, and to extensive developments in the field of differential equations on the other hand. Many results in Differential Geometry, particularly those on conformal transformations, and those in the theory of surfaces associated with Beltrami's equations, may be regarded properly as direct extensions of the traditional theory. Finally, the attempts to extend the theory to three dimensions are reviewed, including recent papers by the author and others.

2. Professor H. Bateman: *Transverse seismic waves on the surface of a semi-infinite solid composed of heterogeneous material.*

This paper will appear in full in an early issue of this Bulletin.

3. Professor E. T. Bell: *On the arithmetic of abelian functions.*

The arithmetic applications of the abelian functions appear to be distinguished from those of the elliptic by this cardinal difference: the abelian functions in general yield properties of *simultaneous* diophantine equations, while the elliptic only exceptionally refer to such. An extremely fertile field for new properties of simultaneous diophantine systems is presented by the singular abelian functions (those whose periods are connected by one or more singular relations in the usual sense) and by the classic biquadratic relations between the thetas of more than one variable. The present note defines a new type of arithmetic invariance relevant to *simultaneous* representations in a set of forms in the *same* indeterminates, and illustrates this invariance for certain systems of quadratic forms in six indeterminates deduced from the singular abelian functions in two variables, the invariant of the singular relation having the value five.

4. Professor Florian Cajori: *Robert Burton's horoscope and the year of his birth.*

Whether Robert Burton was born in 1576, 1577 or 1578, the year of birth on his horoscope being illegible, was determined, for want of regular ephemeris, by comparing the positions of Jupiter and Saturn in his horoscope with those in other horoscopes. The year was found to be 1577.

5. Professor Florian Cajori: *Comparison of methods of determining calender dates by finger reckoning.*

In this paper Professor Cajori compares the procedures in determining calendar dates by use of the fingers, found in an anonymous writer of the fourteenth century, in Anianus, in C. Fernandez Tranquoso, and in B. Francisco de Ossorio.

6. Dr. Vern James: *Primitive linear homogeneous groups of variety two or three.*

The variety of a group is defined and this definition is shown to be consistent with the definition of the variety of an abelian group. Theorem 2 of the paper proves that there are no primitive groups of variety two in more than two variables. Theorem 3 proves that the order of a primitive group of variety three is not divisible by a prime number greater than 7. By two other theorems the order of a primitive group of variety three is reduced to the form $2^a 3^b$ when the number of variables exceeds four.

7. Professor C. H. Langford: *Some features of logical and mathematical propositions.*

Everyone recognizes that the propositions of logic and mathematics differ in certain important respects from propositions which occur in the natural sciences; but there is no very general agreement concerning the precise nature of these differences. This paper considers some views which are current concerning the nature, in certain respects, of the propositions of logic and mathematics; and it is especially concerned with an analysis of propositions which would commonly be said to be necessary propositions, and with an analysis of relations of entailing, incompatibility, and the like, as they occur among propositions and properties. Necessary and sufficient conditions for the truth of propositions of the kinds "p entails q" and "p is incompatible with q" are suggested.

8. Professor W. J. Trjitzinski: *Behavior at infinity of functions determined by their initial values.*

In a paper to appear in the Annals of Mathematics the author gave a representation of functions of a real variable which are determined by their initial values and in particular include quasi-analytic functions, in the form $\sum a_i f(m_i x)$, where m_i and $f(x)$ satisfy certain conditions. With additional restrictions on $f(x)$ the limit of a function so represented exists for x tending to infinity along the real axis. An explicit expression is available which enables us to study at infinity the classes of functions in question.

9. Professor J. V. Uspensky: *On a problem concerning the approximate evaluation of definite integrals.*

The author discusses the evaluation of certain definite integrals by means of continued fractions. Conditions on the integral are obtained to make the continued fraction regular and periodic.

10. Mr. Morgan Ward: *The general theory of recurring series.* Preliminary report.

The theory of linear difference equations of the nth order with coefficients in an arbitrary Galois field and elements matrices over the field-is developed with examples by postulational methods, and the structure and periods of the associated cycles of marks are determined. Connections are established with the Galois field theory, diophantine analysis and the theory of ideals.

11. Mr. Morgan Ward: *On the theory of ideal numbers.* Preliminary report.

A theory of ideal numbers is developed and applied to cubic, quartic and certain cyclotomic fields. Its connection with Zolotareff's and Dedekind's theories and its relation to results in other branches of algebraic arithmetic is given.

12. Miss Marie J. Weiss: *Primitive groups which contain substitutions of prime order p and of degree $6p$ or $7p$.*

In this paper the author proves that the degree of aprimitive permutation group (not alternating or symmetric) which contains a permutation of prime order $p(>7)$ and of degree $6p(7p)$ does not exceed $6p+6(7p+8)$.

B. A. BERNSTEIN,
Secretary of the Section

THE TWENTIETH REGULAR MEETING OF THE SOUTHWESTERN SECTION

The twentieth regular meeting of the Southwestern Section was held at Washington University on Saturday, November 26, 1927. The total attendance was thirty-five including the following twenty-two members:

Nola Anderson, Herman Betz, Florence Black, Brenke, Davisson, Dunkel, Gerst, Gouwens, Byron Ingold, Louis Ingold, Harry Levy, G. O. James, Jaeger, Osborn, Pennell, Rider, Edna Robinson, Roever, Eugene Stephens, Stouffer, J. S. Turner, Wyant.

Professor Roever occupied the chair being relieved during the sessions by Professor Brenke. The morning session was devoted to the reading of the papers listed below. During the afternoon session Professor E. B. Stouffer gave a special lecture by invitation of the program committee on *Some canonical forms and associated canonical expansions in projective differential geometry.*

The titles and abstracts of the papers read are given below. The papers of Dr. Ayres and Professor Whyburn were read by title.

1. Professor W. C. Brenke: *Polynomial solutions of a class of difference equations.*

The author considers linear difference equations of the second order, of the form $y_n=(\alpha_n+\beta_n x)y_{n-1}-\gamma_n y_{n-2}$ where the coefficients α_n, β_n, γ_n are independent of the parameter x. Solutions of this equation which are polynomials in x, when such exist, are obtained in explicit form as special cases of the secular determinant. Many well known polynomials may thus be written as determinants.

2. Professor W. H. Roever: *Some geometric relations concerning the theory of the Eötvös torsion balance.* Preliminary communication.

In the weight field of force of the earth the lines of force have a first curvature which differs slightly from zero, and the level surfaces have, at each point, principal curvatures which differ slightly from each other. Small though these differences be, they determine those horizontal components of the weight which constitute the moments that operate the Eötvös torsion balance. By means of certain geometric theorems, which are proved in this paper, it is possible to determine directly, without any analytic treatment, expressions for the moments which operate the balance, and also to detect a relation which is usually overlooked in the theory of the balance.

3. Professor Florence Black: *A reduced system of differential equations for the invariants of ternary forms.*

A homogeneous function of the coefficients of a ternary form of order n is invariant under the general linear transformation if and only if it is a solution of a certain complete system of nine linear partial differential equations of the first order. By the introduction of a set of new variables, F. Junker has reduced the number of differential equations in the complete system to seven. A comparison with the study made by E. B. Stouffer on invariants of linear homogeneous differential equations suggests a different choice of new variables. The author of the present paper has found that the introduction of these new variables serves to reduce the number of differential equations in the complete system to six. Moreover the resulting equations are far simpler in form than those obtained by Junker's reduction.

4. Professor Louis Ingold: *Surfaces in five-dimensional space.*

In this paper a vector function $f(u, v)$, depending on two scalar parameters, is considered. It is assumed that all third partial derivatives of the vector f are linearly expressible in terms of the five first and second partial derivatives. It is shown how the coefficients in these partial differential equations for the vector f may be expressed in terms of certain fundamental quantities which include the E, F, G of ordinary differential geometry.

5. Professor G. O. James: *Remarks on euclidean approximations to Einsteinian dynamics.*

This paper contains a tentative examination of the assumptions made in approximating the Einsteinian motion of a particle in the solar field by a potential function in euclidean space.

6. Mr. W. O. Pennell: *Operational systems.*

This paper outlines several classes of operators which follow the fundamental laws of algebra, namely, the distributive, associative, and commutative laws and the index laws. The definition of the operators includes the interpretation of fractional exponents, as well as integral exponents. Multiplication and division is an operational process, and an example is given of algebraic division performed by an operational method. The name "gamma operator" is suggested for the operator whose definition equation is $p^{\alpha}x^{n}=\Gamma(n+1)^{1/2}x^{n-\alpha}/\Gamma(n-\alpha+1)^{1/2}$. This operator parallels, in most respects, the operation of differentiation and integration; it also defines fractional integration and differentiation. The important properties of this operator are briefly outlined and discussed.

7. Dr. W. L. Ayres: *On continuous curves having certain properties.*

In this paper it is proved: (1) if every arc of a plane continuous curve M is a subset of a simple closed curve (an open curve) [a ray] of M then M is the entire plane or M is a simple closed curve (every point of M is a cut-point) [every point of M except possibly one is a cut-point]; (2) the boundary B of a plane domain D is uniformly regularly accessible from D with respect to the bounded continua of B if and only if (a) every component of B is a point or a simple continuous curve, (b) if J is any simple closed curve and $\epsilon>0$ is any number, then there are but a finite number of components of B of diameter $>\epsilon$ that have points interior to J; (3) a continuum containing no cut-point is a simple closed curve if every two points of it may be separated by two points. A necessary and sufficient condition in order that a continuous curve contain an open curve is given.

8. Professor G. T. Whyburn: *Concerning Menger regular curves.*

The following are among the results proved in this paper. (1) If no maximal cyclic curve of a continuous curve M contains an infinite collection of mutually exclusive simple closed curves, then (a) M is a Menger regular curve which is a *Baum im kleinen* at all save a countable number of points, (b) the ramification points of M are countable, and (c) every connected subset of M is arcwise connected. (2) A cyclically connected continuous curve is a *Baum im kleinen* if and only if it has only a finite number of ramification points. (3) The ramification points of a continuous curve M are countable if and only if the ramification points of each maximal cyclic curve of M are countable. (4) If the set of all ramification points of a continuous curve M is punctiform, then M is a Menger regular curve. (5) If a bounded continuum M is disconnected by the omission of any one of its countably infinite subsets, then M is a *Baum im kleinen.* (6) A continuous curve M is disconnected by the omission of any two of its points lying together on some simple closed curve in M if and only if every maximal cyclic curve of M is a simple closed curve.

9. Professor G. T. Whyburn; *Accessibility theorems.*

In this paper the author proves the following results. (1) If the limit point P of a point set R is not regularly accessible from R, then there exists a positive number ϵ and an infinite sequence of points of R which has P as a limit point and such that no two of these points can be joined in R by any arc of diameter $<\epsilon$. (2) Every one dimensional (Menger-Urysohn sense) point P of a continuum M in $n(n\geqq 3)$ dimensions is regularly accessible from each complementary domain of M whose boundary contains P. (3) In order that every point of a non-dense continuous curve M in a plane S should be regularly accessible from $S-M$ it is necessary and sufficient that M should contain no infinite collection of mutually exclusive simple closed curves. (4) If M is a plane non-dense continuous curve, G denotes the collection of all the complementary domains of M, and T denotes the limiting set of the collection G, then T is identical with the set of all those points of M at which M is not a *Baum im kleinen.*

10. Professor J. S. Turner: *On residues of iterated powers, modulo m.*

In this paper it is proved that if f belongs to a (modulo m), and if b is prime to f, then the residues of (I) $a, a^b, a^{b^2}, \cdots, a^{b^{f-2}}$, taken modulo m, are the same in the same order, as those of (II) $a, a^2, a^3, \cdots, a^{b-1}$, provided that f is a prime, and b is a primative root of f. In all other cases (I) contains fewer distinct residues than (II). Some consequences are pointed out.

E. B. STOUFFER,
Secretary of the Section

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THE VISITING LECTURESHIP OF THE AMERICAN MATHEMATICAL SOCIETY

At the annual meeting of the Society in Philadelphia last year, the Council approved the plan for creating a Visiting Lectureship of the American Mathematical Society. The primary purpose of the plan is to make available every year lectures by a distinguished mathematician at the mathematical centers which are actively interested. The financial responsibility is to rest upon the colleges and universities concerned, and is to be binding from year to year only. A Committee of the Society, consisting this year of the undersigned, is responsible for the nomination of a lecturer to the Council, for the issuing of the invitation to the Visiting Lecturer after he has been appointed by the Council, and for the oversight of the general arrangements.

By means of this plan it is hoped that knowledge of recent and important mathematical advances can be brought more quickly to the various centers, and that the opportunity for informal contacts will prove of scientific value.

The support of the plan obtained for this year has been notable. Professor Constantin Carathéodory of the University of Munich has accepted the invitation to become the first Visiting Lecturer. He is expected to arrive in America at the beginning of 1928, and to remain through the following summer. He will be on the staff at Harvard University during the entire second semester of this academic year, and on the staff of the University of California during the Summer Session at Berkeley during the summer of 1928. His schedule as Visiting Lecturer is being arranged so as to avoid conflicts with these duties. A schedule of lectures during the month of January, 1928, appears on page 122 of this number of this Bulletin.

G. D. Birkhoff, *Chairman*, G. A. Bliss, E. R. Hedrick.

MATHEMATICAL RIGOR, PAST AND PRESENT*

BY JAMES PIERPONT

1. *Introduction.* The Mengenlehre of Cantor, or the theory of aggregates (sets), has brought to light a number of paradoxes or antinomies which have profoundly disturbed the mathematical community for a quarter of a century. Mathematical reasoning which seemed quite sound has led to distressing contradictions. As long as one of these is unexplained in a final and conclusive manner there is no guarantee that other forms of reasoning now in good standing may not lead to other contradictions as yet unsuspected. For ages the reasoning employed in mathematics has been regarded as a model of logical perfection; mathematicians have prided themselves that their science is the one science so irrefutably established that never in its long history has it had to take a backward step.

No wonder then, that these paradoxes of Burali-Forti (1897), Russell, and others produced consternation in the camp of the mathematicians; no wonder that the foundations on which mathematics rest are being scrutinized as never before. Elaborate attempts are now in progress to give mathematics a foundation as secure as it was thought to have in the days of Euclid or of Weierstrass. Personally we do not believe that absolute rigor will ever be attained and if a time arrives when this is thought to be the case, it will be a sign that the race of mathematicians has declined. However, the aim of this paper is not to show this, but rather to pass in review some typical examples of what were regarded at the time as good mathematical demonstrations, somewhat as a historical pageant presents to our eyes famous persons in chronological sequence.

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* Presented by invitation of the Program Committee at the Annual Meeting of the Society, held in connection with the meetings of the American Association for the Advancement of Science, at Nashville, December 28, 1927.

2. *The English School of the Eighteenth Century.* Let us start with the eighteenth century, a century in which the efforts of mathematicians were largely spent in perfecting the calculus and in applying it to geometry, astronomy, and the natural sciences. The founders, Newton and Leibnitz, have left no clear account of its principles, and those who took up the new science promptly became embroiled in labyrinthine disputes. We who regard the infinitesimal calculus as nothing more than a calculus of limits can for the most part read into the obscure and scanty statements of Newton and Leibnitz what they perhaps wished to say. But their contemporaries and immediate followers had no clear idea of limits as we have, moreover their difficulties were increased by the fact that neither Newton* nor Leibnitz is quite consistent with himself.

Thus two schools arose; the English school, following Newton rested their reasoning on "the motion of bodies accelerated according to various hypotheses" and on "prime and ultimate ratios"; the continental school, following Leibnitz, rested their reasoning on infinitely small, non-archimedian quantities or "little zeros." Let us exhibit a few examples from the two schools; we shall take first the English.

A highly esteemed work was the *Treatise on Fluxions*, by Thomas Simpson (1st edition, 1737; 2d edition, 1776). As is well known, the proof that Newton gave in his *Principia* that $(uv)'=uv'+vu'$ is not satisfactory. This apparently was recognized by Simpson, who gives an alternate proof resting on the derivative of $y=x^2$.

On page 1 we read: "All kinds of magnitudes are to be considered as generated by the continual motion of some of their bounds or extremes; as a line by the motion of a point;

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* As for Newton, see an illuminating article by A. De Morgan, Philosophical Magazine, (4), vol. 4 (1852), p. 321.

a surface by the motion of a line; and a solid by the motion of a surface. Every quantity so generated is called a variable or flowing quantity; and the magnitude by which any flowing quantity would be uniformly increased in a given portion of time with the generating celerity at any proposed position or instant (was it from thence to continue invariable) is the fluxion of the said quantity at that position or instant."

Thus in our language the fluxion of a variable u at the time t is $(du/dt)\Delta t$. Simpson represents this by $\dot{u}$. Let us see now how Simpson gets the fluxion of $y=x^2$. On a straight, let the reader mark points A, r, R, B in order, and on another straight the points C, s, e, S, D. Simpson reasons textually as follows: "Conceive two points m, n to proceed at the same time from two points A, C, along the right lines AB and CD in such sort that $CS=y$ is always equal to the square of $AR=x$, which latter moves uniformly. Furthermore let r, s, R, S, be any contemporary positions of the generating points. If $rR=v$ we have $CS=y=x^2$, $Cs=(x-v)^2=x^2-2xv+v^2$ and hence $Ss=CS-Cs=2xv-v^2$. From whence we gather that while the point m moves over the distance v, the point n moves over the distance $2xv-v^2$. But this last distance (since the square of any quantity is known to increase faster in proportion than the root) is not described with an uniform motion (like the former) but with an accelerated one. It therefore is equal to, and may be taken to express, the uniform space that might be described with the mean celerity at some intermediate point e in the same time. Therefore, seeing the distances that might be described in equal times, with the uniform celerity of m and the mean celerity at e are as v to $2xv-v^2$, or as $\dot{x}$ to $2x\dot{x}-v\dot{x}$, it is evident that in the same time the point m would move uniformly over the distance $\dot{x}$ the other point n with its celerity at e would move uniformly over the distance $2x\dot{x}-v\dot{x}$. This being the case let r, R and s, S be now supposed to coincide, by the arrival of the generating points at R and S, then e (being always between s and S) will likewise coincide with S; and the distance $2x\dot{x}-v\dot{x}$ which might be uniformly described in the aforesaid time with the velocity at e (now at S) will become barely equal $2x\dot{x}$ which (by the definition) is equal to $\dot{y}$, the true fluxion of Cn or x^2."

To avoid the difficulty in Newton's proof, Simpson has introduced the point e, which merely replaces one stumbling block by another. To realize the great difference between this English school and the modern doctrine of limits, one has only to compare this bungling proof with the simple little proof in any calculus of today.

3. *The Continental School of the Eighteenth Century.* We now turn to the continental school founded by Leibnitz. Perhaps the first systematic presentation of Leibnitz's methods to be published was the *Analyse des Infiniment Petits*, by the Marquis de l'Hospital (1696), which enjoyed the most widespread popularity. An English translation *The Method of Fluxions*, by E. Stone, appeared in 1730. De l'Hospital's book was largely founded on a little treatise by John Bernoulli, *Die Differentialrechnung*,* written in 1691–92, but only published in 1922 on the occasion of the tercentenary celebration of the Bernoulli family in Basel.

The treatise begins on page 11 with three postulates, of which the first reads: "A quantity which is diminished or increased by an infinitely small quantity is neither increased nor decreased." On page 12, we find: "The differential of x^2 is $2x\,dx$, which is proved thus: $(x+e)^2=x^2+2ex+e^2$, subtracting x^2 gives $2ex+e^2$ as remainder, and this on account of postulate 1 is $2ex=2xdx$.

"The differential of x/y is $(y\,dx-x\,dy)/y^2$. For if we subtract x/y from $(x+e)/(y+f)$ we get $(ey-fx)/(y^2+fy)=$ by postulate 1, $(ey-fx)/y^2=(ydx-xdy)/y^2$."

The next writer of this school whom we wish to consider is the immortal Euler, one of the greatest figures in the whole history of mathematics. Let us look at his *Institutiones*

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* We quote from *Die Differentialrechnung von Johann Bernoulli*, Ostwald's Klassiker, Nr. 211.

Calculi Differentialis (1775). In the preface he considers the differential of $y=x^2$. He gives x the increment ω, the corresponding increment of y is $2x\omega+\omega^2=\eta$; the ratio η to ω is $2x+\omega$ to 1. Hence this ratio approaches $2x$ the smaller ω is taken. He is thus led to define the differential calculus as the method of determining the ratio of evanescent increments. These evanescent quantities are called differentials "que, cum quantitate destituantur, infinite parva quoque dicuntur, quae igitur sua natura ita sunt interpretanda, ut omnino nulla seu nihilo aequalia reputentur." He admonishes the reader to bear in mind that these differentials are absolutely zero and that nothing can be inferred from them other than their mutual ratio, which is in the end reduced to a finite quantity (verum perpetuo tenendum est, cum haec differentialia absoluta sint nihila, ex iis nihil aliud concludi nisi eorum rationes mutual, quae utique ad quantitates finitas deducuntur).

Thus Euler accepted unqualifiedly the notion that there exist quantities which are absolutely zero, yet whose ratios are finite numbers. The reader who wishes further information regarding Euler's views may consult Chapter III of the above work, entitled *De infinitis atque infinite parvis.* He encourages the reader here by remarking that this notion does not hide so great a mystery as is commonly thought, and which in the mind of many renders the calculus suspect. Any doubts which may have arisen will be shown devoid of foundation as the theory is developed. This reminds one of d'Alembert's statement: "Allez en avant, la foi vous viendra."

Euler's *Differential and Integral Calculus* and his *Introduction to Infinitesimal Analysis* were the standard textbooks of the day; they were in everybody's hands. The fame of the author and their great popularity renders it necessary to give another example of his style of reasoning.

How does he find the differential of $y=\log x$? This is treated in §180 of the *Institutiones*. Replacing x by $x+dx$ gives

$$dy = \log(x+dx) - \log x = \log(1+dx/x).$$

Now in Chapter VII of volume I of the *Introduction to Analysis* he has found

$$\text{(1)} \qquad \log(1+z) = z - z^2/2 + z^3/3 - z^4/4 + \cdots .$$

Replacing here z by dx/x gives

$$dy = dx/x - dx^2/(2x^2) + dx^3/(3x^3) - \cdots .$$

As all the terms of this series beyond the first are evanescent we have

$$d \cdot \log x = dx/x.$$

We turn thus to the *Introductio in Analysin Infinitorum* (1748) to learn how the series (1) is established. We find (§115, seq.) the demonstration rests on Newton's celebrated binomial formula

$$\text{(2)} \qquad (1+u)^m = 1 + mu + \frac{m(m-1)}{1 \cdot 2} u^2 + \cdots .$$

Euler gives no proof of this, which in those days was proved in algebra.* Let us see, however, how he uses (2) to prove (1). Euler starts with the relation $a^\omega = 1 + k\omega$, ω infinitely small; then a being taken as base, $\omega = \log(1+k\omega)$. Hence

$$a^{i\omega} = (1 + k\omega)^i = 1 + \frac{i}{1} \cdot k\omega + \frac{i(i-1)}{1 \cdot 2} k^2\omega^2 + \cdots .$$

Set $i = z/\omega$, z finite; then $\omega i = z$, and

$$a^z = 1 + kz + \frac{(i-1)}{2i} k^2 z^2 + \frac{(i-1)(i-2)}{2i \cdot 3i} k^3 z^3 + \cdots .$$

As i is a "number larger than any assignable quantity"

$$\frac{i-1}{2i} = \frac{1}{2}, \quad \frac{(i-1)(i-2)}{2i \cdot 3i} = \frac{1}{2 \cdot 3}, \text{ etc.},$$

hence

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[*] Judged by modern standards, these demonstrations are quite worthless.

$$(3) \qquad a^z = 1 + kz + k^2z^2/2! + k^3z^3/3! + \cdots .$$

The larger i is taken, the nearer $(1+k\omega)^i$ is to 1. Euler thus sets

$$(1 + k\omega)^i = 1 + x, \quad \therefore k\omega = (1 + x)^{1/i} - 1,$$
$$i\omega = i[(1 + x)^{1/i} - 1]/k.$$

Hence

$$\log (1 + x) = i(1 + x)^{1/i}/k - i/k.$$

Now

$$(1 + x)^{1/i} = 1 + \frac{x}{i} - \frac{(i-1)}{2i^2}x^2 + \cdots ;$$

hence if i is infinitely large

$$i(1 + x)^{1/i} = i + x - x^2/2 + \cdots ,$$

and thus

$$(4) \qquad \log (1 + x) = [x - x^2/2 + x^3/3 - \cdots]/k.$$

Setting $z=1$ in (3) gives

$$a = 1 + k + k^2/2! + k^3/3! + \cdots .$$

As a has been left arbitrary, we may take it so that $k=1$; calling this value e, we have

$$e = 1 + 1 + 1/2! + 1/3! + \cdots ,$$

while (4) goes over into the desired formula (1).

This demonstration from a Weierstrassian standpoint is about as bad as it could be; but then, are we not told now by the intuitionalists that a large part of the Weierstrassian mathematics is devoid of proof, if indeed it is not nonsense? Let us therefore be charitable. We have taken space to give this proof because it is entirely typical.

We have not space to follow the further history of "the little zeros" of Leibnitz and Euler. As Americans, it may interest us sufficiently to note that our greatest mathematician of earlier days, B. Peirce, used them without hesitation in his remarkable treatise *Curves, Functions, and Forces* (2 vols., 1841), as may be seen in Book II, Chapter 2, page 172 of volume 1.

4. *The Method of Lagrange.* Another eddy in the current of mathematical thought was produced by Lagrange's *Théorie des Fonctions Analytiques* (1st edition, 1797; 2d edition, 1813). He is dissatisfied with the little zeros of Leibnitz, Bernoulli, and Euler "which although correct in reality are not sufficiently clear to serve as foundation of a science whose certitude should rest on its own evidence." He is as little satisfied with the fluxions of Newton, which introduces a foreign notion, that of motion; moreover "one can see by the learned *Treatise on Fluxions* by Maclaurin how difficult it is to demonstrate the method of fluxions and how many ingenious artifices we must employ to demonstrate the different parts of this method."

Lagrange therefore proposes to get rid of little zeros and of limits at one stroke by founding the calculus on the development of a function in a power series (Taylor's or Maclaurin's). With becoming modesty he remarks that it is strange that this method of establishing the calculus did not occur to mathematicians earlier, especially to Newton, the inventor of the method of series and of fluxions. Let us see what this grand idea is.

If in $f(x)$ we replace x by $x+h$, "it becomes $f(x+h)$ and by the theory of series one can develop it in a series

$$(1) \qquad f(x+h) = f(x) + ph + qh^2 + rh^3 + sh^4 + \cdots ."$$

"Not to make gratuitous assumptions," Lagrange begins by examining the form of the series (1), and shows that the exponents of h are in fact integers and not fractions. He next studies the coefficients $p, q, r, \cdots$. To this end he observes that if we replace h by $h+k$ in (1) we get

$$(2) \quad \begin{cases} f(x+h+k) = f(x) + (h+k)p + (h+k)^2q + \cdots \\ \qquad\qquad = f(x) + hp + h^2q + h^3r + \cdots \\ \qquad\qquad\quad + kp + 2hkq + 3h^2kr + \cdots \text{ etc.} \end{cases}$$

Also, if we replace x by $x+k$ in (1), we get

$$f(x+h+k)=f(x+k)+hp(x+k)+h^2q(x+k)+\cdots. \tag{3}$$

Now

$$\begin{aligned} f(x+k) &= f(x)+kf_1(x)+\cdots, \\ p(x+k) &= p(x)+kp_1(x)+\cdots, \\ q(x+k) &= q(x)+kq_1(x)+\cdots, \text{ etc.} \end{aligned}$$

These in (3) give

$$\begin{aligned} f(x+h+k) = f(x)+hp+h^2q+h^3r+\cdots \\ +kf_1(x)+hkp_1(x)+h^2kq_1(x)+\cdots. \end{aligned} \tag{4}$$

Comparing (2) and (4) gives

$$p=f_1(x),\quad q=\tfrac{1}{2}p_1(x),\quad r=\tfrac{1}{3}q_1(x),\cdots. \tag{5}$$

Lagrange now observes that we get q from p in the same way that we get p from $f(x)$, and a similar remark holds for the other coefficients $r, s, \cdots$ in (1).

He calls $f_1(x)$ the first derivative of $f(x)$; it is merely the first coefficient in the development (1), and not a differential coefficient obtained by a passage to the limit. He denotes it by $f'(x)$. This function also has a first derivative which he denotes by $f''(x)$ etc. Thus the relations (5) give

$$p=f'(x),\quad q=\frac{1}{2!}f''(x),\quad r=\frac{1}{3!}f'''(x),\cdots,$$

which in (1) give

$$f(x+h)=f(x)+hf'(x)+\frac{1}{2!}f''(x)+\cdots.$$

Lagrange now observes: "This new expression has the advantage of showing how the terms of the series depend on each other, and especially how when one knows how to form the first derivative function, one can form all the derivative functions which enter the series." Then a little later he adds: "For one who knows the rudiments of the (traditional) differential calculus, it is clear that these derivative functions coincide with

$$dy/dx, \quad d^2y/dx^2, \cdots."$$

This then is the grand scheme: no more infinitesimals, no more prime and ultimate ratios whose principles have given rise to endless disputes. Differential coefficients are merely the coefficients in the development of a function.

When a modern reader looks over reasoning like this and bears in mind that Lagrange was one of the greatest mathematicians of all time, he is amazed. The great gulf that separates mathematical reasoning of to-day from that of date 1813 is brought home very clearly to him. The consoling feature about the work that we have seen thus far is this: the results are right although the reasoning is faulty. The intuition of these great men is far in advance of their logic. Is it not quite likely that a similar state of affairs holds in the theory of aggregates today?

Lagrange's method of development of the calculus free from the knotty questions regarding infinitesimals and limits was received with considerable favor. It suffers however a mortal defect. It rests upon the assumption that a given function can be developed in a power series, and there is no known method of deciding this question independently of the thing he wishes to avoid, namely limits.

5. *The Standpoint of Cauchy.* With Euler and Lagrange, a well defined period in the history of mathematics is closed. Euler is the last great formalist. We have just seen that Lagrange is fully alive to the objectionable reasoning of many of his contemporaries, and we have witnessed his vain efforts to give the calculus "toute la rigueur des démonstrations des Anciens," as he says. The true method was not Newton's, nor Euler's, nor Lagrange's, and yet it lay close at hand and had already been clearly stated by

d'Alembert in the *Encyclopédie*,* in his article *Différentiel.* Here he writes: "What we most need to treat here is the metaphysics of the differential calculus. This metaphysics about which one has written so much is more important and perhaps more difficult to develop than even the rules of this calculus." He remarks that certain mathematicians cannot admit the suppositions which are made concerning infinitesimals, as they are false in principle and capable of leading to wrong results. He then goes on to say: "But when one observes that all the facts that have been discovered by ordinary geometry may be established by the calculus much more easily, one cannot help concluding that this calculus furnishes certain simple and exact methods, and rests on principles just as simple and certain."

These he thinks form the metaphysics of Newton, which Newton has revealed only in part. D'Alembert reads into Newton what Newton ought to have said but which (if we are not mistaken) no one had seen before; and this we believe is d'Alembert's great discovery. He says: "Newton has never regarded the differential calculus as a calculus of infinitesimals, but as a method of prime and ultimate ratios, that is to say, a method of finding the limit of these ratios. Newton has never differentiated quantities, but only equations, since every equation embraces a relation between two variables, and the differentiation of equations is merely finding the limit of the ratios between finite differences of the two variables which the equation involves."

In another article, on *Limits*, he says: "The theory of limits is the true metaphysics of the differential calculus."

The first treatise to adopt this standpoint to the exclusion of any other was (as far as I have ascertained) the *Traité Élémentaire* of Lacroix (2d edition, 1806; I have not seen the first edition). It is, however, with Cauchy that the new era begins. In his *Cours d'Analyse* (1821), his *Résumé des Leçons sur le Calcul Infinitésimal* (1823), his *Leçons sur le Calcul Différentiel* (1829), and in the *Leçons de Calcul Différentiel et de Calcul Intégral* (2 vols., 1840), by his pupil L'Abbé Moigno, we behold for the first time the foundations of the calculus developed with a rigor which is near to that of our own today. De Moigno in his *Introduction* observes: "Before Cauchy published his treatises, the demonstrations of the fundamental theorems of the calculus rested too often on the consideration of certain series which were used without discernment, without having examined their convergence, or if indeed they represented the functions which gave them birth. It was a veritable abuse against which Cauchy never ceased to protest. Never did he employ the development of a function in a series without first establishing its possibility, its form, its convergence, in a word its right to represent the given function." This we think is rather exaggerated, but it is true on the whole.

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* *Encyclopédie ou Dictionnaire Raisonné des Sciences*, vol. 10, Geneva, 1770. This is, I believe, a pirated copy of the original French edition. I have not seen this latter; probably d'Alembert's article on the calculus was published for the first time some years earlier.

If one asks what is one of Cauchy's most obvious oversights as to rigor, we would say that it is overlooking the care that one must take in a very common process in analysis, i.e., the interchange in the order of passing to the limit in a double limit. Thus in his *Cours d'Analyse* (p. 131), he states that $F(x)=\sum u_n(x)$ is continuous if F is convergent and the u_n are continuous. Under the same conditions (*Résumé des Leçons*, Œuvres (2), vol. 4, p. 237),

$$\int_a^b F\,dx = \sum \int_a^b u_n\,dx.$$

Also (ibid., p. 195)

$$\frac{\partial}{\partial u}\int_a^b f(x,u)\,dx = \int_a^b \frac{\partial f}{\partial u}\,dx,$$

etc.

The justification of these and similar interchanges of limits rests on the notion of uniform convergence, which was not discovered till a later date (Stokes, 1847; Seidel, 1848; Cauchy, 1853).

Cauchy's standard of rigor was immeasurably in advance of his contemporaries (we except Gauss of course); it served as model for a generation of mathematicians. Abel,* writing to Holmboe from Paris (1826), says of Cauchy: "He is at present the only one who knows how mathematics should be treated." In another letter,† speaking of Taylor's development, he says: "I have found only a single rigorous demonstration, that of M. Cauchy."

6. *The Standards of Weierstrass.* However, even after making various minor improvements relative to double limits, etc., the last word on rigor had not been said; there was a still higher standard for which to strive. Indeed, so vastly superior was this new standard in the minds of some of its proselytes that they looked on the rest of their fellow mathematicians as living in utter darkness.

What was the new doctrine and who was its founder? Both questions can be answered by one word, it is a name revered and honored the world around, Weierstrass.

What the Weierstrassian doctrine is, is too well known to you for me to dwell upon. I may be allowed, however, to mention one or two matters which will come up for discussion when we reach the next era, the era of to-day. The great step in advance that Weierstrass took was to arithmetize analysis. Before then, many analytical facts were accepted as self evident. For example: if $f(x)$ is continuous in (a, b) and has opposite signs at a and b, then $f(x) = 0$ at some point within the interval. One has merely to picture the graph of this function to see intuitively the truth of this theorem. That this type of reasoning is inadmissible rests on the following observation. Let us define with Cauchy: The function $f(x)$ is continuous for $x = c$ if $\lim f(x) = f(c)$ for $x \to c$. One assumes now that such continuous functions are coextensive with the class of functions as pictured by our geometric intuition. Many examples show us the contrary.

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* Œuvres, vol. 2, p. 250.

† Ibid., p. 257

Perhaps the one that startled the easy going world the most was Weierstrass' continuous function having at no point a derivative. Another hardly less remarkable example was Peano's curve which passes through every point of a square. Finally we note a fundamental problem: What is the meaning of the limit of a sequence $a_1, a_2, a_3, \cdots$?

To arithmetize analysis one had to ban geometric reasoning; all proof must rest entirely on pure analysis. This forced Weierstrass to find an arithmetic equivalent of our geometric notion of an interval or segment of a straight line, i.e., to lay down an arithmetic theory of irrational numbers. Only then was he able to prove arithmetically the simplest properties of continuous functions, properties which had been accepted previously as too obvious to require even a passing remark. To meet the requirements of the new standards of rigor a great deal of the reasoning which had been regarded as valid had to be supplemented, or when too bad, to be replaced by considerations of a different type. If one wishes to form an idea how great these changes were, one cannot do better than to compare attentively the first edition of C. Jordan's *Cours d'Analyse* (1882–1887) with the second (1893–1896). In the calculus of variations the devastation produced by the teaching of Weierstrass was even greater, and the work of repair far more difficult.

By the end of the last century the Weierstrassian standards of rigor had won over practically the whole world. It was believed that absolute rigor had been reached, that on the foundations of Weierstrass our mathematical edifice, whose giddy summits seem to reach the clouds, rests so securely that nothing can disturb its massive repose. Illustrious names can be cited to support this statement; one will suffice. In his address before the Paris Congress (1900), Poincaré,* reviewing the arithmetization of mathematics, asks: "Have we at last attained absolute rigor? At each stage of its evolution our forerunners believed they too had attained it. If they were deceived, are we not deceived like them?" After discussing the question he gives his verdict: "One may say to-day that absolute rigor has been attained." Alas, to-day, a scarce quarter of a century after these memorable words were uttered some there are who think we have been living in a fool's paradise. The mighty edifice is tottering, they think, perhaps soon to fall leaving a horrible ruin in its place. I personally do not believe this to be the case and this view is shared by most of the mathematicians of to-day. And yet when one hears one of the greatest living mathematicians calmly telling the world that a considerable part of our analysis is devoid of proof, if it is not nonsense, and when one beholds the mighty efforts which the champions of Weierstrass are making to repel these attacks, it is only reasonable, in view of such facts, to ask ourselves, "Is all well?"

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* Compte Rendu du Deuxième Congrès International des Mathématiciens, p. 115.

We shall use the remainder of this address to give an account of this last stage in the evolution of mathematical rigor. It goes back to Kronecker, who confessed one day to Netto* that he had spent far more time thinking in philosophy than he had in mathematics. Philosophy and mathematics are not good companions.

If I had entered more fully into the early history of our theme, you would doubtless have been bored by a great deal of talk about its metaphysics. When the mathematician has not been clear in his own mind he has had recourse in the past to metaphysics. Bertrand Russell once humorously defined philosophy as the science that makes simple things complicated. Now the mathematician trained in the school of Weierstrass was fond of referring to his science as the absolutely clear science. Any attempt to drag in metaphysical speculation was resented with indignant energy. How the times have changed! O tempora, o mores! Now see what philosophy did to Kronecker; later we shall see what it has done to Brouwer and Weyl, and what a revolution it wishes to bring about.

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* E. Netto, Mathematical Papers, International Mathematical Congress, Chicago, 1893.

7. *Kronecker's Position.* To make Kronecker's position clear we may illustrate it by an example taken from algebra. Every one knows how fundamental the notion of reducibility is. Before Kronecker, algebraicists were content to say a polynomial $f(x)$ might have a rational factor, in which case $f(x)$ is reducible; in the contrary case it is irreducible. This is merely a logical definition of these terms; it is a case of "either," "or" with no means in sight as far as the definition is concerned to decide whether the given $f(x)$ is reducible or is not.

In his great Festschrift *Grundzüge einer arithmetischen Theorie der algebraischen Grössen,* Journal für Mathematik, vol. 92 (1882), Kronecker announces for the first time the new doctrine: "The definition of reducibility given in §1 is devoid of a sure foundation until a method is given by means of which it can be decided whether a given function is irreducible or not by the definition." In a footnote, he calls attention to the fact that "a similar need (often indeed unnoticed) is manifest in many other cases, in definition as well as in proof, and I shall take this up on another occasion in a general and careful manner." Only to a very minor degree did Kronecker carry out his intention, and that only in the field of algebra and algebraic numbers. In the great field of analysis he did nothing except to criticize in lectures and conversation the work of his contemporaries. Let us see how deep the new program cuts. The foundation of exact analysis is the real number system, whether defined as by Weierstrass, or as by Dedekind, or as by Méray and Cantor. Of two real numbers a and b, either $a=b$ or $a>b$ or $a<b$, and definitions are given for each case. According to Kronecker they are only definitions in appearance, since these definitions do not give the means of deciding in each case which of these three alternatives holds.

Another example is the fundamental theorem of Bolzano-Weierstrass relative to the upper and lower limits of a limited function. The well known proof consists in establishing the logical existence of a sequence of intervals one within the other, but it does not in general give the means of calculating these limits. The definition, therefore, is not a valid definition and must be rejected, from Kronecker's standpoint.

In the same year (1882) that Kronecker promulgated his new doctrine, Lindemann succeeded in showing that π is not algebraic, and so showed the futility of trying to square the circle. In a conversation with Lindemann,* Kronecker asks "Of what use is your beautiful investigation regarding π? Why study such problems, since irrational numbers are nonexistent?" Kronecker's attitude is made still clearer by the following extract of a letter of Weierstrass† to Mme. Kowalevski (1885): "I say that a so-called irrational number has as real an existence as any other object of the mind; Kronecker on the contrary regards it now as an axiom that only equations between whole numbers exist . . . .

"It makes the matter worse that Kronecker uses his authority to support the view that all who have labored on the foundations of the function theory are sinners before the Lord. When a person like Christoffel says that in twenty or thirty years the present function theory will be buried, and that analysis will be reduced to a theory of forms, one answers with a shrug of the shoulders; but when Kronecker makes this assertion which I reproduce word for word: 'If I still have the time and the energy, I will myself show the mathematical world that not only geometry but also arithmetic can point the path to analysis, and certainly a more rigorous one. If I cannot do this, then another will who comes after me, and the world will recognize the inexactitude of the types of proof now employed in analysis,' such a statement I say,

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* H. Poincaré, *Wissenschaft und Hypothese*, 3d edition, 1914, p. 252.

† G. Mittag-Leffler, *Une page de la vie de Weierstrass*, Paris Congress (1900), pp. 150–151.

. . . *is not only humiliating to those who are thus recommended to acknowledge their error and abjure what has been the object of their unremitting thoughts and efforts, but it is also a direct invitation to the younger generation to leave their present leaders and to group themselves about him as the apostle of a new doctrine which, it is true, has yet to be established. Really this is sad and fills me with bitter pain."* The revolutionary movement inaugurated by Kronecker in 1882 apparently died an early death from sheer inanition. It was easy for him dogmatically to assert: "Definitions must contain the means of reaching a decision in a finite number of steps and existence proofs must be conducted so that the quantity in question can be calculated with any required degree of accuracy," but outside of algebra he took no steps to realize his program, nor did any one for a quarter of a century seriously make the attempt. We therefore leave this movement for the present and consider briefly quite another subject.

8. *Rigor in Geometry.* So far we have said nothing of rigor in geometry. The reason is obvious; more than two thousand years before, geometry went through the process which we have just sketched for analysis. In the *Elements* of Euclid we have the results of a long period of critical analysis of the most acute Greek minds. After the downfall of the antique world, the intellectual world of Western Europe lay buried in darkness till the great awakening in the sixteenth and seventeenth centuries. No wonder that the achievements of Greek learning seemed almost superhuman to a people who had outgrown the futile subtleties of scholasticism. To the mathematicians of the Renaissance and long after, the *Elements* of Euclid was a work of superlative excellence. Let the following serve as an example.* It is "a work whose propositions have such an admirable connection and dependence that take away but one, and the whole falls; whose method is the most just, admitting and advancing nothing

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* Benjamin Martin, *Biographia Philosophica,* London, 1764, p. 56.

without a demonstration, and no demonstration but from what foregoes, and these so convincing, elegant, and perspicuous that it is beyond the skill of man to contrive better. Here the most artful and diligent carpers have never been able to set a footing."

However the "artful and diligent carpers" have found first one "footing" and then another, so that to-day it is generally recognized that the *Elements* are far, very far, from being a perfect work. Definitions are given which do not define, axioms are implied but not stated, certain proofs are needlessly complicated. Perhaps the most unsatisfactory feature relates to congruence and the implication of motion or displacement. On the other hand there is much that we to-day admire not with the blind idolatry of the past but with the critical knowledge of a connoisseur. We have a great advantage over the Greeks therein that a study of various non-euclidean geometries has enabled us to realize the difficulties which beset an entirely rigorous treatment of euclidean geometry.

There is no space in this address to discuss the history of the critical movement in geometry; we must however mention one aspect of it on account of its great importance in the thought of to-day relative to the foundations of analysis. We refer to the axiomatic treatment of the foundations of geometry which reached its highest excellence in Hilbert's *Grundlagen der Geometrie* (1899). Instead of trying to define a point, a straight, a plane, this method introduces three sets of things and subjects them to certain relations called axioms. One has no mental picture of these things; the reasoning makes no call on our geometric intuition, it is purely formal. On this account it is absolutely necessary to give a proof of freedom of contradiction and this is done by showing that any contradiction would involve a contradiction in arithmetic.

9. *Arithmetic.* The non-contradiction of the axioms of geometry is thus referred to the axioms of our number

system. Are the laws of arithmetic non-contradictory? However far we carry our developments, can we ever arrive at a result $1=0$, for example? No one in his senses has ever believed this; yet mathematics is not a science which rests on faith, but on proof. We are thus led back to our main theme, the foundations of analysis. A most important part of this theory relates to the number system.

We may begin our discussion by asking "What are the integers 1, 2, 3, · · · ?" The question is as elusive as the question, "What is a straight line?" We do not expect the philosophers to agree as to what a number is, and indeed they have not agreed; but we do expect the mathematicians to be unanimous. Do not numbers form the very basis of all analysis? E. G. Husserl in his *Philosophie der Arithmetik* (1891) observes "Not one significant question do I know which those concerned have answered with even tolerable harmony."

The first attempt to establish rigorously the laws of arithmetic was, as far as I know, by R. Dedekind in *Was sind und was sollen die Zahlen* (1887). He bases his development on infinite aggregates, i.e., as he defines them, on sets which can be put in uniform correspondence with a subset. It was incumbent on him to prove the existence of such sets. His reasoning (page 17) rests on the set of all things and is thus open to the objection of Burali-Forti's paradox.

G. Frege's *Grundgesetze der Arithmetik* (vol. 1, 1893; vol. 2, 1903) also makes use of the aggregate theory. At the close of vol. 2 (p. 253) he remarks "A scientist can hardly meet with anything more undesirable than to have the foundation give way just as the work is finished. In this position I was put by a letter from Mr. Bertrand Russell as the work was nearly through the press." In his despair Frege's only comfort is "Solatium misereris, socios habuisse malorum," if indeed this is a comfort. His criticism of his predecessors was merciless; what a bitter pill it must have been for him to illustrate the old adage, "Errare humanum est!" Russell's criticism has to do with the paradox named after him, namely, the set of all sets which are not members of themselves.

These unsuccessful attempts of Dedekind and Frege to found the number system on the notion of infinite aggregates have brought us now face to face with the notorious contradictions in Cantor's Mengenlehre which we referred to at the beginning of this address. Paradoxes were no new thing in philosophy. Did not Epimenides, the Cretan, say that all Cretans are liars, and did not Zeno the Eleate show that Achilles could not overtake the tortoise? They were something very new, however, in mathematics. Burali-Forti's paradox (1897) was like a bomb, soon to be followed by other bombs of like nature. Consternation spread among mathematicians; what was to be done? The greatest authorities were called in consultation; their diagnoses were poles apart: quot homines, tot sententiae. Poincaré* at the Rome Congress (1908) went so far as to say "Later generations will regard the Mengenlehre as a disease from which one has recovered." To get rid of these paradoxes, various lines of action have been adopted. One may do as Cantor, Dedekind and Frege did: give up, quit the field. For such we would recall a saying of DeMorgan† prompted by the contradictions arising from divergent series: "The history of algebra shows us that nothing is more unsound than the rejection of any method which naturally arises on account of one or more apparently valid cases in which such method leads to erroneous results. Such cases should indeed teach caution, but not rejection; if the latter had been preferred to the former, negative quantities, and still more their square roots, would have been an effectual bar to the progress of algebra · · · and those immense fields of analysis over which even the rejectors of divergent series now range without fear, would have been not so much as discovered, much less cultivated and settled."

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* O. Hölder, *Die mathematische Methode*, Berlin, 1924, p. 556.

† A. DeMorgan, *Differential and Integral Calculus*, London, 1842, p. 566.

Other lines of action rest on the answer to the query: when is a mathematical object or process defined? Some say, its definition can employ but a finite number of words. For example Poincaré, discussing Zermelo's well ordering of the continuum said:* "There are two cases; either one asserts that the method of well ordering is expressible in finite terms ('endlich aussagbar'), in which case the assertion is not proved; · · · or we allow the possibility that the method is not 'endlich aussagbar'? In this case I can attach no meaning to the procedure; for me it is only empty words."

In another place,† Poincaré asks: "Is it possible to reason on objects which cannot be defined by a finite number of words? Is it even possible to speak of them, knowing what one is speaking of, pronouncing only empty words? Or on the contrary should one not regard them as unthinkable? For my part, I do not hesitate to respond that they are pure nonentities." Mathematicians of this stamp are called finitists; their position is hotly attacked by the Cantorians.

Poincaré holds that the paradoxes of the theory of aggregates are due to the use of non-predicative definitions. In his *Science et Méthode* (Paris, 1912), page 207, he says: "Thus the definitions which should be regarded as non-predicative are those which contain a vicious circle." On page 208 he declares: "A definition which contains a vicious circle defines nothing." This view is also held by B. Russell. The Cantorians claim that this position cannot be maintained, since without such definitions modern analysis would be robbed of some of its most valuable results.‡

10. *The Logistic Group.* Without making too fine distinctions we may say that those engaged in laying the new foundations of analysis fall into three groups:

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* H. Poincaré, *Sechs Vortrage*, Leipzig, 1910, p. 48; they were held at Göttingen in 1909.

† *Dernières Pênsées*, Paris, 1913, p. 132.

‡ E. Zermelo, *Neuer Beweis fur de Moglichkeit einer Wohlordnung*, Mathematische Annalen, vol. 65 (1907), p. 107.

(1) The logistic: the Italian school is led by Peano, the English by Russell and Whitehead.

(2) The axiomatic: led by Hilbert.

(3) The intuitionist: led by Brouwer.

The first point to note about the logistic group is the fact that their work is written in a sign-language invented by Peano and extended by Russell. Relatively few people are willing to learn this language and these works therefore are for the most part unknown. We may summarize some of their leading ideas as follows. Their cardinal idea is that mathematics is a part of logic. The concepts and processes of mathematics on careful analysis are found to be few in number and to admit a symbolic treatment analogous to algebraic manipulations. By giving each symbol a precise unique meaning one hopes to avoid pitfalls that beset ordinary mathematical reasoning due to the ambiguity of common language. Moreover the symbolism itself is an aid in reasoning in the same way that algebraic symbols help us in ordinary arithmetic. The logistic program is to reduce all mathematics to symbolic logic; this has already been partly accomplished.* One of the great difficulties in this program is to avoid the contradictions and paradoxes of the theory of aggregates. Russell observed that these contradictions could all be avoided by using his vicious circle principle.† "Whatever involves all of a collection must not be one of the collection." To make this observation effective he has invented a theory of types. In this theory propositional functions and classes form a hierarchy according to their possible arguments, also a distinction is made between the various functions belonging to the same argument. By this means all known contradictions of the aggregate theory are avoided, and mathematical induction is established. In order to establish other parts of analysis, for example the irrational numbers, Russell is obliged to introduce a certain axiom, the axiom of reducibility. This axiom states that any combination or disjunction of predicates is equivalent to a single predicate. By this means the order of a nonpredicative function can be lowered by one, so that after a finite number of steps we reach an equivalent predicative function. This axiom of reducibility seems to have excited universal opposition. One author states that its introduction is an act of harikari. Ramsey* holds that there is no reason to suppose that it is true, and if it were, it would be a happy accident and not a logical necessity. It has no place in mathematics, and what cannot be proved without it cannot be regarded as proved at all. Ramsey believes he has discovered how the work of Wittgenstein† can be utilized so as to free the *Principia* from the objections which have caused its rejection by the majority of German authorities.

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* Formulaire de Mathématique, vol. 1, 1895; vol. 2, 1897–99; vol. 3, 1901; vol. 4, 1902–03. The different parts are written by a variety of persons, Peano, Burali-Forti, Vailati, Padoa, Vivanti, etc. See also A. N. Whitehead and B. Russell, *Principia Mathematica* (3 vols.), 2d edition, 1925–27.

† B. Russell, American Journal of Mathematics, vol. 30 (1908), p. 225. Whitehead and Russell, *Principia Mathematica* (2d edition, 1925), p. 58.

11. *The Axiomatic Group.* The great leader of this group is Hilbert. His masterly treatment of the foundations of geometry already referred to quite prepared the mathematical public to lend a willing ear to his address *Ueber die Grundlagen der Logik und der Arithmetik* at the Heidelberg Congress, 1904. After referring to the difficulties which beset a rigorous development of the number system due in part to paradoxes of the aggregate theory, Hilbert announces his program as follows: "I believe that all the difficulties which I have touched upon may be overcome and an entirely satisfactory foundation of the number concept can be reached by a method which I call the axiomatic method, and whose leading idea I wish now to develop."

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* F. P. Ramsey, *The foundations of mathematics*, Proceedings of the London Mathematical Society, (2), vol. 25 (1926), p. 338.

† L. Wittgenstein, *Tractatus Logico-Philosophicus*, London, 1922.

Hilbert has treated this subject in five other papers, and I do not believe he has said his last word yet. These papers are:

Ueber den Zahlbegriff, Jahresbericht der deutschen Mathematiker Vereinigung, vol. 8 (1900).

Axiomatisches Denken, Mathematische Annalen, vol. 78 (1918), p. 405.

Neubegründung der Mathematik, Abhandlungen des Mathematischen Seminars der Hamburgischen Universität, vol. 1 (1922), p. 157.

Die Logischen Grundlagen der Mathematik, Mathematische Annalen, vol. 88 (1922–23), p. 151.

Ueber das Unendliche, Mathematische Annalen, vol. 95 (1926), p. 161.

As in the days of Newton and Leibniz, so now the notion of infinity is our greatest friend; it is also the greatest enemy of our peace of mind. We may compare it to a great waterway bearing the traffic of the world, a waterway however which from time to time breaks its bounds and spreads devastation along its banks. Weierstrass taught us to believe we had at last thoroughly tamed and domesticated this unruly element. Such however is not the case; it has broken loose again. Hilbert and Brouwer have set out to tame it once more. For how long? We wonder.

Hilbert thinks it can become our useful servant and preserve all its powers uncurtailed. Brouwer thinks this is impossible; we can at most build a canal through our territories and allow a fraction of the infinite to pass through it.

We use the notion infinity in two ways illustrated by the integers 1, 2, 3, 4, · · · . In one sense, this is an endless sequence, such that after each element there follows another; in the other, we regard them in their totality, as a finished product. We have two other common infinite notions, space and time; neither has a bound. After each moment of time there is another; after each point on a straight there is another. We never think of these things as closed or finished.

Now Gauss wished* only such a conception of the infinite in any part of mathematics. For him there was no actual infinity, only an endless growing or becoming. It is this notion of an actual infinity which lies at the basis of Cantor's theory of transfinite numbers. Of this theory Hilbert says: "This seems to me the most admirable fruit of the mathematical mind and in fact one of the highest achievements of man's purely intellectual processes." After remarking on the catastrophic effects that the paradoxes of Russell etc. have caused, he declares: "No one shall drive us out of the paradise that Cantor created for us." We have seen what Poincaré thought of this paradise: that it is a non-entity. We will presently examine Brouwer's ideas about it.

Hilbert tells us that for a quarter of a century the questions relating to the foundations of mathematics have never been out of his thoughts, yet in his last paper *Ueber das Unendliche*, he is forced to admit that the present state of affairs relative to the paradoxes of the aggregate theory is intolerable. However, let us be comforted, for Hilbert assures us: "There is an entirely satisfactory way to escape the paradoxes without betraying our science." This goal can be reached only when the notion of the infinite has been made entirely clear.

To this end he turns to the physical sciences. He finds that the tendency in modern science is an emancipation from the infinite. Matter is not continuous, but atomic; so also is energy. The infinite space of our forefathers has shrunk to a finite volume, and has become elliptic.

How does this accord with our mathematical concepts? Does not nature show us that we are on the wrong track in dealing with the infinite? In opposition to Frege and Dedekind, who thought to develop the number system independently of intuition or experience by employing an actual infinity, Hilbert finds that a prerequisite of scientific knowledge is a fund of intuitive ideas; pure logic is not sufficient. Operations on the infinite can be rendered certain by an appeal to the finite. In Kant's philosophy, an idea is a concept of the intellect which transcends all experience, and by which the concrete of our senses is made complete as a totality. This is the role left to infinity: it is an idea.

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* Werke, vol. 8, p. 216.

Although Hilbert is quite opposed to Russell and Whitehead in the belief that mathematics is a part of logic, it is interesting to note how the methods of symbolic logic creep into Hilbert's work. Kepler found the conic sections of the ancients ready at hand. Einstein found the tensor analysis of Ricci and Levi-Civita likewise. Behold the same pre-established harmony is again made manifest. The symbolic logic of Peano, Russell, and Whitehead lies before him like a ripe fruit ready to be picked.

Hilbert takes this logic but reinterprets it. For him it is a sign-language which puts mathematical theorems into formulas and which expresses logical reasoning by formal processes. Signs and symbols of operation are freed from their significance with respect to content. The axioms of mathematics merely express the rules by which formulas follow one another. Beyond these meaningless signs and formulas which constitute mathematics there is a "meta-mathematics" which deals only with the concrete and never employs but a finite number of logical steps of a kind universally admitted.

The first axioms laid down therefore relate to the finite; for them the laws of ordinary logic hold; their freedom of contradiction is intuitive. We must however pass to the transfinite and use "all," "there is," "the excluded middle," "complete induction" etc. To this end Hilbert introduces the axiom

$$A(\tau A) \rightarrow A(a)$$

which he says means in ordinary language "If a predicate A applies to the object τA, it applies to all objects a." To illustrate this Hilbert supposes A is the predicate "is venal." Then $A(\tau A)$ is to be regarded as a definite person of such uncorruptible sense of justice that should it turn out that he were venal, it would follow that all mankind is venal.

Hilbert regards this transfinite axiom as adjoined to the finite axioms just as in algebra we adjoin to the real number system the imaginary numbers, and in geometry we adjoin to real space the ideal plane at infinity. Still more striking he finds the analogy with the ideals in the theory of algebraic numbers. How strange it comes about, Hilbert exclaims, that this (transfinite) form of reasoning, which Kronecker so passionately opposed, proves to be the exact analog of Kummer's ideals, these numbers which Kronecker so ardently admired and praised as the highest mathematical achievement.

Having found a system of axioms of sufficient generality for all the needs of modern analysis, there remains the final and most important step: the placing of the keystone to the arch, which in this case is the proof of the freedom from contradiction of these axioms. Having outlined how this is to be accomplished, Hilbert illustrates the power of his methods of proof by disposing (rather summarily) of such oft-debated problems as complete induction, the Weierstrass theorem of the existence of upper and lower limits, Zermelo's axiom.

12. *Intuitionalism.* The chief figure in this group is L. E. J. Brouwer, professor of mathematics at the University of Amsterdam. He early attained international eminence for his extraordinarily subtle and far reaching papers in the theory of point sets (analysis situs). His doctor's dissertation (1907) was *Over de grondslagen der wiskunde* (*On the foundation of mathematics*). His inaugural address as professor at the Amsterdam university (1912) was on *Intuitionism and formalism.* A translation of this by Professor A. Dresden appeared in this Bulletin, vol. 20 (1913–14). Beginning with 1918, Brouwer has expounded his theory in a series of papers in the Proceedings of the Amsterdam Academy of Sciences, in the Mathematische Annalen, and in the Journal für Mathematik.

It appears that Brouwer, like Kronecker and H. Weyl, is very much of a philosopher; his view of mathematics we would expect to be much influenced by his philosophical speculations. If he holds certain mathematical tenets radically different from the usual ones of to-day, his justification will probably be of a philosophical nature.* We recall that Kronecker declared that if he did not live to carry out his program of overhauling mathematics from the bottom up, another would come who would. The fulfillment of this prophecy seems to be Brouwer.

Brouwer not only agrees with Kronecker that a definition must contain the means of constructing the object defined, but he carries this idea to its logical end and declares the use of the logical form known as the *tertium non datur*, or excluded middle, is legitimate only for finite sets. Brouwer believes† this law arose from the consideration of finite sets in mathematics; after being adopted by logic it was given an a priori existence independent of its mathematical origin, and then by virtue of its a priori character it was unjustly extended by mathematicians to infinite sets. By adopting these two restrictions Brouwer robs himself of two of the most powerful aids of research in modern analysis. Now no one would have an objection to any mathematician's trying to see what he could do under certain restrictions; one might regret such efforts as a waste of time, or look at the affair as a sporting proposition, like swimming the English Channel with one's hands tied. This is not Brouwer's attitude, however. Like Kronecker, he does not hesitate to tell his contemporaries that they are wrong. Thus in his *Intuitionistische Mengenlehre*,‡ he says that the Komprehensionsaxiom, by virtue of which all things which have a certain property are united to form an aggregate (Menge), even in the limited form used by Zermelo, is inadmissible in founding the aggregate theory, or is at least unserviceable. Only in a constructive definition of aggregates is a trustworthy foundation of mathematics to be found. Further the principle of the excluded middle is an unpermissible means of proof, which can be allowed only a scholastic or heuristic value; so that theorems which cannot be proved otherwise are devoid of any mathematical content. There is no sufficient ground for admitting this principle; logic rests on mathematics, and not conversely. In his paper* *Ueber die Bedeutung des Satzes vom ausgeschlossen Dritten*, Brouwer remarks: "On this foundation, particularly in the last half century, extensive false theories have been erected. The contradictions which have been repeatedly encountered have brought to life the formalistic criticism." He grants that their axiomatic treatment will avoid contradictions, "but nothing of mathematical value will be attained in this manner; a false theory which is not halted by a contradiction is none the less false, just as a criminal policy unhindered by a reprimanding court is none the less criminal." These are strong words, and strong words are usually met by others. This is Hilbert's counterblast.† "What Weyl and Brouwer are doing, is mainly following in the path of Kronecker; they are trying to establish mathematics by throwing overboard everything which does not suit them and dictatorially promulgating an embargo. The effect of this is to dismember and cripple our science and to run the risk of losing a large part of our most valuable possessions. Weyl and Brouwer condemn the general notions of irrational numbers, of functions, even number theoretic functions, Cantor numbers of higher classes etc., the theorem that an infinite set of positive integers has a least, and even the 'tertium non datur', so for example the statement: either there is only a finite number of primes or there are infinitely many. These are examples of forbidden theorems and modes of reasoning. I believe that powerless as Kronecker was to abolish irrational numbers (Weyl and Brouwer do allow us to retain a torso), no less powerless will their efforts prove to-day. No, Brouwer's program does not signify a revolution, but merely the repetition of a vain coup de main with former means, but which then was undertaken with greater dash, yet totally failed. To-day the State is thoroughly armed and strengthened through the labors of Frege, Dedekind, and Cantor. Weyl and Brouwer's efforts are doomed in advance to futility."

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* The reader may consult a paper by A. Dresden, *Brouwer's contribution to the foundations of mathematics*, this Bulletin, vol. 30 (1924), p. 31.

† Jahresbericht der Vereinigung, vol. 28 (1919), p. 203.

‡ Proceedings, Amsterdam Academy, vol. 23 (1922), p. 949.

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* Journal für Mathematik, vol. 154 (1925), p. 1.

† *Neubegründung der Mathematik*, Abhandlungen des Mathematischen Seminars der Hamburgischen Universität, vol. 1 (1922), p. 157.

We have stated the two main theses of Brouwer: the rejection of the comprehension axiom as used by Cantor and Zermelo, and the law of the excluded middle.

In general the rejection of the law of the excluded middle produces great complication, which we are inclined to believe represents an element of strength rather than of weakness in Brouwer's theory. To go further into Brouwer's theory would require fresh definitions, and space does not permit this.

All new theories have to struggle for existence and recognition. Brouwer's theory is not easy to read. So far his theory has not made matters simpler; quite the reverse. It may be that this is necessary. Cauchy's methods forced the reasoning of his predecessors into innocuous desuetude; Weierstrass showed Cauchy's reasoning was far from perfect. Is Brouwer destined to lay down the standards of the next generation?

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A FURTHER NOTE ON THE CONVERSE OF FERMAT'S THEOREM

BY D. H. LEHMER

In a previous paper* the writer had discussed the converse of Fermat's theorem as a means of establishing the primality or non-primality of a large integer. Use was made chiefly of the following theorem:

THEOREM 3. *If* $a^x \equiv 1 \pmod{N}$ *for* $x = N-1$ *and if* $a^x \equiv r \not\equiv 1$ *for* $x = (N-1)/p$ *and if* $r-1$ *is prime to* N, *then all the factors of* N *belong to the form* $np^\alpha + 1$ *where* α *is the highest power of the prime* p *contained in* $N-1$.

It is the purpose of this note to give a more general theorem in which the third part of the hypothesis of Theorem 3 is removed.

THEOREM 4. *If* $a^x \equiv 1 \pmod{N}$ *for* $x = N-1$ *and* $a^x \equiv r \not\equiv 1$ *for* $x = (N-1)/p$, *then all the factors of* N/δ *are of the form* $np^\alpha + 1$, *where* α *is the highest power of the prime* p *contained in* $N-1$ *and where* δ *is the G.C.D. of* $r-1$ *and* N.

Let k be a prime factor of N/δ and let ω be the exponent to which a belongs modulo k. Then ω divides $N-1$ and $k-1$ but not $m = (N-1)/p$; for if ω divided m we would have $a^m \equiv 1 \pmod{k}$ so that $r-1$ would divide by k. But this is impossible, since k divides N/δ which is prime to $r-1$. From here on, the proof is the same as in Theorem 3 with the result that $k = np^\alpha + 1$.

Ordinarily, we have $\delta = 1$ so that the two theorems become identical. An example in which this is not the case is the following: Let $N = 16{,}046{,}641$. $N-1 = 2^4 \times 3^3 \times 5 \times 17 \times 19 \times 23$. It will be found that

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* This Bulletin, vol. 33 (1927), pp. 327–340.

$$2^{N-1} \equiv 1 \pmod{N},$$
$$2^{(N-1)/19} = 8708025 = r.$$

If we take the G.C.D. of $r-1$ and N, we get $\delta = 35113$, so that $N/\delta = 457$ and we are able to say that the factors of 457 are of the form $19n+1$. The factors of δ may be obtained by applying Theorem 4 in which a new value of m is chosen. The modulus may be taken as δ.

Theorem 4 has been applied to the number

$$N = (2^{61}+1)/3 = 768614336404564651.$$

This number has been listed as a prime but the writer could not find any account of its investigation.*

The factorization of $N-1$ is

$$N - 1 = 2(2^{60}-1)/3 = 2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 41 \cdot 61 \cdot 151 \cdot 331 \cdot 1321.$$

It was found that

$$\begin{aligned} 3^{(N-1)/5} &\equiv 754529885435533861 = r_1, \\ 3^{(N-1)/3} &\equiv 243065045915817725 = r_2, \\ 3^{N-1} &\equiv 1. \qquad\qquad \pmod{N}. \end{aligned}$$

It happens that r_1-1 and r_2-1 are both prime to N. Hence the factors of N belong to the forms

$$\left.\begin{aligned} 3n &+ 1 \\ 25n &+ 1 \\ 61n &+ 1 \end{aligned}\right\} 4575n + 1.$$

By making use of the fact that N has no factor $<300{,}000$ and by seeking to represent N as the difference of two squares it was easily shown to be a prime.

Two more numbers dividing $10^n \pm 1$ have also been tested recently. The first of these is $(10^{37}-1)/9$ or

$$N = 1{,}111{,}111{,}111{,}111{,}111{,}111{,}111{,}111{,}111{,}111{,}111{,}111{,}111.$$

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* Cunningham and Woodall, *Factorisation of* $y^n \pm 1$, London, 1925, p. 1.

It was found that

$$7^{N-1} \equiv 618{,}117{,}398{,}624{,}349{,}204{,}361{,}513{,}620{,}865{,}505{,}749 \pmod{N}.$$

Hence N is composite. This number furnishes another example of the scarcity of primes of this form. The next such number which has any chance of primality consists of 47 of the digits 1.

The second number tested is $(10^{41}+1)/11$ or

$$N = 9{,}090{,}909{,}090{,}909{,}090{,}909{,}090{,}909{,}090{,}909{,}090{,}909{,}091.$$

In this case it was found that

$$3^{N-1} \equiv 763{,}287{,}007{,}500{,}473{,}474{,}161{,}903{,}784{,}495{,}157{,}879{,}509 \pmod{N}.$$

It follows, then, that N is also composite. This result represents the sixth attempt and failure to discover a larger prime than $2^{127}-1$ found by Lucas in 1877.

THE UNIVERSITY OF CALIFORNIA

ON THE APPROXIMATE REPRESENTATION OF ANALYTIC FUNCTIONS*

BY DUNHAM JACKSON

The purpose of this paper is to discuss the convergence of approximating polynomials determined by a least-square criterion, together with certain auxiliary conditions. Let $f(x)$ be a given function over the interval $a \leqq x \leqq b$. For each positive integral value of n, let p_n be a positive integer $\leqq n$. A polynomial of the nth degree may be required, for example, to coincide in value with $f(x)$ at p_n specified points of the interval; and among the infinitely many polynomials of the

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* Presented to the Society, September 8, 1927.

nth degree satisfying this requirement, an approximating polynomial may be determined by the condition that the integral of the square of the error shall be a minimum.* Or it may be required that the polynomial and its first $p_n - 1$ derivatives shall agree in value with $f(x)$ and its first $p_n - 1$ derivatives at a single specified point. Or, more generally, contact of more or less high order may be prescribed at each of a number of different points, in such a way that the total number of auxiliary conditions adds up to p_n.

In carrying through the proof of convergence, it will be supposed that $f(x)$ is analytic over a certain region of the complex plane containing the interval (a, b) of the real axis in its interior. Some of the preliminaries, however, can be dealt with just as easily under less restrictive hypotheses.

The existence of a minimum in each of the cases to be considered is an almost immediate corollary of well known existence proofs for the problem without auxiliary conditions. For the main point of the proof there is to show that if an upper bound is assigned for the integral of the square of the error (with polynomials of given degree) the coefficients must belong to a bounded region, which may be taken as closed,† and the imposition of the new conditions merely restricts the choice of the coefficients to a closed manifold in this region. And the uniqueness of the minimizing polynomial is proved by the usual argument that if two different polynomials give equally good approximations their average gives a better one.‡ So these questions may be dismissed without further consideration.

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* For the case of trigonometric approximation, a problem of this character with auxiliary conditions independent of n has been treated in a recent paper by the writer: *Note on a problem in approximation with auxiliary conditions*, this Bulletin, vol. 32 (1926), pp. 259–262. The method there was quite different, the highly specialized character of the auxiliary conditions being compensated by greater generality in the hypotheses on $f(x)$.

† See, for example, D. Jackson, *On functions of closest approximation*, Transactions of this Society, vol. 22 (1921), pp. 117–128; pp. 118–121.

‡ See Transactions, loc. cit., pp. 121–122.

To proceed with a closer characterization of the approximating polynomials to be studied, let $f(x)$ be a given real function, defined for $a \leqq x \leqq b$, and n a given positive integer. For simplicity of statement, let it be supposed that $f(x)$ has $n+1$ continuous derivatives throughout (a, b). Then, if $P(x)$ is any polynomial, and if $P(x_1) = f(x_1)$ for a value of x_1 in the interval, the difference $f(x) - P(x)$, expressed by means of Taylor's theorem with the remainder, is either of the form $(x-x_1)^\nu \psi(x)$, where $1 \leqq \nu \leqq n$ and $\psi(x)$ is continuous and different from zero for $x = x_1$, or else it is of the form $(x-x_1)^{n+1}\psi(x)$, where $\psi(x)$ is bounded. In the former case the root is of multiplicity ν, and in the latter case, while not necessarily of determinate multiplicity, it may be said to be of multiplicity $\geqq n+1$.

Let $x_1, x_2, \cdots, x_q$ be q distinct points of the interval (a, b), where q is a positive integer $\leqq n$. With each of these points x_k, let a positive integer ν_k be associated, so that $\nu_1 + \nu_2 + \cdots + \nu_q = p \leqq n$. (If in particular $q = n$, of course each ν_k must be equal to 1.) Let $P_n(x)$ be a polynomial of the nth degree with real coefficients such that, for each x_k,

$$(1) \qquad P_n(x_k) = f(x_k), \quad P_n'(x_k) = f'(x_k), \cdots, \\ P_n^{(\nu-1)}(x_k) = f^{(\nu-1)}(x_k) \qquad (\nu = \nu_k).$$

There would be one and just one polynomial of degree $p-1$ satisfying these conditions,* and since $p-1 < n$, they are satisfied by infinitely many polynomials of the nth degree. In particular, let $P_n(x)$ be determined among all such polynomials of the nth degree by the requirement that

$$\int_a^b [f(x) - P_n(x)]^2 dx$$

shall be a minimum. *Then* $f(x) - P_n(x)$ *must be orthogonal to every polynomial* $\pi_n(x)$ *of the nth degree which vanishes with its first* $\nu_k - 1$ *derivatives at each of the points* x_k. For if there were a polynomial $\pi_n(x)$ satisfying these conditions, and not orthogonal to $f(x)-P_n(x)$, the polynomial $P_n(x)+h\pi_n(x)$ would satisfy the auxiliary conditions imposed on $P_n(x)$, for arbitrary h; the integral

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* See, for example, Markoff, *Differenzenrechnung*, Leipzig, 1896, pp. 1–3.

$$\int_a^b [f(x) - P_n(x) - h\pi_n(x)]^2 dx,$$

which satisfies the conditions for differentiating with regard to h under the sign of integration, would have a derivative reducing for $h=0$ to

$$-2\int_a^b [f(x) - P_n(x)]\pi_n(x)\,dx \neq 0\,;$$

and the integral would not have a minimum for $h=0$, as it must under the hypothesis that $P_n(x)$ is the minimizing polynomial.

From this it follows further that $f(x)-P_n(x)$ *must have roots of aggregate multiplicity* $\geqq n+1$ *in* (a, b). For if this were not the case, it would be possible to construct a polynomial $Q_n(x)$, of the nth degree or lower, having roots at the same points, with exactly the same multiplicities. Since the roots of $f(x)-P_n(x)$ must include those prescribed by (1), each with at least the specified multiplicity, $Q_n(x)$ would belong to the class of polynomials denoted by $\pi_n(x)$ in the preceding paragraph. But the supposed $Q_n(x)$ is not orthogonal to $f(x)-P_n(x)$, because $[f(x)-P_n(x)]Q_n(x)$ is not identically zero, and never changes sign, having all its roots of even order. So denial of the assertion in italics results in a contradiction.

The property expressed in this assertion can be made the basis of a convergence proof. Let it be supposed from now on that $f(x)$ is analytic throughout the interior and on the boundary of a circle of the complex plane with center on the axis of reals at the middle point of the interval (a, b), and with radius $R>2r$, where $r=(b-a)/2$, the function $f(x)$ being real for real values of x. Let a polynomial $P_n(x)$ be defined as above for each positive integral value of n, the numbers $q=q_n$, x_k, and ν_k being variable with n in any way, subject to the relations of inequality that have been imposed on them. *Then* $P_n(x)$ *converges toward* $f(x)$ *throughout the interior and on the boundary of any circle of radius* $\rho < R-2r$, *concentric with the first.* The theorem is stated and proved in this form, in spite of the fact that the region of convergence thus obtained does not necessarily include the whole interval (a, b); it will be possible to choose ρ so as to include the entire interval, if $R > 3r$.

Let $y_1, y_2, \cdots, y_{n+1}$ be the $n+1$ roots, or $n+1$ of the roots, of $f(x)-P_n(x)$, multiple roots being indicated by repetition; it will not be necessary to specify what roots are distinct, and what roots are coincident. It is a matter of elementary algebra to see that*

$$\begin{aligned}\frac{1}{t-x} &= \frac{1}{t-y_1} + \frac{x-y_1}{t-y_1}\cdot\frac{1}{t-x}\\ &= \frac{1}{t-y_1} + \frac{x-y_1}{t-y_1}\cdot\frac{1}{t-y_2} + \frac{x-y_1}{t-y_1}\cdot\frac{x-y_2}{t-y_2}\cdot\frac{1}{t-x}.\\ &= \cdots,\end{aligned}$$

or, with the notation

$$g_k(t) = (t-y_1)(t-y_2)\cdots(t-y_k),$$

$$\frac{1}{t-x} = \frac{1}{g_1(t)} + \frac{g_1(x)}{g_2(t)} + \cdots + \frac{g_n(x)}{g_{n+1}(t)} + \frac{g_{n+1}(x)}{g_{n+1}(t)}\cdot\frac{1}{t-x}.$$

For any value of x inside the circle of radius R, by Cauchy's formula,

$$f(x) = \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{t-x},$$

the integral being extended around the circumference of the circle. Hence

$$f(x) = c_0 + c_1 g_1(x) + \cdots + c_n g_n(x) + R_n(x) g_{n+1}(x),$$

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[*] For the entire convergence proof, see Runge, *Theorie und Praxis der Reihen*, Leipzig, 1904, pp. 126–142; Hermite, *Sur la formule d'interpolation de Lagrange*, Journal fur Mathematik, vol. 84 (1878), pp. 70–79; pp. 70–72.

where the c's are constants, and

$$R_n(x) = \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{g_{n+1}(t)(t-x)},$$

a function analytic throughout the interior of the circle. The terms $\sum c_k g_k(x)$ constitute a polynomial of the nth degree. Since $R_n(x)\ g_{n+1}(x)$ vanishes at the points $y_1, \cdots, y_{n+1}$, the polynomial agrees in value with $f(x)$ at these points, the roots of the difference between $f(x)$ and the polynomial having (at least) the multiplicities indicated by the repetitions among the y's. But these conditions determine a polynomial of the nth degree uniquely; *the polynomial* $\sum c_k g_k(x)$ *must be identical with* $P_n(x)$. Consequently

$$f(x) = P_n(x) + \int_C \frac{(x-y_1)(x-y_2)\cdots(x-y_{n+1})}{(t-y_1)(t-y_2)\cdots(t-y_{n+1})}\cdot\frac{f(t)dt}{t-x}.$$

Now let it be understood that x is in the interior or on the boundary of the circle of radius ρ, where, as already specified, $\rho < R-2r$. Then $|x-y_k| \leqq \rho+r$, for all values of k. On the other hand, $|t-y_k| \geqq R-r$, for all values of k and for all points t on the path of integration; and $|t-x| \geqq R-\rho$. So, if M is the maximum of $|f(t)|$ on this path,

$$|f(x) - P_n(x)| \leqq \frac{2\pi RM}{R-\rho}\cdot\left(\frac{\rho+r}{R-r}\right)^{n+1}.$$

Since $(\rho+r)/(R-r) < 1$, the right-hand member approaches zero as n becomes infinite, and $f(x)-P_n(x)$ converges uniformly toward zero for all the values of x in question.

The foregoing discussion can be supplemented in various ways. The case $q_n = n+1$, which falls just outside the hypotheses, being a straightforward problem of interpolation with no place for a least-square condition, is the one treated by Runge in the passage cited. The case $q_n \leqq n$, $p = p_n = n+1$, also leaves no room for a least-square condition, but comes within the scope of the convergence proof.* In this case, as well as in the preceding, it makes no essential difference if the points x_k, instead of being on the axis of reals, are anywhere in the circle of radius r. At the other extreme, the problem for $q_n=0$ is that of the Legendre series, and while the part of the discussion that relates to auxiliary conditions is irrelevant, the proof of convergence is still valid, as far as it goes; it shows, for example, that the Legendre series for an entire function converges everywhere.* In one of the most interesting special cases under the earlier hypotheses, that in which $q_n=1$, $p_n=n$ (or, a little less narrowly, $p_n=n-l$, where l is $\geqq 0$ and independent of n), the point x_1 being taken for simplicity as the middle point of the interval, Dr. D. V. Widder, with whom the writer discussed the substance of the paper in its preliminary stages, noted at once that the convergence takes place throughout the interior of the large circle of radius R; and instead of the requirement that $R>2r$ it is sufficient to assume that $R>r$. More generally, if l is independent of n, and if there are for each value of n at least $n-l$ auxiliary conditions attached to points distant by not more that r' from the middle point of (a, b), where $r'<r$, it is allowable in the convergence proof to let ρ be any number less than $R-2r'$, while the assumption with regard to R is that R is greater than the larger of the numbers r, $2r'$. To generalize in another direction, it would be possible to replace the square of the error by the mth power of its absolute value, $m>1$, and to admit a weight-function in the integral to be minimized. An exhaustive discussion of questions of this sort would be outside the scope of the present paper.

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* See Hermite, loc. cit.

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* See Whittaker and Watson, *Modern Analysis*, third edition, Cambridge University Press, 1920, pp. 322–323.

GENERALIZATIONS OF THE THEOREM OF FERMAT AND CAUCHY ON POLYGONAL NUMBERS*

BY L. E. DICKSON

1. *Introduction and Summary.* We seek the least l such that every integer $A \geqq 0$ is a sum of l values of

$$(1)\qquad p_{m+2}(x-k) = \tfrac{1}{2}(x-k)[m(x-k-1)+2]$$

for integers $x \geqq 0$. When $k=0$, (1) is a polygonal number of order $m+2$, and Fermat stated that $l=m+2$. This was first proved by Cauchy, who found that all but four of the $m+2$ polygonal numbers may be taken as 0 or 1. A simpler proof was given by the writer in this Bulletin (vol 33 (1927), pp. 719–726) that paper will be cited as I.

When all but four of the l values of (1) are 0 or 1, we shall prove here that

$l=m-1$ if $k=1$, $m \geqq 7$; $l=m-2$ if $k=2$, $m \geqq 8$, or if $k \geqq 3$, $m \geqq 7$; $l=6$ if $k=1$, $m=6$, or $k=2$, $m=7$; $l=5$ if $k=1$, $m=4$ or 5, or if $k \geqq 2$, $m=6$; $l=4$ if $k \geqq 1$, $m=3$, or if $k \geqq 2$, $m=4$ or 5.

When all but s of the l values of (1) are 0 or 1, where $5 \leqq s \leqq l$, we shall prove that, for every $k \geqq 1$,

$$l = m-2 \quad \text{if} \quad m > 7, \qquad l = s \quad \text{if} \quad m \leqq 7.$$

But A is a sum of l values of (1) if and only if†

$$(2)\qquad 8mA + l(m-2)^2 = \sum_{l} (2mx+2-m-2mk)^2,$$

summed for l integers $x \geqq 0$. We saw that $l=4$ when $k=1$, $m=3$, and when $k=2$, $m=4$ or 5. Hence (2) shows that for every integer $A \geqq 0$,

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* Presented to the Society, September 9, 1927.

† This implies the like formula with k replaced by any larger integer $k+p$. The values $x \geqq p$ give the same squares as occur in (2) for $x \geqq 0$. Hence we use the least permissible k.

$$(3)\qquad \begin{gathered} 24A+4=\sum_4 (6x-7)^2,\quad 40A+36=\sum_4 (10x-23)^2,\\ 8A+4=\sum_4 (4x-9)^2. \end{gathered}$$

By Theorem 6 and (2), we have

$$(4)\qquad 3A+4=\sum_4 (3x-16)^2 \quad \text{if } A\not\equiv 4 \pmod 8.$$

For $s=5$, $m\leqq 7$, we saw that $l=5$. Then (2) for $k=1$ is

$$(5)\qquad 8mA+5(m-2)^2=\sum_5 (2mx+2-3m)^2.$$

For $m=3$ this result is obtained by adding 1 to each member of (3_1) and is discarded. When m is 4 or 6, we cancel factors 4 or 16. We get

$$(6)\quad 8A+5=\sum_5 (4x-5)^2,\quad 40A+45=\sum_5 (10x-13)^2,$$

$$(7)\quad 3A+5=\sum_5 (3x-4)^2,\quad 56A+125=\sum_5 (14x-19)^2.$$

The results obtained when $s=4$, $l=5$ are consequences of these.

When $k=1$, $m\geqq 8$, Theorem 2 states that $l=m-2$ except for $A=54m+16$. This is a remarkable fact since the single exceptional value may be made as large as we please by taking m sufficiently large. Again, if $k=5$, $m=6$, we find that $l=4$ fails first for $A=980$; by increasing k, the first failure will occur for a value of A exceeding any assigned number.

In later papers I shall prove that the value of l obtained when $s=3$ is never less than the minimum l found here, and I shall give an improved method valid when $s>4$, but not when $s=4$.

2. *Formulas.* Give (1) the notation $\frac{1}{2}mx^2+\frac{1}{2}nx+c$. Thus

$$(8)\qquad n=2-m-2mk,\quad c=\tfrac{1}{2}m(k^2+k)-k.$$

For $k\geqq 1$ the reduced minor conditions in I now follow from

$$(9)\qquad A\geqq 4c+4E,\quad A\geqq 4c+\tfrac{2}{3}m.$$

For, from the inequality (9_2) we deduce (11) of I, since the sum $(-n)(2m+n)/m$ of its last two terms is negative, because $-n$ is positive and $2m+n$ is negative. By (8_1), we may write (4) of I in the form

$$A = mg + 4c + b + r, \quad g = \tfrac{1}{2}(a - b - 2kb). \tag{10}$$

3. *Excess E when $k=1$.* The values of (1) for $x \geqq 0$ are $m-1$ and all the polygonal numbers of order $m+2$.

TABLE II.

SUMS BY FOUR OF $m-1$ AND POLYGONAL NUMBERS

0–4, $m-1$, m, $m+1$–5. $2m-2$, $2m-1$, $2m$, $2m+1$–6, $3m-3$, $3m-2$, $3m$, $3m+1$, 3–7, $4m-4$, $4m-1$, $4m+2$–8, $5m+1$, 2, 4, 5, 7, 8, $6m$, $6m+3$–9, $7m+3$–9, $8m+2$–10, $9m+1$, 4, 7–10, $10m+5$–11, $11m+4$–9, 11, $12m+3$. 4, 6–12, $13m+2$, 5, 7–12, $14m+6$–12, $15m+6$–9, 11–13, $16m+5$–13, $17m+4$, 5, 7–13, $18m+3$, 6, 7, 9–14, $19m+8$, 9. 11–14, $20m+7$, 10–14, $21m+7$–13, 15, $22m+6$–15, $23m+5$, 6, 8, 9, 11–15, $24m+4$, 7, 10–16, $25m+9$–13, 15, 16, $26m+8$, 10, 11, 13–16, $27m+9$, 11–16, $28m+8$–17, $29m+7$–13, 15–17, $30m+6$, 7, 9, 10, 12–17, $31m+5$, 8, 11–17, $32m+10$–18, $33m+9$, 10, 12, 13, 15–18, $34m+12$–18, $35m+11$–17, $36m+9$–19, $37m+8$–13, 15–19, $38m+7$, 8, 10, 11, 13–15, 17–19, $39m+6$, 9, 12–17, 19, $40m+11$, 12, 14–20, $41m+10$, 13, 15–20, $42m+13$–20, $43m+12$–17, 19, 20, $44m+11$–20, $45m+10$–13, 15–21, $46m+9$–21, $47m+8$, 9, 11–17, 19–21, $48m+7$, 10, 12–21, $49m+12$, 13. 15–21, $50m+11$, 14–22, $51m+13$–17, 19–22, $52m+13$–22, $53m+12$, 13, 15–21, $54m+17$–19, 21, 22.

If $m \geqq 6$, $E(9m+6)=2$, since neither $9m+5$ nor $9m+6$ is equal to a number of Table II. If $m=5$, Table II lacks $17=2m+7=3m+2=4m-3$. If $m=4$, it lacks $26=4m+10 = 5m+6 = 6m+2 = 7m-2$. Also,* $E(54m+16) = m-5$. Hence E is not smaller than the value shown in

THEOREM 1. *For $k=1$, $E=m-5$ if $m \geqq 7$, $E=2$ if $m=6$, $E=1$ if $m=4$ or 5, $E=0$ if $m=3$.*

By (10) we require that the values of $b+r$ shall include a complete set of residues modulo m when r takes the values $0, 1, \cdots, E$, and b takes certain consecutive odd values. When $m=3$ or $m \geqq 5$, this will be true if b takes the values

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* And independently of Table II, since the only partitions of 53 into 0, 1, 3, 6, 10, 15, 21, 28, 36, 45 are $1+1+6+45$, $1+1+15+36$, $1+6+10+36$, $1+3+21+28$, $0+10+15+28$, $1+10+21+21$.

β, $\beta+2$, $\beta+4$, whence $d=6$. But when $m=4$, we need only β, $\beta+2$, whence $d=4$. Conditions (9) hold if $A\geqq 8m$.

Let $m\geqq 7$. By (6)–(15) of I,

$$U = 24mA - 63m^2 + 12m + 36, \quad V = 2mA - m^2 + 6m + 4,$$
$$P = 7m + 2, \quad W = 3mA - 14m^2 - 2m + 4,$$
$$F = m^2A^2 - 74m^3A - 28m^2A + 193m^4 + 70m^3 - 68m^2 - 24m > 0.$$

The last evidently holds if

$$A \geqq 74m + 28. \tag{11}$$

Next, let $m=6$. Then

$$U = 36(4A - 60), \; V = 12A - 8, \; P = 44, \; W = 18A - 512,$$
$$F = 36[(7A - 88)^2 - (12A - 8)(4A - 60)] = 36(A^2 - 480A + 7264),$$

and $F>0$ if $A\geqq 465$ and hence if (11) holds.

If $m\geqq 6$, Theorem 1 now follows from Lemma 3 of I and the following lemma.

Lemma 1. *If* $m\geqq 5$ *in Table* II, $E(A)\leqq m-5$ *when* A *is between any consecutive blocks, while* $E(A)\leqq 2$ *when* A *is in any block.*

For, the difference between consecutive entries in any block is always $\leqq 3$. If r is the term free of m in the leader $qm+r$ of any block, then $r+4$ is found to be the term free of m of a number of Table II within the preceding block. Hence $qm+r-1$ is the sum of $m-5$ and the number $(q-1)m+r+4$ in the table.

When $m=5$, the last two sentences hold after we suppress from Table II all entries down to and including the last entry which differs by 3 from the next entry of the block. The only exception is the leader $9m+7$, while $9m+6$ exceeds $8m+10$ by 1. Hence $E(A)\leqq 1$ if $A\leqq 54m+22$.

$$\text{For } m = 5, d = 6, n = -13, c = 4,$$
$$U = 120A - 1479, \; V = 10A - 1, \; P = 37, \; W = 15A - 356,$$
$$F = 25A^2 - 10150A + 126685 > 0 \text{ if } A > 393 = 74m + 23.$$

These facts with Lemma 3 of I prove Theorem 1 for $m=5$.

$$\text{For } m = 4, d = 4, n = -10, c = 3,$$

$$U = 4(24A - 231),\ V = 4(2A-1),\ P = 14,\ W = 12A - 140,$$

$$F = 16[(7A - 37)^2 + (2A-1)(24A - 231)]$$
$$= 16(A-16)^2 + 16 \cdot 882 > 0.$$

Finally, let $m=3$. Then $d=6$, $n=-7$, $c=2$,

$$U = 9(8A - 55),\ V = 6A + 1,\ P = 23,\ W = 9A - 128,$$

$$F = 9[(7A - 42)^2 - (6A+1)(8A-55)]$$
$$= 9(A^2 - 266A + 1819) > 0,$$

if $A \geqq 259$. For $A < 259$, Theorem 1 was verified by Tables I and II.

4. THEOREM 2. *If* $k=1$, $m \geqq 8$, $E(A) \leqq m-6$ *except for* $A = 54m+16$.

Within every block of Table II the difference of any two consecutive entries is $\leqq 3$. If f is the term free of m in the leader $qm+f$ of any block having $q \neq 16, 46, 52, 54$, then $f+5$ is that of a number t occurring explicitly in the table. Hence $qm+f-1$ is the sum of $m-6$ and $t=(q-1)m+f+5$. For $q=16$, 46, and 52, a like result holds if we replace $f+5$ by $f+6$ and hence replace $m-6$ by $m-7$. Hence the theorem is true for $A < 54m+16$. For $54m+17 \leqq A \leqq 199m+37$, it is true by Lemmas 3 and 4 of I.

For $A \geqq 199m+38$, $E=m-6$, we have $d=8$,

$$U = 24mA - 63m^2 + 12m + 36,\quad V = 2mA - m^2 + 8m + 4,$$

$$P = 11m + 2,\qquad W = 3mA - 23m^2 - 4m + 4,$$

$$F = m^2A^2 - 200m^3A - 48m^2A + 562m^4 - 84m^3$$
$$- 264m^2 - 48m.$$

Then $F>0$, in fact for $A = 198m+44+t$, $t \geqq 0$.

5. *Excess E_s when $k=1$.*

THEOREM 3. *When $k=1$, $E_s=0$ if $m\leqq 7$, $s\geqq 5$, and if $m>7$, $s\geqq m-2$, while $E_s=m-s-2$ if $m>7$, $5\leqq s\leqq m-2$.*

If $m\leqq 5$, this follows from Theorem 1. If $m>7$, $E_5\leqq m-7$ by Theorem 2 and the fact that $54m+16$ is the sum of the polygonal number $3m+3$ and the entry $51m+13$ of Table II. Since the polygonal numbers >1 exceed $m-1$, the summands yielding $m-2$ are all 1, whence $E_s(m-2)=m-2-s$. Hence $E_5=m-7$. If $s>5$, regard $s-5$ of the values 1 as polygonal numbers; hence $E_s\leqq m-7-(s-5)$. This proves Theorem 3 except for $m=6$ and 7.

For $m=6$ or 7, we seek a constant C_m such that $E_4(A)\leqq 1$ for every $A\geqq C_m$. When r takes the values 0 and $E=1$, and b takes 3 or 4 consecutive odd values according as $m=6$ or $m=7$, the values of $b+r$ include a complete set of residues modulo m, whence $d=6$ or 8, respectively.

In the discussion in § 3 for $m\geqq 7$, we had $d=6$, $E=m-5$. Hence it is valid here for $m=6$, $d=6$, $E=1$, and shows that $E_4(A)\leqq 1$ if $A\geqq 74m+28$. Likewise, the work at the end of § 4 is valid here for $m=7$, $d=8$, $E=1$, and shows that $E_4(A)\leqq 1$ if $A\geqq 198m+44$.

It remains to treat the values of A below these two limits. From Table II we readily deduce a list of the sums by five of $m-1$ and polygonal numbers and then verify for $m=6$ and $m=7$ that the list includes all positive integers $\leqq 54m+25$. From thence to $74m+28$, $E_4(A)\leqq 1$ if $m=6$ or 7 by Lemma 3 of I. If $m=7$ and $74m+20\leqq A\leqq 199m+37$, $E_4(A)\leqq 1$ by Lemma 4 of I. This completes the proof of Theorem 3.

6. *Excess E when $k=2$.* The values of (1) for $k=2$, $x\geqq 0$ are $m-1$, $3m-2$, and all the polygonal numbers of order $m+2$. Evidently $E(m-2)=m-6$. For $m=7$ the sums by four of the values mentioned were tabulated to 395; the only consecutive integers missing from the list are 393–4; hence $E(394)=2$, $E(A)\leqq 1$ if $A<394$. For $m=6$, $E(60)=1$. Hence E is not smaller than the value shown in

THEOREM 4. *For* $k=2$, $E=m-6$ *if* $m\geqq 8$, $E=2$ *if* $m=7$, $E=1$ *if* $m=6$, $E=0$ *if* $m=3$, 4, *or* 5. *If* $m=7$, $E(A)\leqq 1$ *except for* $A=394$.

Conditions (9) are satisfied if $A\geqq 16m$. For $r=0, 1, \cdots$, E, we require that $b+r$ shall include a complete set of residues modulo m. When $m\geqq 8$, this will be true if b takes the values β, $\beta+2$, $\beta+4$, $\beta+6$; when $m=5$, also $\beta+8$; when $m=6$ or 7, only the first three.

First, let $m\geqq 8$. Then $d=8$,

$$U = 24mA - 135m^2 + 36m + 36, \qquad P = 9m + 2,$$
$$V = 2mA - m^2 + 8m + 4, \quad W = 3mA - 27m^2 + 4,$$
$$F = m^2A^2 - 112m^3A - 40m^2A + 706m^4 + 188m^3 - 152m^2 - 48m.$$

Evidently $F>0$ if $A\geqq 112m+40$. For smaller A's exceeding $54m+16$, Lemmas 3 and 4 of I show that $E(A)\leqq m-6$ if $m\geqq 7$.

LEMMA 2. *If* $m\geqq 7$ *in Table* II, $E(A)\leqq m-6$ *except for* $A=9m+6$, $20m+9$, *and* $54m+16$.

From each block we suppress all entries down to and including the last entry which differs by 3 from the next entry. Proceed as in § 4. We now have the further exceptions $q=9$ and 20. Also

$$9m + 3 = m - 6 + t_1, \quad 9m + 5 = 1 + t_2,$$
$$20m + 8 = m - 6 + t_3, \quad 54m + 15 = m - 6 + t_4,$$

where the t_i occur in the table.

For $m\geqq 8$, Theorem 4 now follows since

$$9m + 6 = 1 + 2(3m + 3) + 3m - 2 + 1, \quad 20m + 9 = m + 2 + 6m + 4 + 10m + 5 + 3m - 2,$$
$$54m + 16 = 2(3m + 3) + 45m + 10 + 3m - 2 + 2.$$

For $m=7$, $E=2$, $d=6$, $P=37$,

$$U = 168A - 6327, \quad V = 14A - 3, \quad W = 21A - 962,$$
$$F/7 = 7A^2 - 826A + 131149,$$

and $F>0$ for every $A>0$. But if we attempt to use $E=1$, we have $d=8$; the discussion for $m\geqq 8$ now applies also for $m=7$ except when $A=54m+16=394$. This proves both parts of Theorem 4 if $m=7$.

Next, let $m=6$. Then $E=1$, $d=6$, $P=32$,

$$U = 9(16A - 512),\quad V = 4(3A + 1),\quad W = 18A - 704,$$

$$F/36 = (7A - 116)^2-(3A + 1)(16A - 512) = (A - 52)^2 + 11264.$$

The minor conditions in I are all satisfied if $A\geqq 44$. For $m=6$, Table II includes all numbers less than 44 except $33=1+2(3m-2)$.

Let $m=5$. Then $E=0$, $d=10$, $P=67$,

$$U = 120A - 3159,\quad V = 10A + 9,\quad W = 15A - 956,$$

$$F/25 = (A - 703)^2 - 457878 > 0 \text{ if } A \geqq 1380.$$

It was verified that every integer $\leqq 1380$ is a sum by four of 4, 13, and polygonal numbers of order 7, whence $E=0$.

Let $m=4$. Then $n=-18$, $c=10$, $A=2a-9b+40$. Our general method requires that a and b be odd and applies only when A is odd. By (10) with $r=0$, $b\equiv A \pmod 4$. Hence $d=4$, $P=6$,

$$U = 4(24A - 495),\quad V = 4(2A + 1),\quad W = 4(3A - 63),$$

$$F/16 = (7A - 61)^2 - (2A + 1)(24A - 495) = A^2 + 112A + 4216,$$

whence $F>0$ for every $A\geqq 0$. The minor conditions in I are all satisfied if $A\geqq 28$. For $m=4$, Table II includes all integers $\leqq 28$ except $26=m+2+2(3m-2)$.

Next, let A be even. Since b must be even, our previous method is not applicable.

LEMMA 3.* *If $b=2B$, a is even, $a-B^2$ is a sum of three squares, and*

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* Legendre, *Théorie des Nombres*, ed. 3, II, No. 628, with omission of denominator 2 in No. 629.

(12) $$4B^2 + 2B + 1 > 3a,$$

then $a = \sum\alpha^2$, $b = \sum\alpha$ *have solutions* $\alpha, \beta, \gamma, \delta$ *in integers* $\geqq 0$.

Write $A = 2S$. Then $a = S + 9B - 20$. Inserting this value of a into (12) and multiplying by 16, we see that (12) holds if

(13) $$8B > 25 + u^{1/2}, \quad u = 48S - 351 \geqq 0.$$

Similarly, $B^2 < a$ if

(14) $$2B < 9 + v^{1/2}, \quad v = 4S + 1.$$

Take $B \equiv 2 - S \pmod 4$. Then $a \equiv 2 \pmod 4$, $a - B^2 \equiv 1$ or $2 \pmod 4$, according as B is odd or even. Hence $a - B^2$ is a sum of three squares, and all conditions in Lemma 3 are satisfied.

To insure the choice of B as a prescribed residue modulo 4, we require that the difference between the limits for B, determined by (13) and (14), shall exceed 4. The resulting inequality reduces as usual to

$$S^2 - 340S + 2440 > 0, \quad S \geqq 333.$$

It remains only to verify Theorem 4 for $m = 4$ when A is even and < 666. This was done above when $A \leqq 28$. Table II includes all integers from 28 to 238 except 236; Table I includes all from 235 to 666.

7. *Excess when* $k \geqq 3$. Evidently $E(m-2) = m - 6$. If $m = 7$,

$$394 = 54m + 16 = 6m - 3 + 2(10m + 5) + 28m + 8 + 1.$$

Hence Theorem 4 implies the cases $m \geqq 7$ of

THEOREM 5. *For* $k \geqq 3$, $E = m - 6$ *if* $m \geqq 7$, $E = 1$ *if* $m = 6$.

That $E = 1$ if $m = 6$ follows from Theorem 4 and a result to be proved in a later paper on extended polygonal numbers. It was verified that 116 (or 980) is the least positive integer which is not a sum by four of the values of (1) for $x \geqq 0$, $k = 4$ (or $k = 5$). For $x \geqq 0$, the values > 1 of (1) are all $\leqq m - 1$. Hence the proof in § 5 shows that Theorem 3 holds also when $k > 1$.

8. THEOREM 6. *If* $m=6$, $k=5$, $E(A)=0$ *except for* $A\equiv 4 \pmod 8$.

If A is odd, our general method applies with

$$d \doteq 6,\ U = 144(A-119),\ V = 12A+16,\ P = -4.$$

Then (14) and hence (13) of I holds. Also, (9) hold if $A \leqq 344$, which is below the limit 980 for which $E=0$ (§ 7).

Next, let $A\equiv 2 \pmod 4$. Take $b=2B$ and determine B modulo 3 to make a an integer. Then $a\equiv 2 \pmod 4$. Apply Lemma 3. Then (12) holds and $B^2<a$ if

$$4B > 31 + u^{1/2},\quad 3B < 32 + v^{1/2},\quad u = 4A - 403,$$
$$v = 3A + 4.$$

The difference between these limits for B exceeds d if

$$4v^{1/2} - 3u^{1/2} > R \doteq 12d - 35. \tag{15}$$

Here $d=3$ and (15) holds for every A such that $u>0$.

Finally, let $A\equiv 0 \pmod 8$. Take an odd B such that a is an integer, whence $d=6$. Then $a\equiv 4 \pmod 8$ and Lemma 3 applies. The square of (15) holds if

$$(A-491)^2 > 65712,\quad A \geqq 748.$$

THE UNIVERSITY OF CHICAGO

A RATIONAL NORMAL FORM FOR CERTAIN QUARTICS*

BY RAYMOND GARVER

A quartic equation with rational coefficients, irreducible in the field R of rational numbers, will have for its Galois group (for the field R) one of the following:

I. The symmetric group of order 24.

II. The alternating group of order 12.

III. A group of order 8 such as the one whose substitutions are 1, (12), (34), (13)(24), (12)(34), (14)(23), (1324), (1423).

IV. A cyclic group of order 4 such as 1, (1324), (12)(34), (1423).

V. The 4-group 1, (12)(34), (13)(24), (14)(23).

It is the purpose of this note to give what seems to be a more direct proof of a theorem of Wiman:†

THEOREM. *Any rational quartic, irreducible in R, whose Galois group for R is either* III, IV, *or* V *above, can be transformed by a rational Tschirnhausen transformation into the form $y^4+py^2+q=0$, where p and q are rational.*

It will then follow as a corollary that p and q can be made integral; this can be accomplished by a simple additional transformation.

We may consider the given quartic in the reduced form

$$x^4 + a_2x^2 + a_3x + a_4 = 0, \quad (a_1 = s_1 = 0, a_3 \neq 0). \tag{1}$$

Apply the quadratic transformation

$$y = x^2 + k_1x + k_2. \tag{2}$$

Now the transformed equation will lack its second and fourth terms provided $\sum y=0$, $\sum y^3=0$.‡ The first of these equations, by (2), will be satisfied if $k_2=a_2/2$. To obtain $\sum y^3$ we cube both sides of (2), sum over the four roots of (1), and use Newton's relations applied to (1), namely

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* Presented to the Society, October 29, 1927.

† Arkiv för Matematik, Astronomi, och Fysik, vol. 3 (1906–7); No. 28.

‡ The summations extend over the 4 roots of (1).

$$(3)\quad \begin{cases} s_2 = -2a_2, \quad s_3 = -3a_3, \quad s_4 = 2a_2^2 - 4a_4, \\ s_5 = 5a_2a_3, \quad s_6 = 6a_2a_4 - 2a_2^3 + 3a_3^2. \end{cases}$$

The equation $\sum y^3=0$ then takes the form

$$(4)\quad a_3k_1^3 - (a_2^2 - 4a_4)k_1^2 - 2a_2a_3k_1 - a_3^2 = 0.$$

Now it is known* that if x_1, x_2, x_3, x_4 denote the roots of (1), then $y_1=x_1x_2+x_3x_4$, $y_2=x_1x_3+x_2x_4$, $y_3=x_1x_4+x_2x_3$ are the roots of the resolvent cubic

$$(5)\quad y^3 - a_2y^2 - 4a_4y + 4a_2a_4 - a_3^2 = 0.$$

And if the group of (1) is III, IV, or V, equation (5) will have at least one rational root. In this case (4) will also have a rational root, since the roots of (4) can be expressed rationally in terms of those of (5). It is easily verified that they are

$$(6)\quad \frac{-a_3}{a_2-y_1}, \quad \frac{-a_3}{a_2-y_2}, \quad \frac{-a_3}{a_2-y_3}.$$

These expressions for the roots of (4) were not, however, by any means obvious; the problem was to exhibit them in some form so that it would be possible to show that (4) has a rational root provided the group of (1) is III, IV, or V. For using this rational value of k_1, together with $k_2=a_2/2$, we have exhibited the rational transformation which leads to the result of the theorem.

Bucht† has pointed out, and it is not difficult to show that if the group of (1) is V, the q of the transformed equation will be not only rational, but the square of a rational number.

THE UNIVERSITY OF ROCHESTER

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* Dickson, *Elementary Theory of Equations*, 1914, p. 39.

† Arkiv för Matematik, Astronomi, och Fysik, vol. 6 (1910–11), No. 30.

ON THE LAW OF EXCLUDED MIDDLE

BY ALONZO CHURCH*

1. *Introduction.* The purpose of this paper is to discuss the possibility of a system of logic in which the law of excluded middle is not assumed, and also to point out what seem to be errors in a recent paper† in which the conclusion is reached that such a system of logic is self-contradictory.

The law of excluded middle is the logical principle in accordance with which every proposition is either true or false. This principle is used, in particular, whenever a proof is made by the method of *reductio ad absurdum.* And it is this principle, also, which enables us to say that the denial of the denial of a proposition is equivalent to the assertion of the proposition.

The simplest alternative to the inclusion of the law of excluded middle among the principles of logic is its bare omission without assertion of any contrary principle. The effect of such an omission is, of course, to reduce the number of theorems which can be proved and also to render of interest certain theorems otherwise vacuous. We cannot derive theorems which contradict theorems obtained with the aid of the law of excluded middle unless we make some assertion of a principle which contradicts the law of excluded middle.

It is not possible, as an alternative to the law of excluded middle, to assert that some proposition is neither true nor false, because by so doing not only the law of excluded middle would be denied but also the law of contradiction. In fact, to assert that a proposition p is not true and is also not false is to assert at once *not-p* and *not-(not-p)* and consequently to assert that *not-p* is both true and false.

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* National Research Fellow in Mathematics.

† M. Barzin and A. Errera, *Sur la logique de M. Brouwer,* Académie Royale de Belgique, Bulletins de la Classe des Sciences, (5), vol. 13 (1927), pp. 56–71.

It may be, however, that, by introducing the middle ground between true and false as an undefined term, let us say "tiers" (adopting from the French the word used by Barzin and Errera), and making an appropriate set of assumptions about the existence and properties of tiers propositions, we can produce a system of logic which is consistent with itself but which becomes inconsistent if the law of excluded middle be added.

2. *The Position of L. E. J. Brouwer.* L. E. J. Brouwer proposes that the law of excluded middle should not be regarded as an admissible logical principle,* and expresses, as a basis for his proposal, doubts concerning the truth of this law. He says, for example, that the law of excluded middle has been extended to the mathematics of infinite classes by an unjustified analogy with that of finite classes. He says also that to assert the law of excluded middle is equivalent to asserting the doubtful proposition that every proposed theorem can be either proved or disproved if the proper method be found. The latter point depends, of course, on identifying the truth of a proposition with the possibility of proving the proposition. But it seems more in accord with our usual ideas to think of truth as a property of a proposition independent of our ability to prove it. Consequently we prefer to take the truth of a proposition merely as an undefined term subject to certain postulates, among them, if we choose to include it, the law of excluded middle.

In connection with geometry and other branches of mathematics it is commonly recognized that it is meaningless to ask about the absolute truth of a postulate and that the choice between one of two contrary postulates must be made on the basis of simplicity and serviceability. It seems reasonable to recognize the same thing with regard to the postulates of logic, in particular the law of excluded middle, and to say on this basis that it is meaningless to ask about the truth of the law of excluded middle.

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* See, for example, *Intuitionistische Mengenlehre*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 28 (1919), pp. 203–208.

Taking this point of view, we may accept a system of logic in which the law of excluded middle is assumed, a system in which the law of excluded middle is omitted without making a contrary assumption, and a system which contains assumptions not in accord with the law of excluded middle as all three equally admissible, unless one of them can be shown to lead to a contradiction. If we had to choose among these systems of logic, we could choose the one most serviceable for our purpose, and we might conceivably make different choices for different purposes.

3. *Barzin and Errera's Paper.* Barzin and Errera, however, reach the conclusion (loc. cit.) that the system of logic proposed by Brouwer leads to contradiction. This conclusion we believe to be erroneous, for the following reasons.

The method of the argument is the method of *reductio ad absurdum.* It is assumed that if the law of excluded middle is not accepted then it must be explicitly denied by asserting the existence of tiers propositions, and on this basis contradictory results are obtained. This argument is clearly not effective against one who merely omits the law of excluded middle from his system of logic without assuming any contrary principle, because the insistence that one who refuses to accept a proposition must deny it can be justified only by an appeal to the law of excluded middle, the very principle in doubt. The method of *reductio ad absurdum,* in fact, necessarily employs the law of excluded middle and cannot be used against one who does not admit this law.

The argument of Barzin and Errera would not be effective, however, even against one who asserted the existence of tiers propositions, because in the course of the argument it is necessary to assume what the authors name the principle of excluded fourth, that every proposition is either true, false, or tiers, a principle which seems to be a restricted form of the law of excluded middle. This assumption is defended on the basis of the definition of a tiers proposition as one which is neither true nor false, but this definition, as pointed out above, is not consistent with itself and consequently cannot be used. If the concept *tiers* is introduced at all, it must be as an undefined term.

If we admit the concept *tiers* and the principle of excluded fourth, the argument of Barzin and Errera is then quite correct. They show that "p is tiers" together with "p obeys the principle of excluded fourth" implies a contradiction; hence that p is not tiers; and hence, by an appeal to the principle of excluded fourth, that p is either true or false. But one who asserts the existence of tiers propositions need not assert the principle of excluded fourth; and one who merely omits the law of excluded middle without introducing the concept *tiers* certainly could not assert the principle of excluded fourth.

It would not be permissible to say that the word *tiers* is to cover all possibilities other than true and false, however many such possibilities there may be. This language is, in fact, equivalent to defining tiers to mean neither true nor false, and is open to the same objections. But the following objection could also be raised. Using p' to stand for "p is tiers," it seems likely that $(p')'$, or p'', would constitute a fourth possibility, $(p'')'$ a fifth possibility, and so on. Uniting these possibilities into one under a new name, let us say p^*, we then have to reckon with the likelihood that $(p^*)'$ gives still another possibility which we may designate by the transfinite ordinal ω. And after this we have to reckon with an $(\omega+1)$th possibility, and so on, so that ultimately we reach any given transfinite ordinal whatever. Any attempt to unite all these possibilities into one must be regarded as extremely doubtful, because of its connection with Burali-Forti's paradox† concerning the ordinal number of the sequence of all transfinite ordinals. This paradox, in fact, compels us to regard as illegitimate the consideration of this sequence as a whole.

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† C. Burali-Forti, *Una questione sui numeri transfiniti*, Rendiconti del Circolo Matematico di Palermo, vol. 11 (1897), pp. 154–164. Whitehead and Russell, *Principia Mathematica*, vol. III, pp. 73–75.

NOTE ON INTERCHANGE OF ORDER OF LIMITS*

BY T. H. HILDEBRANDT

There exist a number of theorems giving necessary and sufficient conditions for the relation

$$\lim_m \lim_n a_{mn} = \lim_n \lim_m a_{mn},$$

$\{a_{mn}\}$ being a double sequence of real numbers. Most of these theorems are not symmetric in m and n, which is only natural, because as a rule wherever the interchange of order of limits is in question, there is information about the iterated limit in one order, and the existence of the limit in reverse order is desired. Nevertheless it may be of interest to deduce a condition which is symmetric in m and n.

By way of notation, we shall assume that the symbol $\lim_m \lim_n a_{mn}$ implies that for every m, $\lim_n a_{mn}$ exists. On the other hand, we shall use the symbol $\lim_m \overline{\underline{\lim}}_n a_{mn}$ with the implication that

$$\lim_m \overline{\lim}_n a_{mn} = \lim_m \underline{\lim}_n a_{mn}.$$

Then, obviously, if we deduce necessary and sufficient conditions for the relation

$$\lim_m \overline{\underline{\lim}}_n a_{mn} = \lim_n \overline{\underline{\lim}}_m a_{mn},$$

those for the equality mentioned at the outset require in addition the assumption of the existence of $\lim_n a_{mn}$ for every m and of $\lim_m a_{mn}$ for every n.

Let us deduce necessary conditions. Let $\overline{\lim}_m a_{mn} = b_n$ and $\underline{\lim}_m a_{mn} = c_n$, and $\lim_n b_n = \lim_n c_n = d$. Then for every $\epsilon > 0$, there exists an n_ϵ such that when $n \geqq n_\epsilon$, we have

$$|b_n - d| \leqq \epsilon \text{ and } |c_n - d| \leqq \epsilon.$$

From the definition of b_n and c_n we have, for every n, and every $\epsilon > 0$, an $m_{\epsilon n}$ (depending on ϵ and n) such that when $m \geqq m_{\epsilon n}$, we have

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* Presented to the Society, September 8, 1927.

$$c_n - \epsilon \leqq a_{mn} \leqq b_n + \epsilon.$$

By combining these two statements, we obtain the following. For every $\epsilon > 0$, there exists an n_ϵ such that for every $n_1 \geqq n_\epsilon$, there exists an $m_{\epsilon n_1}$ such that for every $m_2 \geqq m_{\epsilon n_1}$, we have

$$d - 2\epsilon \leqq a_{m_2 n_1} \leqq d + 2\epsilon,$$

i. e.,
$$\left| a_{m_2 n_1} - d \right| \leqq 2\epsilon.$$

For future reference we shall call this Statement A. Similarly, from the fact that $\lim_m \overline{\underline{\lim}}_n\, a_{mn} = d$, we have: For every $\epsilon > 0$ there exists an m_ϵ such that for every $m_1 \geqq m_\epsilon$, there exists an $n_{\epsilon m_1}$ such that for every $n_2 \geqq n_{\epsilon m_1}$, we have

$$\left| a_{m_1 n_2} - d \right| \leqq 2\epsilon.$$

By combining these two statements into one, and replacing 4ϵ by ϵ, we get the necessary condition desired, viz.: For every $\epsilon > 0$, there exists an n_ϵ and an m_ϵ such that for every $n_1 \geqq n_\epsilon$ and $m_1 \geqq m_\epsilon$, there exists an $m_{\epsilon n_1}$ and an $n_{\epsilon m_1}$ such that for every $m_2 \geqq m_{\epsilon n_1}$ and $n_2 \geqq n_{\epsilon m_1}$, we have

$$\left| a_{m_1 n_2} - a_{m_2 n_1} \right| \leqq \epsilon.$$

This condition is also sufficient. For suppose the condition satisfied. Then for a particular m_1 and n_2 chosen in accordance with the specifications, and for every $n_1 \geqq n_\epsilon$ and every $m_2 \geqq m_{\epsilon n_1}$, we have

$$a_{m_1 n_2} - \epsilon \leqq a_{m_2 n_1} \leqq a_{m_1 n_2} + \epsilon.$$

From this we conclude that for every $n_1 \geqq n_\epsilon$, the greatest and the least of the limits b_n and c_n satisfy the conditions

$$a_{m_1 n_2} - \epsilon \leqq c_{n_1} \leqq b_{n_1} \leqq a_{m_1 n_2} + \epsilon.$$

This has as consequence that if n_1 and n_0 are greater than or equal to n_ϵ, then

$$\left| b_{n_0} - b_{n_1} \right| \leqq 2\epsilon, \quad \left| c_{n_0} - c_{n_1} \right| \leqq 2\epsilon, \quad \text{and} \quad \left| c_{n_1} - b_{n_1} \right| \leqq 2\epsilon.$$

Hence $\lim_n c_n$ and $\lim_n b_n$ exist and are equal, i.e. $\lim_n \overline{\lim}_m a_{mn}$ exists. From the symmetry of the condition, we conclude that $\lim_m \overline{\lim}_n a_{mn}$ exists also. The identity of the two limits is then a consequence of the condition of our theorem and Statement A.

We note finally that the Cauchy condition for convergence of the double limit, $\lim_{mn} a_{mn}$, is the special case of our condition in which $m_{\epsilon n_1}$ and $n_{\epsilon m_1}$ are independent of n_1 and m_1 respectively, and can therefore be taken as m_ϵ and n_ϵ, respectively.

UNIVERSITY OF MICHIGAN

ON BOUNDED REGULAR FRONTIERS IN THE PLANE*

BY W. A. WILSON

1. *Introduction.* The term *regular frontier* has been introduced by P. Urysohn† to designate a continuum which is the frontier of two or more components of its complement. Regular frontiers in the plane have been discussed by various authors. A. Rosenthal‡ has shown that a continuum which is the union of two bounded continua that are irreducible between the same pair of points and have no other common points is a regular frontier. R. L. Moore§ has given necessary and sufficient conditions that a bounded continuum be a regular frontier whose complement has exactly two components. C. Kuratowski‖ has given necessary conditions for a continuum to be a regular frontier which is the frontier of *every* component of its complement.

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* Presented to the Society, October 29, 1927.

† P. Urysohn, *Mémoire sur les multiplicités Cantoriennes*, Fundamenta Mathematicae, vol. 7, p. 98.

‡ A. Rosenthal, *Teilung der Ebene durch Irreduzible Kontinua*, Sitzungsberichte der Münchener Akademie, 1919.

§ R. L. Moore, *Concerning the common boundary of two domains*, Fundamenta Mathematicae, vol. 6, pp. 203–213.

‖ C. Kuratowski, *Sur les coupures du plan*, Fundamenta Mathematicae, vol. 6, pp. 130–145.

In the present article Rosenthal's theorem is generalized to cover the case of two continua having a finite or enumerably infinite set of closed sets in common and irreducible between each pair. With this result and a theorem by the author† given elsewhere, it is possible to formulate necessary and sufficient conditions for a bounded continuum to be a regular frontier. The principal theorems are to be found in §§5–7.

2. *Notation.* Besides the ordinary notation of the aggregate theory the following special notation and terminology will be used.

The whole plane will be denoted by Z. If A is a sub-set of C, the set of inner points of A relative to C will be denoted by A.* If A and B are two sets in the plane Z without common points and C separates A from B (i.e., $C \cdot A = C \cdot B = 0$ and every continuum in the plane which contains points of both A and B also contains one or more points of C), we say that C is an $S(A, B)$. If F is a frontier set, a component of $Z - F$ which has the frontier F will be called a *principal* component.

The statement "C is a continuum irreducible between the sets A and B" will apply not only to the case that $A + B \subset C$, but also to the case that C is irreducible between each point of $C \cdot A$ and each point of $C \cdot B$.

3. *Some Auxiliary Theorems.* Let $F = H + K$ be the union of two bounded continua and let $H \cdot K$ be the sum of a finite set of closed sets $\{\alpha\}$, such that both H and K are irreducible between each pair. The following properties of F are either well known or so easily established that the proofs are omitted.

(a) *If the number of sets* $\{\alpha\}$ *is greater than two, both H and K are indecomposable.*

(b) *If C is a sub-continuum of H or of K, $F - C$ is connected.*

(c) $H^* = F - K$ *and* $K^* = F - H$ *are connected; also* $\overline{H}^* = H$ *and* $\overline{K}^* = K$.

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† W. A. Wilson, *On irreducible cuts of the plane between two points.* (To appear soon in Annals of Mathematics.)

If F lies in a plane Z, certain properties expressing relations between F and the rest of the plane can be proved without difficulty.

(d) *If $H \cdot K$ has at least n components, there are at least n components of $Z-F$ which have frontier points on both H^* and K^*. If $H \cdot K$ has an infinity of components, the components of $Z-F$ having frontier points on both H^* and K^* are enumerably infinite.*

These statements are proved by adding to H and K those components of $Z-F$ which do not have frontier points on both H^* and K^* and applying a theorem of S. Straszewicz.†

(e) *There is a bounded continuum P which is an irreducible $S(H^*, K^*)$ and contains $H \cdot K$.*

To establish this we observe that, since H^* and K^* are connected, there is a bounded continum Q which is an $S(H^*, K^*)$ by a theorem of Knaster and Kuratowski.‡ But by a theorem of Mazurkiewicz,§ Q contains a subcontinuum P which is an irreducible $S(H^*, K^*)$. Moreover $P \supset H \cdot K$, since every point of $H \cdot K$ is a common limiting point of H^* and K^*.

It may be added that every bounded irreducible $S(H^*, K^*)$ is a continuum containing $H \cdot K$.

4. LEMMA. *Let $F=H+K$ be the union of two bounded continua and let $H \cdot K$ be the sum of a finite set of closed sets $\{\alpha\}$ between each pair of which both H and K are irreducible. Let R be a component of $Z-F$ such that $\overline{R}$ contains a bounded continuum containing points of two or more sets α but no other points of F. Then F is the frontier of R.*

PROOF. The principles of inversion permit us to demonstrate the lemma on the assumption that R is bounded. Let

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† S. Straszewicz, *Über die Zerschneidung der Ebene*, Fundamenta Mathematicae, vol. 7, p. 174.

‡ B. Knaster and C. Kuratowski, *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2, p. 236.

§ S. Mazurkiewicz, *Sur un ensemble G*, etc., Fundamenta Mathematicae, vol. 1, p. 63.

C be the continuum referred to above. Then R contains a closed polygon J (whose interior is I) so large that $C-C\cdot I$ has no component containing points of more than one α. This is easily seen since the number of sets α is finite. On the other hand, if α_1 and α_2 are two sets met by C, there are sub-continua A_1 and A_2 of $C-C\cdot I$ irreducible between α_1 and α_2, respectively, and J, and $A_1\cdot A_2=0$. Let M be an arc of J irreducible beween A_1 and A_2. Then $Q=A_1+M+A_2$ is irreducible between α_1 and α_2 and does not meet $F-(\alpha_1+\alpha_2)$.

Since Q and H are both irreducible between α_1 and α_2 and have only points of these sets in common, while Q is not the union of two indecomposable continua, it follows by an extension of Rosenthal's theorem† that $Z-(Q+H)$ has two principal components S_1 and S_2. Similarly, let T_1 and T_2 be the principal components of $Z-(Q+K)$.

As K^* is connected, it lies in but one component of $Z-(Q+H)$; suppose that $K^*\cdot S_1=0$. Since $(F+Q)\cdot S_1=0$ and S_1 has frontier points on M, $R\cdot S_1\neq 0$ and $S_1\subset R$. Likewise either T_1 or T_2, say T_1, is a part of R. As $F=H+K$, every point of F is a frontier point of either S_1 or T_1, and a fortiori of R.

5. THEOREM. *Let $F=H+K$ be the union of two bounded continua and let $H\cdot K$ be the sum of a finite number n of closed sets $\{\alpha\}$ between each pair of which both H and K are irreducible. Then the number of principal components of $Z-F$ is at least n.*

PROOF. Let R_i be any component of $Z-F$ having frontier points on both H^* and K^*, let P be a bounded irreducible $S(H^*, K^*)$, and let $P_i=P\cdot R_i$. Suppose that $\overline{P}_i$ has a sub-continuum containing points of more than one set α for only k values of i, say $i=1, 2, \cdots, k$, where $k<n$. We first show that $k\geqq 1$. For, if α' denotes any α and α'' the sum of the remaining sets α, the continuum P contains a connected

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† W. A. Wilson, *On the separation of the plane by irreducible continua*, this Bulletin, vol. 33, pp. 733–744, §5.

set having no points of α' or α'', but limiting points on both. This connected set must be a part of some P_i.

In some work done elsewhere[†] it was shown that there is a continuum Q, which is a bounded irreducible $S(H^*, K^*)$, constructed as follows: Q is the union of k arcs $\{A_i\}$, each A_i lying in R_i, $i=1, 2, \cdots, k$, and k continua $\{B_i\}$, such that $B_i \cdot B_{i'}=0$ if $i \neq i'$, each B_i is irreducible between an end of A_i and one of A_{i+1} ($i+1=1$, if $i=k$) and meets no other points of $\sum_1^k A_i$, and $H \cdot K \subset \sum_1^k B_i \subset P$. Moreover, in the demonstration referred to, the circle used may be replaced by a polygon so large that, if I_i is its interior, no component of $\overline{P}_i - \overline{P}_i \cdot I_i$ has points on two sets α. Consequently any connected sub-set of any B_i having limiting points on two sets α, but no points on F, must lie in some R_i, $i>k$.

Since $n>k$, some B_i contains points of two or more sets α. Hence for some $i>k$, there is a connected sub-set j of B_i having limiting points on two sets α, but containing no points of F.[‡] Then j lies in some P_i, where $i>k$ by the last part of the previous paragraph. Therefore $\overline{P}_i$ contains points of more than one set α, contrary to the assumption at the beginning of the proof.

Therefore, for at least n values of i, $\overline{P}_i$, and consequently $\overline{R}_i$, contains a continuum having points on more than one set α. Then §4 shows that F is the frontier of each such R_i. Hence the theorem is proved.

COROLLARY. *Let $F=H+K$ be the union of two bounded continua and for every integer n let $H \cdot K$ be the sum of n closed sets such that both H and K are irreducible between each pair. Then the number of principal components of $Z-F$ is infinite.*

[†] See reference to paper by the author under §1, §§6 and 7. The hypothesis in these sections requires R to be a principal component of $Z-F$, but the demonstration only requires that R have frontier points on both H^* and K^*.

[‡] Anna M. Mullikin, *Certain theorems relating to plane connected sets*, Transactions of this Society, vol. 24, Theorem 1.

6. THEOREM. *For the bounded decomposable continuum F to be the frontier of exactly n components of its complement, it is necessary and sufficient that F be the union of two continua H and K such that $H \cdot K$ is the sum of n, but of no finite number greater than n, closed sets between each pair of which both H and K are irreducible.*

PROOF. If F is decomposable and $Z-F$ has n principal components, it has been shown elsewhere† that F is the union of two continua H and K such that $H \cdot K$ is the sum of a finite number, greater than or equal to n, of closed sets between each pair of which both H and K are irreducible. On the other hand, if $H \cdot K$ can be expressed as the sum of n closed sets between each pair of which both H and K are irreducible, there are at least n principal components of $Z-F$ by §5.

The combination of these statements gives the theorem.

REMARKS. The decomposition of $H \cdot K$ into n closed sets given above is unique. For, if there were two different decompositions into n closed sets with the properties assigned, say $H \cdot K = \sum_1^n \alpha_i$ and $H \cdot K = \sum_1^n \beta_i$, some β_i, say β_1, would contain points of more than one set α_i. Let $\beta_1 \cdot \alpha_1 \neq 0$ and let $\beta_{11} = \beta_1 \cdot \alpha_1$ and $\beta_{12} = \beta_1 \cdot (H \cdot K - \alpha_1)$. Then $H \cdot K = \beta_{11} + \beta_{12} + \sum_2^n \beta_i$ is a decomposition into $n+1$ closed sets between each pair of which both H and K are irreducible. We then have the contradiction that $Z-F$ has at least $n+1$ principal components.

7. THEOREM. *For the bounded decomposable continuum F to be the frontier of an infinity of components of its complement, it is necessary and sufficient that F be the union of two continua H and K such that for every integer n the set $H \cdot K$ is the sum of n closed sets between each pair of which H and K are irreducible.*

PROOF. The necessity of these conditions was shown in the proof of the theorem referred to in §6. That they are sufficient follows from §5, Corollary.

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† See reference to paper by the author under §1.

8. *Two Examples.* It might be inferred that the statements of the theorems in §§6 and 7 are unnecessarily complicated and that, if $H \cdot K$ is the sum of an infinite set of closed sets between each pair of which both H and K are irreducible, then $Z-F$ has an infinite set of principal components. This is not in general true. The following examples show the existence of two continua H and K having these properties, but such that $Z-F$ has no principal component in Ex. I and but two principal components in Ex. II.

EXAMPLE I. Let $Q=OABC$ be a closed rectangle, such that the lengths of OA and AB are 1 and 1/2 unit, respectively. Let q be the frontier of Q. Let M be a Cantor set extending from O to A, whose complementary open intervals $\{I_n\}$ are ordered according to size.

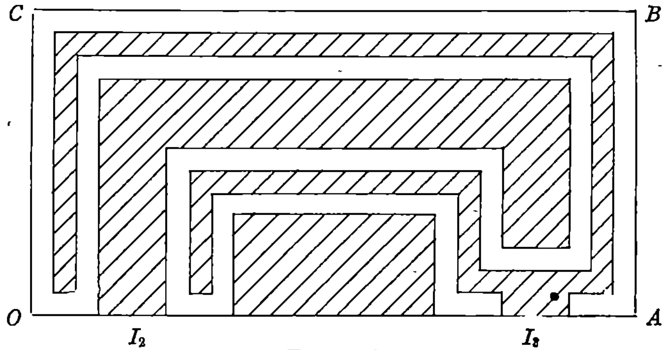

FIGURE 1

Now let a finite ordered set of closed rectangles, each of width d and of length greater than d, such that each rectangle has in common with the one preceding and the one following two different squares of side d but no point in common with any other rectangle, be called a *band*† of width d. It is easily seen that there is in Q a unique band which is contiguous to all points of $q-I_1$ ar q_1 is the union of $q-I_1$ and a broken line b in the end-points of I_1. Then $Q-B_1$ con simply connected region E_1 whose fronti $G_1=Q-B_1$.

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† For a complete discussion of the "method of bands" as applied to indecomposable continua see B. Knaster, *Un continu dont tout sous-continu est indécomposable*, Fundamenta Mathematicae, vol. 3, pp. 247–287. See also a memoir by K. Yoneyama, Tôhoku Mathematical Journal, vol. 12, pp. 60–62.

Likewise, in B_1 there is a unique band whose frontier q_2 is the union of q_1-I_1 and meeting q_1 only in the end-points of I_2. consists of I_2 and a region E_2 whose frontie construction can be repeated indefinitely; shaded area is $G_1+G_2+G_3$; the unshaded wit

Set $H=Q-\sum_1^\infty G_n=\prod_1^\infty B_n$. Obviously H The Cantor set M may be regarded as the s of closed sets without common points. T $O+A$, an enumerable set of sets each consi points of an interval I_n, and a non-enume each of which is one of the other points of of these sets. It is comparatively easy to irreducible between each pair of sets α.

Let K be the continuum symmetrical to OA. Then K is also irreducible between ea and $H\cdot K=\sum\alpha$. But each component of frontier a pair of continua of condensation

EXAMPLE II. This is a variation of Ex. I rectangle Q whose length and width are 1 unit respectively. Let ab be one side ar c, d, e and f divide ab into five intervals of 1 $cd=2/27$; $de=1$; $ef=2/27$; $fb=1/9$. Let $\{I$ able set of open intervals divided into three c $\{I_{3n}\}$ is the set of complementary intervals M in the interval de ordered as in Ex. I; $\{I$ are two other sets, of which the first are and the others will be defined later. Let q be

For $k=1, 2, 3$, let B_k, q_k, b_k, E_k, and G_k Ex. I. For $n=2$, let $I_k=I_{3n-2}=I_4$ be an length 1/9 at the center of one of the lon

b_1 perpendicular to, but not meeting, ab. Then B_4 is a band of width $1/3^4$ contiguous to q_3-I_4. This gives q_4, b_4, E_4, and G_4. In this case, we note that E_1+G_4 is a simply connected region whose frontier is $I_1+(b_1-I_4)+b_4$. In like manner, we let I_5 be an open interval of length 1/9 at the center of one of the longest segments of b_2 perpendicular to and not meeting ab and define B_5, q_5, b_5, E_5, and G_5. Here E_2+G_5 is a simply connected region whose frontier is $I_2+(b_2-I_5)+b_5$.

We now return to I_6, which has been defined above. Let this process be repeated indefinitely. Each I_{3n} lies on de; each I_{3n-2} is an open interval of length 1/9 at the center of one of the longest segments of b_{3n-5} perpendicular to and not meeting ab; and similarly for I_{3n-1}. In the figure the shaded areas are E_1+G_4 and E_2; the narrower white area is B_4; and the other white area is E_3. For every value of k, Q is the union of B_k and a finite number of the sets G_k. No two of the sets G_{3n} or of the corresponding regions E_{3n} have common points. For each n, $E_1+G_4+\cdots+G_{3n-2}$ and $E_2+G_5+\cdots+G_{3n-1}$ are simply connected regions whose frontiers

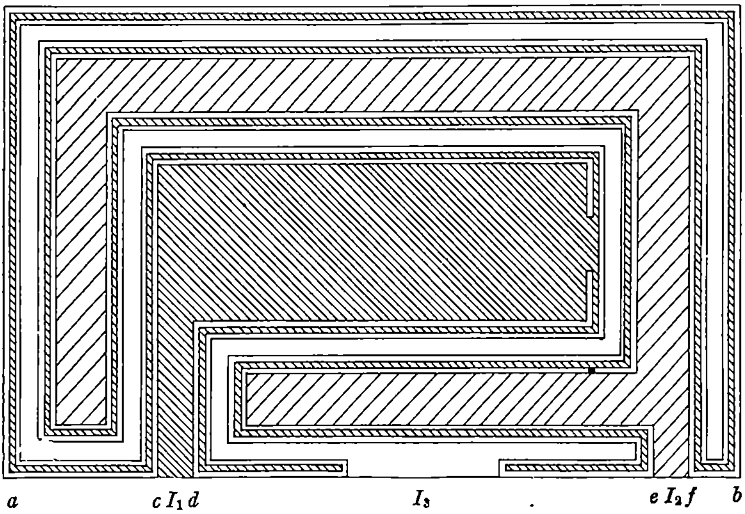

FIGURE 2

are the union of I_1 and I_2, respectively, with a broken line interior to Q.

Let $H=Q-\sum_1^\infty G_k=\prod_1^\infty B_k$. Obviously H is a continuum. The set $ac+fb+M$ is a closed set which may be regarded as the sum of an infinity of closed sets without common points. These are the set $ac+fb$, an enumerable set of sets each consisting of the end-points of an interval I_{3n}, and a non-enumerable set of sets each of which is one of the other points of M. Let α denote any of these sets. It is comparatively easy to show that H is irreducible between each pair of sets α.

Reflect the above figure about ab and denote corresponding sets by primes. Then $K=H'$ is a continuum irreducible between each pair of sets α and $H\cdot K=\sum\alpha$. The components of $Z-F$, where $F=H+K$, are the exterior of the rectangle $Q+Q'$, the enumerable set of regions $G_{3n}+G'_{3n}$, each of which has as its frontier a pair of continua of condensation of F, and two other regions. These are $\sum_1^\infty G_{3n-2}+\sum_1^\infty G'_{3n-2}$ and $\sum_1^\infty G_{3n-1}+\sum_1^\infty G'_{3n-1}$. Each of these has the frontier F and is therefore a principal component of $Z-F$.

The essential difference between the two examples is that in Ex. I, $H\cdot K$ cannot be divided into any finite number of closed sets between each pair of which H and K are irreducible, while in Ex. II, $H\cdot K$ can be divided into precisely two, but no more, such sets, namely the sets $ac+fb$ and M. Thus Ex. II is in strict accordance with §§6 and 7.

YALE UNIVERSITY

ON THE BEHAVIOR OF INTEGRAL FUNCTIONS IN DISTANT PORTIONS OF THE PLANE*

BY W. B. FORD

1. *Introduction.* In a series of remarkable papers extending over the years 1904–08 Mr. E. W. Barnes† has determined the asymptotic behavior in the neighborhood of the point at infinity of various types of analytic functions. The types considered are distinguished from each other by the forms of their Maclaurin developments, these being regarded as given and forming the starting point of the investigation in each case.‡ In point of method, Barnes makes extensive use of contour integrations and the calculus of residues. In point of results, his emphasis falls rather upon the detailed study of each individual function-type considered than upon the attainment of one or more general theorems which may be regarded as central in character to the subject as a whole. In my book on divergent series (see footnote below) I have already indicated on page 60 a general theorem through which the asymptotic character of a wide variety of special functions, including some of those considered by Barnes, may be readily determined§ and I desire in the present paper to indicate another such theorem, this being of a nature supplementary to the former one. It may be applied

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* Presented to the Society, September 8, 1927.

† Now the Bishop of Birmingham (England).

‡ A list of the papers referred to may be found in my book entitled *Studies on Divergent Series and Summability* (Michigan Science Series, vol. II, 1916) page 184. The various function-types considered by Barnes are summarized in his paper entitled *The asymptotic expansion of integral functions defined by Taylor's series* appearing in the Philosophical Transactions of the Royal Society of London, vol. 206 (1906), pp. 249–297.

§ I take this occasion to note that the relation $w=0, 1, 2, 3, \cdots$, occurring in line 6 of the theorem should be corrected to read $w=0, \pm 1, \pm 2, \pm 3, \cdots$.

in particular to the special function designated by Barnes as $G_\beta(z, \theta)$ and defined as follows:

$$G_\beta(z,\theta) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)(n+\theta)^\beta}, \tag{1}$$

where β and θ are any constants, real or complex, except that $\theta \neq 0, -1, -2, \cdots$. In §4 I shall show, in fact, how the theorem may be applied to this special function.

As to method, I find that the elementary theory of differential equations suffices to establish one of the two parts of the theorem. For the other part, I employ the calculus of residues in the manner common to such investigations.

Wherever the symbol $R(z)$ occurs it is to be understood as meaning *the real part of z.*

2. THEOREM.

Part I. $R(z)>0$. *If, when considered for large (positive integral) values of n, the function g(n) occurring in the coefficient of the power series*

$$\sum_{n=0}^{\infty} \frac{g(n)}{\Gamma(n+k)} z^n \text{ ; } k = \textit{any given constant (real or complex)} \tag{2}$$

may be developed in either a convergent or asymptotic series of the form

$$g(n) \doteq c_0 + \frac{c_1}{n+k} + \frac{c_2}{(n+k)(n+k+1)} + \cdots + \frac{c_p + \delta(p,n)}{(n+k)(n+k+1)\cdots(n+k+p-1)} \tag{3}$$

where $c_0, c_1, c_2, \cdots$ *are constants and* $\lim_{n\to\infty}\delta(p, n) = 0$, *then the function f(z) defined by the series* (2), *when considered for values of z for which* $R(z)>0$, *may be developed asymptotically in the form*

$$f(z) \sim e^z z^{1-k}\left[c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots\right]. \tag{4}$$

Part II. $R(z) < 0$. *If the function* $g(n)$ *may be regarded as a function* $g(w)$ *of the complex variable* $w = x + iy$ *and as such satisfies the following two conditions*: (a) *it is single valued and analytic throughout the finite w-plane and* (b) *it remains less than a constant when considered for all values of w of large modulus and of real part greater than any arbitrary fixed value (positive or negative), then the function* $f(z)$ *defined by* (2), *when considered for values of z for which* $R(z) < 0$, *may be developed asymptotically in the form*

$$f(z) \sim -\frac{g(-1)}{\Gamma(k-1)}\frac{1}{z} - \frac{g(-2)}{\Gamma(k-2)}\frac{1}{z^2} - \frac{g(-3)}{\Gamma(k-3)}\frac{1}{z^3} - \cdots. \tag{5}$$

It is to be observed at once that under the conditions imposed in either part I or part II of the theorem, the series (2) necessarily has an infinite radius of convergence, thus rendering $f(z)$ a so-called integral function. In this particular the theorem is in contrast to the former one referred to in §1.

In order to prove the theorem, let us begin by considering the special function defined by (2) in case $g(n) = 1$; that is, the function which we shall call $F(z)$, defined as follows:

$$F(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+k)}. \tag{6}$$

Upon multiplying both sides of this relation by z^{k-1} and subsequently taking the derivative of each member of the result, making use of the relation

$$\Gamma(n+k) = (n+k-1)\Gamma(n+k-1),$$

we find that

$$\begin{aligned}\frac{d}{dz}[z^{k-1}F(z)] &= \frac{z^{k-2}}{\Gamma(k-1)} + z^{k-1}\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n+k)}\\ &= \frac{z^{k-2}}{\Gamma(k-1)} + z^{k-1}F(z).\end{aligned}$$

Thus it appears directly that $F(z)$ satisfies the differential equation

$$F'(z) + \frac{k-1-z}{z}F(z) = \frac{1}{z\Gamma(k-1)}\cdot$$

This equation, being linear and of the first order, is readily solved by elementary methods giving as its general solution

$$F(z) = \frac{e^z z^{1-k}}{\Gamma(k-1)}\int_c^z e^{-z}z^{k-2}dz, \tag{7}$$

where c is an arbitrary constant. In order that this $F(z)$ shall coincide with the particular $F(z)$ defined by (6) we need only to determine c in such a manner that the limit approached by (7) as z approaches zero shall be the first term of (2); that is, shall be $1/\Gamma(k)$. Moreover, this will be the case if we take $c=0$, it being assumed for the moment (in order that the integral involved may assuredly have a meaning) that $R(k)>1$. In fact, with $c=0$ in (7) we have as desired

$$\lim_{z\to 0} F(z) = \frac{1}{\Gamma(k-1)}\lim_{z\to 0}\frac{\int_0^z e^{-z}z^{k-2}dz}{z^{k-1}} = \frac{0}{0}$$

$$= \left[\frac{e^{-z}z^{k-2}}{\Gamma(k-1)(k-1)z^{k-2}}\right]_{z=0} = \frac{1}{\Gamma(k)}\cdot$$

Whenever $R(k)>1$ we may therefore express the given $F(z)$ of (6) as follows:

$$F(z) = \frac{e^z z^{1-k}}{\Gamma(k-1)}\int_0^z e^{-z}z^{k-2}dz. \tag{8}$$

If we now confine ourselves to values of z for which $R(z)>0$, we may write (8) in the form

$$F(z) = \frac{e^z z^{1-k}}{\Gamma(k-1)}\left[\int_0^\infty e^{-z}z^{k-2}dz - \int_z^\infty e^{-z}z^{k-2}dz\right],$$

where it is understood that the integration in the first integral takes place along the positive real axis from $z=0$ to $z=+\infty$, while in the second integral it takes place in the direction of the positive real axis from $z=z$ to $z=\infty$.

We have now but to recall that

$$\int_0^\infty e^{-z}z^{k-2}dz = \Gamma(k-1)$$

in order to have

$$(9)\quad F(z) = e^z z^{1-k}\left[1 - \frac{1}{\Gamma(k-1)}\int_z^\infty e^{-z}z^{k-2}dz\right];\ R(z) > 0.$$

The expression here appearing is seen to have a meaning not only under the tentative assumption $R(k)>1$, which was introduced above, but for all values (real or complex) of k. In fact, it is analytic throughout the finite k-complex plane, and, inasmuch as the same is true of the expression occurring on the right in (6), it follows that, whatever the value of k, $F(z)$ is given by (9) so long as $R(z)>0$. Moreover, with $R(z)$ thus limited, we see that as mod z increases indefinitely the integral appearing in (9) becomes an infinitesimal of order higher than (mod $z)^{-m}$, where m is any assigned positive integer. Hence, if $R(z)>0$, we may write

$$(10)\qquad F(z) = e^z z^{1-k}[1+\eta(z)];\quad \lim_{|z|\to\infty} z^m\eta(z) = 0,$$

where $m=0, 1, 2, 3, \cdots$. To state this in the language of the theory of asymptotic series, we thus may write when $R(z)>0$,

$$F(z) \sim e^z z^{1-k}\left[1 + \frac{0}{z} + \frac{0}{z^2} + \frac{0}{z^3} + \cdots\right],$$

or simply

$$(11)\qquad F(z) \sim e^z z^{1-k}.$$

This preliminary result having been established, we proceed to establish part I of the theorem. In order to avoid unnecessary repetition of statement, let it be henceforth understood until the contrary is stated that the only values of z under consideration are those for which $R(z)>0$.

Having chosen an arbitrarily large positive integer m, we may write the function $f(z)$ defined by (2) in the form

$$f(z) = \phi(z,m) + \psi(z,m), \tag{12}$$

where

$$\phi(z,m) = \sum_{n=0}^{m-1} \frac{g(n)}{\Gamma(n+k)} z^n, \quad \psi(z,m) = z^m \sum_{n=0}^{\infty} \frac{g(m+n)}{\Gamma(n+k+m)} z^n.$$

By choosing m sufficiently large, the coefficient $g(m+n)$ may be developed for all values of n in the form (3). If this be done, $\psi(z, m)$ takes the following form:

$$\begin{aligned} \psi(z,m) &= c_0 z^m \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+k+m)} \\ &\quad + z^m \sum_{r=1}^{p} c_r \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+k+m+r)} + R(z,m,p), \end{aligned} \tag{13}$$

where

$$R(z,m,p) = z^m \sum_{n=0}^{\infty} \frac{\delta(m+n,p)}{\Gamma(n+k+m+p)}. \tag{14}$$

Considering now the first two expressions appearing on the right in (13), if we apply (11) to the first of these, using the appropriate value of k, namely, $k=k+m$, we see that for values of z of large modulus it becomes asymptotic to $c_0 z^m e^z z^{1-k-m} = c_0 e^z z^{1-k}$. Similarly, the second of the expressions in question consists of p series of which the rth is seen to be asymptotic to $c_r e^z z^{1-k-r}$, when mod z is large. Hence, $\psi(z, m)$ is of the form

$$\psi(z,m) = \Omega(z,m,p) + R(z,m,p) \tag{15}$$

where

$$\Omega(z,m,p) \sim e^z z^{1-k} \left[c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots + \frac{c_p}{z^p} \right]. \tag{16}$$

We shall now show that when mod z is large $\psi(z, m)$ may be developed asymptotically in the analogous series (instead of polynomial) form as follows:

$$(17)\qquad \psi(z,m) \sim e^{z}z^{1-k}\left[c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \cdots\right].$$

In order to do this, it suffices in view of (15) and (16) to show that, having chosen an arbitrarily small positive quantity ϵ, we may determine m such that, by subsequently taking $\rho =$ mod z sufficiently large, we may write

$$(18)\qquad |R(z,m,p)| < \frac{K\epsilon e^{\rho}\rho^{1-k}}{\rho^{p}} = K\epsilon e^{\rho}\rho^{1-p-k}\ ;\ K = \text{const}.$$

This relation, in fact, may be brought about as follows: For the given ϵ choose m so large that $|\delta(m, p)| < \epsilon$. Then we shall likewise have necessarily $|\delta(m+n, p)| < \epsilon$; $n = 0, 1, 2, 3, \cdots$. From the definition (14) of $R(z, m, p)$, we may then write

$$|R(z,m,p)| < \rho^{m}\epsilon \sum_{n=0}^{\infty} \frac{\rho^{n}}{\Gamma(n+k+m+p)}\,.$$

Applying (10) with $k+m+p$ used for k, we thus have

$$\begin{aligned} |R(z,m,p)| &< \rho^{m}\epsilon e^{\rho}\rho^{1-(k+m+p)}[1+\eta(m,p,z)] \\ &= \epsilon e^{\rho}\rho^{1-p-k}[1+\eta(m,p,z)], \end{aligned}$$

where $\lim_{|z|\to\infty}\eta(m, p, z) = 0$. This result at once yields the desired relation (18), thus establishing (17).

We have now but to observe that $\phi(z, m)$, being a polynomial in z, is such that

$$(19)\qquad \phi(z,m) \sim e^{z}z^{1-k}\left[0 + \frac{0}{z} + \frac{0}{z^2} + \frac{0}{z^3} + \cdots\right],$$

in order to obtain from (12), (17) and (19) the desired result (4) of part I of the theorem.

We therefore proceed to the proof of part II of the theorem. For this we shall employ the following fundamental statement from the calculus of residues:*

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* The statement may be regarded as an immediate corollary of the fundamental theorems of Cauchy in the theory of functions of a complex variable. Details of a proof may be found, however, on page 9 of my *Divergent Series,* etc. already mentioned.

"If $P(w)$ and $Q(w)$ are any two functions of the complex variable w both of which are single valued and analytic throughout a region A of the w-plane and of which $Q(w)$ vanishes within A only at the points $w=\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$ which are zeros of the first order, and if C_n denotes any closed contour lying within A and including the points $w=\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$, then we may write

$$\frac{1}{2\pi i}\int_{C_n}\frac{P(w)}{Q(w)}dw = \sum_{n=1}^{n}\frac{P(\lambda_n)}{Q'(\lambda_n)}, \tag{20}$$

where the indicated integration is taken about the entire contour C_n in the positive direction."

Formula (20) being premised, let us take for our present purposes

$$P(w) = \frac{\pi g(w)(-z)^w}{\Gamma(w+k)}, \qquad Q(w) = \sin \pi w,$$

where $g(w)$ is the function described in part II of the theorem, z being regarded for the present as having any fixed value. Moreover, for the contour C_n, let us take the rectangle in the w-plane formed by the straight lines

$$w = a - \tfrac{1}{2} + iy, \quad w = \tfrac{1}{2} + 2n + iy, \quad w = x \pm ij,$$

where a is any arbitrarily large negative integer, n is an arbitrarily large positive integer and j is any arbitrarily large positive quantity. Applying (20), we then obtain the relation

$$\begin{aligned}\frac{1}{2i}\int_{C_n}\frac{g(w)(-z)^w dw}{\Gamma(w+k)\sin\pi w} &= \sum_{n=a}^{2n}\frac{g(n)z^n}{\Gamma(n+k)} \\ &= \sum_{n=a}^{-1}\frac{g(n)z^n}{\Gamma(n+k)} + \sum_{n=0}^{2n}\frac{g(n)z^n}{\Gamma(n+k)},\end{aligned} \tag{21}$$

the last summation on the right thus becoming the sum of the first $2n$ terms of the given series (2). As to the integral on the left, we shall assume that the function $(-z)^w$ appearing in the integrand is rendered precise in meaning for any given value of $z=\rho(\cos\phi+i\sin\phi)$ through the following convention:

$$(22)\qquad (-z)^w = e^{w\log(-z)} = e^{w[\log\rho+i(\phi+\pi)]}\ ;\quad -2\pi<\phi\leqq 0.$$

Supposing at first that z is real and negative; that is, that $\phi=-\pi$, we now proceed to study in detail the integral appearing in (21).

First, along the side of the rectangle upon which $w=x+ij$, we have $dw=dx$ and $\sin\pi w=\sinh\pi j(\sin\pi x \operatorname{ctnh}\pi j+i\cos\pi x)$, so that, if we designate by I the contribution to the integral from this side, we may write

$$(23)\qquad I=\frac{(-z)^{ij}}{2i\sinh\pi j}\int_{2n+1/2}^{a-1/2}\frac{g(x+ij)(-z)^x dx}{\Gamma(x+k+ij)(\sin\pi x\operatorname{ctnh}\pi j+i\cos\pi x)}.$$

Now, as j is allowed to increase indefinitely, $\sinh\pi j$ becomes infinite like $e^{\pi j}$, while $|\Gamma(x+k+ij)|$ vanishes to as high an order as that of $e^{-\pi j/2}$;* also we have $\lim_{j\to\infty}\operatorname{ctnh}\pi j=1$. Hence, recalling the condition (b) of part II of the theorem, we see that $\lim_{j\to\infty}I=0$.

Similarly, the contribution to the contour integral in (21) arising from the side of C_n upon which $w=x-ij$, is seen to be the same as (23), except that $-j$ occurs throughout instead of j. We have then but to recall that as j increases indefinitely $\sinh(-\pi j)$ becomes infinite like $e^{\pi j}$ while $|\Gamma(x+k-ij)|$ vanishes to as high an order as that of $e^{-\pi j/2}$, to see that this contribution also vanishes in the limit as $j\to\infty$.

Next, let us consider the contribution from the side of C_n upon which $w=1/2+2n+iy$. Here we have $dw=idy$ and $\sin\pi w=\cos i\pi y=\cosh\pi y$, so that, having taken $j=\infty$ as indicated above, the contribution in question becomes

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* See for example formula (10), page 24 of Nielsen's *Handbuch der Theorie der Gammafunction*. The formula there occurring, however, contains a typographical error inasmuch as the letter u which appears under the radical should be v.

$$J = \frac{(-z)^{1/2+2n}}{2}\int_{-\infty}^{\infty}\frac{g(2n+\frac{1}{2}+iy)(-z)^{iy}}{\Gamma(2n+k+\frac{1}{2}+iy)\cosh\pi y}dy$$
$$= \frac{z^{1/2+2n}}{2}\int_{-\infty}^{\infty}\frac{g(2n+\frac{1}{2}+iy)e^{iy\log\rho}}{\Gamma(2n+k+\frac{1}{2}+iy)\cosh\pi y}dy.$$

That the improper integral here appearing has a meaning follows when we recall that as y increases indefinitely in the positive sense, $\Gamma(2n+k+1/2+iy)$ vanishes to as high an order as that of $e^{-\pi y/2}$, and that as y increases indefinitely in the negative sense, the same function vanishes to as high an order as that of $e^{\pi y/2}$; also that, by virtue of (b), $g(2n+1/2+iy)$ remains less than a constant whatever the value of y. Moreover, we may now show that $\lim_{n\to\infty}J=0$. In fact, this appears by writing J in the form

$$J = \frac{g(2n)z^{2n}}{\Gamma(2n+k)}\cdot\frac{z^{1/2}}{2}\int_{-\infty}^{\infty}\frac{g(2n+\frac{1}{2}+iy)}{g(2n)}$$
$$\cdot\frac{\Gamma(k+2n)}{\Gamma(2n+k+\frac{1}{2}+iy)}\cdot\frac{e^{iy\log\rho}}{\cosh\pi y}dy,$$

and recalling that, inasmuch as the series (2) has an infinite radius of convergence, we have

$$\lim_{n\to\infty}\frac{g(2n)z^{2n}}{\Gamma(2n+k)} = 0,$$

whatever the value of z; also recalling our hypothesis (b) and the well known properties of the Γ function.

If we now take account of the contribution to the integral in (21) arising from the remaining side $w=a-1/2+iy$ of C_n, noting that we here have $dw=idy$, $\sin\pi w=(-1)^{a-1}\cosh\pi y$ while the integration takes place from $y=+\infty$ to $y=-\infty$, the fundamental relation (21) takes the following form (n having been allowed to increase indefinitely as indicated above):

(24) $$\frac{(-1)^a(-z)^{a-1/2}}{2}\int_{-\infty}^{\infty}\frac{g(a-\frac{1}{2}+iy)(-z)^{iy}}{\Gamma(a+k-\frac{1}{2}+iy)\cosh\pi y}dy$$
$$= \sum_{n=a}^{0}\frac{g(n)z^n}{\Gamma(n+k)}+f(z).$$

Thus far we have confined z to values which are real and negative. Suppose now that z as appearing in (24) is allowed to take on complex values. The right side of (24) is evidently analytic for all such values of z. We shall now show that the left side is likewise an analytic function of z so long as $R(z) < 0$; that is, so long as $-3\pi/2 < \phi < -\pi/2$. To establish this it suffices to show that the improper integral in (24) converges uniformly for values of z confined to such a region.* Now, the denominator of the integrand becomes infinite to as high an order as that of $e^{\pi y/2}$ when $y \to +\infty$, and it becomes infinite to as high an order as that of $e^{-\pi y/2}$ when $y \to -\infty$. As to the factor $(-z)^{iy}$, we have by (22)

$$(-z)^{iy} = e^{iy[\log \rho + i(\phi+\pi)]} = e^{iy \log \rho} e^{-(\phi+\pi)y},$$

and under the proposed restriction as regards ϕ, that is, $-3\pi/2 < \phi < -\pi/2$, we have $-3\pi/2 + \epsilon \leqq \phi \leqq -\pi/2 - \epsilon$, $\epsilon > 0$, and hence $-\pi/2 + \epsilon \leqq \phi + \pi \leqq \pi/2 - \epsilon$. We now have but to recall again hypothesis (b) of part II of the theorem to see that, under the indicated restrictions as regards ϕ, the integrand of (24) vanishes to as high an order as that of $e^{-\pi y/2} e^{-(-\pi/2+\epsilon)y} = e^{-\epsilon y}$ when $y \to +\infty$, and vanishes to as high an order as that of $e^{\pi y/2} e^{-(\pi/2-\epsilon)y} = e^{\epsilon y}$ when $y \to -\infty$. The uniform convergence property in question now follows immediately.

Inasmuch as (24) holds true, as we have now shown, for all negative values of z and at the same time each side is an analytic function of z so long as $R(z) < 0$, it follows that it is likewise true for all such values of z. The proof of the theorem is thus complete.

3. *Generalizations and Remarks.* In order to secure simplicity of statement in the theorem the conditions (a) and (b) of part II were made unnecessarily restrictive. In fact, if one examines the proof of this part it appears readily that the following generalizations and remarks can be made.

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* See for example Osgood, Encyklopädie der mathematischen Wissenschaften, II, 2, §6.

(α) In condition (b) the statement "remains less than a constant" may be replaced by the following stronger statement: "is such that, to an arbitrarily small positive quantity η there corresponds a positive constant K (independent of x and y) such that for all large values of $|x|$ and for all sufficiently large positive values of y one may write

$$\left| \frac{g(x \pm iy)}{g(x)} \right| < Ke^{\eta y}."$$

(β) In case condition (a) is not satisfied but instead the function $g(w)$, while still single valued throughout the finite w-plane, has m ($m \geqq 1$) singularities situated at the points $w = w_1, w_2, w_3, \cdots, w_m$, then (5) continues to hold true provided one subtracts from the right member the sum of the residues of the function

$$\frac{\pi g(w)(-z)^w}{\Gamma(w+k)\sin \pi w} \tag{25}$$

at these points.

(γ) In case condition (a) is not satisfied but instead the function $g(w)$, while still finite throughout the finite w-plane, is multiple valued owing to the presence of p ($p \geqq 1$) branch points at $w = W_1, W_2, W_3, \cdots, W_m, \cdots, W_p$, then (5) continues to hold true provided that one adds to the right member the sum of the p loop integrals of (25) of which the mth is described as follows: Draw the straight line extending from the point W_m to infinity in the direction of the positive (upper) half of the pure imaginary axis and let this line be regarded as a cut in the w-plane. Now let the loop consist of the two lines drawn on either side of and close to this cut, the ends of these lines near to the point W_m being joined by a circular arc of small radius drawn about this point. Moreover, let the integration over the loop as thus formed be understood to be in the positive (counter-clockwise) direction.

In case the exceptions described in (β) and (γ) both occur in $g(w)$ at the same time, the alterations to be made in (5) will then consist of both those described above.

Finally, it is to be noted that the theorem gives no information concerning the behavior of $f(z)$ in case $R(z)=0$. Naturally, this case calls for further investigation. We shall, however, omit it here, reserving it for possible discussion in a later paper.

4. *Application to the Function* $G_\beta(z, \theta)$. As stated in §1, we shall show in conclusion how the foregoing theorem, subject to the remarks in §3, may be applied in particular to determine the asymptotic behavior of the function $G_\beta(z, \theta)$ studied by Barnes and defined by (1).

Wishing first to apply part I, we write (1) in the form

$$G_\beta(z,\theta) = \sum_{n=0}^{\infty} \frac{g(n)z^n}{\Gamma(n+\beta+1)},$$

where

$$g(n) = \frac{\Gamma(n+\beta+1)}{\Gamma(n+1)(n+\theta)^\beta}\cdot \tag{26}$$

Now, it is a familiar fact immediately deducible from Stirling's formula for $\log\Gamma(x)$, that for values of x whose real part is positive, we have

$$\Gamma(x) \sim (2\pi)^{1/2}x^{x-1/2}e^{-x}\left(1+\frac{a_1}{x}+\frac{a_2}{x^2}+\cdots\right),$$

where $a_1, a_2, \cdots$ are constants. Hence

$$\Gamma(n+\beta+1) \sim (2\pi)^{1/2}e^{-(n+\beta+1)}(n+\beta+1)^{n+\beta+1/2} \cdot\left(1+\frac{a_1}{n+\beta+1}+\frac{a_2}{(n+\beta+1)^2}+\cdots\right),$$

$$\Gamma(n+1) \sim (2\pi)^{1/2}e^{-(n+1)}(n+1)^{n+1/2}\left(1+\frac{a_1}{n+1} + \frac{a_2}{(n+1)^2}+\cdots\right),$$

and therefore

$$\frac{\Gamma(n+\beta+1)}{\Gamma(n+1)} \sim e^{-\beta}\frac{(n+\beta+1)^{n+\beta+1/2}}{(n+1)^{n+1/2}}\left(1+\frac{b_1}{n}+\frac{b_2}{n^2}+\cdots\right),$$

where b_1, b_2, b_3, etc. are constants whose values in terms of a_1, a_2, a_3, etc. may be readily determined. It follows that we may write

$$\frac{\Gamma(n+\beta+1)}{\Gamma(n+1)(n+\theta)^\beta} \sim e^{-\beta}\left(1+\frac{\beta-\theta+1}{n+\theta}\right)^\beta \cdot\left(1+\frac{\beta}{n+1}\right)^{n+1/2}\left(1+\frac{b_1}{n}+\frac{b_2}{n^2}+\cdots\right).$$

The factor

$$\left(1+\frac{\beta-\theta+1}{n+\theta}\right)^\beta$$

here appearing is developable for large values of n in a convergent series proceeding according to ascending powers of $1/n$ while the factor

$$\left(1+\frac{\beta}{n+1}\right)^{n+1/2}$$

is developable for similar values of n in a convergent series of the form

$$e^\beta\left(1+\frac{c_1}{n}+\frac{c_2}{n^2}+\frac{c_3}{n^3}+\cdots\right).$$

It follows that we may write

$$\frac{\Gamma(n+\beta+1)}{\Gamma(n+1)(n+\theta)^\beta} \sim 1+\frac{d_1}{n}+\frac{d_2}{n^2}+\frac{d_3}{n^3}+\cdots,$$

where d_1, d_2, d_3, etc. are assignable constants. Hence, we may also write

$$\frac{\Gamma(n+\beta+1)}{\Gamma(n+1)(n+\theta)^\beta} \sim 1+\frac{e_1}{n+\beta+1} + \frac{e_2}{(n+\beta+1)(n+\beta+2)}+\cdots, \tag{27}$$

where e_1, e_2, e_3, etc. are assignable constants. But (27) is the form (3) wherein $k=\beta+1$.

Applying (4), *we therefore may write when* $R(z)>0$

$$G_\beta(z,\theta) \sim e^z z^{-\beta}\left(1+\frac{e_1}{z}+\frac{e_2}{z^2}+\frac{e_3}{z^3}+\cdots\right). \tag{28}$$

As to the values of the coefficients e_1, e_2, e_3, etc., they may evidently be computed to any desired point as follows, it being understood that $g(n)$ is defined by (26) and that the symbol Δ has its customary significance as used in the theory of finite differences; namely, $\Delta u(n)=u(n+1)-u(n)$:

$$\left\{\begin{aligned} e_1 &= -\lim_{n\to\infty}(n+\beta+1)(n+\beta+2)\Delta g(n),\\ e_2 &= +\lim_{n\to\infty}(n+\beta+3)(n+\beta+4)\Delta\big[(n+\beta+1)\\ &\qquad\cdot(n+\beta+2)\Delta g(n)\big],\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right. \tag{29}$$

Moreover, these limits are not difficult to evaluate when one makes repeated use of the relation $\Gamma(n+1)=n\Gamma(n)$.

Wishing now to apply part II of the theorem to the present function $G_\beta(z,\theta)$, we note that we here have the series (2) in which $k=1$ and $g(n)=(n+\theta)^{-\beta}$. As the function $g(w)$ to be used in applying the theorem, we therefore naturally take

$$g(w) = (w+\theta)^{-\beta}. \tag{30}$$

This function at once satisfies condition (b) of the theorem so long as β is a number having a positive, or zero, real part. In case β has a negative real part the function

$$\frac{g(x\pm iy)}{g(x)} = \left(\frac{x+\theta\pm iy}{x+\theta}\right)^{-\beta} = \left(1\pm\frac{iy}{x+\theta}\right)^{-\beta}$$

evidently satisfies the condition mentioned in (α) of §3. In either case, therefore, the theorem may be applied so far as condition (b) is concerned. As regards condition (a), this will not be satisfied inasmuch as the function (30) will have a singular point in general at $w=-\theta$. However, to meet this situation we need only to apply either remark (β) or (γ) of §3 according as the singular point in question is a pole (in which case β is a positive integer) or a branch point.

Applying (5), subject to the remarks (α), (β), (γ) of §3, and noting also that inasmuch as $k=1$ in the present instance, all the coefficients of (5) are equal to zero, *we conclude that when* $R(z)<0$ *we may write*

$$G_\beta(z,\theta) \sim L(z) + 0 + \frac{0}{z} + \frac{0}{z^2} + \cdots, \textit{ or simply } G_\beta(z,\theta) \sim L(z),$$

where $L(z)$ *represents the negative of the residue of the function*

$$\frac{\pi(-z)^w}{\Gamma(w+1)(w+\theta)^\beta \sin \pi w}$$

at the point $w=-\theta$ *in case* β *is an integer, or represents the loop integral of this function about the same point in the manner described in* (γ) *of* §3 *in case* β *is not an integer.*

Results (28) and (29) are essentially those arrived at by Barnes for the special function $G_\beta(z, \theta)$.* However, he obtains the coefficients e_1, e_2, e_3, etc. under a different form of definition from (29), and through an elaborate investigation he evaluates the loop integral referred to above as $L(z)$.

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* See result (A), page 265, vol. 206A (1906) of the Philosophical Transactions of the Royal Society of London.

A CORRECTION

BY W. L. AYRES

Dr. G. T. Whyburn has kindly called to my attention the fact that Theorems IV and V of my paper, *Concerning the boundaries of domains of a continuous curve*,* are incorrect. The following example constructed by Whyburn shows that theorem IV is false and a somewhat similar example may be used for Theorem V. Let K denote the rectangle with vertices $(1, 1)$, $(1, -1)$, $(-1, 1)$ and $(-1, -1)$ together with its interior. Let C_{-1} and C_0 be the circles with centers $(0, -1)$ and $(0, 1)$ and radii 1 and $\frac{1}{3}$ respectively. For each positive integer n let C_n be the circle with center $(1/2^n, (-1)^n)$ and radius $5/2^{n+3}$. Let C' and C'' be the set of all points (x, y) of $\sum_{i=-1}^{\infty} C_i$ for which $y>1$ and $y<-1$ respectively. Let $M=K+C'+C''$. Let D_{-1} be the set of all points (x, y) of K for which $x<0$ and $y<0$. Let D_0 be the set of all points (x, y) of K for which $x>\frac{1}{2}$. For each positive integer n let D_n be the set of all points (x, y) of K such that $1/2^{2n}>x>1/2^{2n+1}$. Let

$$D = C'' + \sum_{i=-1}^{\infty} D_i$$

and let P be the point $(-1, 1)$. The hypothesis of theorem IV is satisfied but one of the maximal connected subsets of the M-boundary of D with respect to P consists of the interval from $(0, 1)$ to $(0, -1)$ together with the interval from $(-1, 0)$ to $(0, 0)$.

I have found that Theorems IV and V are true if *either* of the following conditions be added to the hypotheses:

(4) *If B_1 is any maximal connected subset of B containing more than one point, then B_1 contains two non-cut-points which are accessible from both D and R.*†

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* This Bulletin, vol. 33 (1927), pp. 565–571.

† The maximal connected subset of $M-\bar{D}$ containing P is denoted by R.

(4′) *If* B_1 *is any maximal connected subset of* B *containing more than one point, then the set of non-cut-points of* B_1 *is accessible from either* D *or* R.

If condition (4) be added, then these theorems may be proved by practically the same argument that I used. If condition (4′) be added, these theorems may be established by the following argument. The argument follows my "proof" of Theorem IV through line 7 page 569. As x and y are accessible from either D or R (we will suppose D), there is an arc xuy which lies in D except for x and y. Let T be a maximal connected subset of $B_1 - xzy$. Suppose T has a limit point t on $<xzy>$. It is not difficult to see that T contains a non-cut-point s of B_1. There exists an arc st which lies in T except for the point t.* By (4′) there exists an arc rs which lies in D except for s and has only the point r in common with xuy. Now the set $xzy+xuy+rs+st$ divides the plane into three domains, one of which contains R. But if H is the domain containing R there is a point Q of B_1 which belongs neither to H nor to its boundary. Then Q cannot be a limit point of R. Hence every limit point of T belongs to the set $x+y$ and, since neither x nor y is a cut-point of B_1, both x and y are limit points of T. Then there is an arc xvy which lies in T except for x and y. Since every point of $xvy+xzy$ is a limit point of R, R lies in the complementary domain of $xuy+xvy+xzy$ which has $xvy+xzy$ as its boundary. Then the exterior of $xzy+xvy$ contains either D or R and the interior contains the other. Then $B = xzy+xvy$.

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* See R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), Theorem 1, p. 342.

SEVERI ON ALGEBRAIC GEOMETRY

Trattato di Geometria Algebrica. Volume I, Part I: *Geometria delle Serie Lineari.* By F. Severi. Bologna, 1926. viii+358 pp.

It is impossible to understand the principles on which a serious mathematical work is constructed unless one is in possession of a key to the author's thought. This should be found in the introduction, that is what an introduction is for, and in the present case Professor Severi sets forth his guiding principle in no uncertain terms.

"Con questo volume inizio la pubblicazione di un ampio Trattato, nel quale,—se il tempo e le forze mi basteranno—vorrei raccogliere, coordinare, completare dove occorra, tutto quanto vi è d'importante nel campo della geometria algebrica."*

To collect, coordinate and complete, when necessary, everything of importance in the field of algebraic geometry. This is the vast design which the author has set before himself. Consequently in the choice of subject matter and method, preference is given to that which has most connection with the general plan rather than to that which is immediately applicable to special problems.

The sub-heading of this first volume tells us that it deals with linear series, which means linear series of point groups on an algebraic curve. Why give so much attention to this particular topic? One is tempted to answer at once, "Because Professor Severi is an Italian, and this is an Italian specialty, almost a monopoly." That is no answer at all. A second answer would be because of the close connection with the study of algebraic functions and their integrals. But in the whole volume of 350 pages there is not a sign of integration, and scarcely any reference to a Riemann surface. All that will be very fully treated in subsequent volumes, but it is still far distant. What then? We must consider the group. The author in the present volume is concerned with algebraic curves, and with those properties of such curves which are unaltered by a general birational transformation. Such a transformation will carry a linear series into a linear series, it will preserve the genus of the curve and the structure of the Weierstrass points. The theory of linear series is the trunk whence all the branches of this birational geometry emanate. For instance, any non-hyperelliptic curve of genus p can be birationally transformed into a non-singular curve of order $2p-2$ in a space of $p-1$ dimensions, and this again can be carried into a plane curve if desired. Consequently in studying the birationally invariant properties of a curve, we may disregard the dimensionality of the space wherein it lies, and imagine that it is plane, or else skew and non-singular as we prefer. Herein we have a fundamental point of divergence from Professor Severi's first book which concerned itself primarily with plane curves.*

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* P. vii.

The first chapter is devoted to a study of linear systems of algebraic plane curves. This is a very large subject, but the author is concerned only with linear systems of curves as a means of reaching linear systems of point groups, and so leaves most of the topics on one side. What is essential is to establish the following theorem: An irreducible r-parameter system of algebraic curves is linear when, and only when, a single curve passes through r generic points, and the general curve is counted only once. An equal importance attaches to the beautiful theorem of Bertini: The general curve of a linear system will have a singular point only when this is a fixed singularity for all curves of the system.

Linear systems of curves lead to linear systems of point groups. Here, however, we may rise to as many dimensions as we please. Take an algebraic curve in a space of N dimensions. Cut it by a system of hypersurfaces linearly dependent on $r+s+1$ of their number, which are linearly independent, but of which s contain the curve, the result will be a linear r parameter series of point groups, a g_n^r, on our curve. We may, once for all, leave the s hypersurfaces out of our consideration. The most general algebraic curve in a space of $r+s$ dimensions may be written

$$\rho x_i = f_i(x, y), \qquad F(x, y) = 0.$$

If the system of hypersurfaces be written

$$\sum_{i=0}^{i=r} \psi_i(x_0 \cdots x_N) = 0$$

we have effectively

$$F(x,y) = 0, \quad \sum_{i=0}^{i=r} \phi_i(x,y) = 0,$$

that is to say, the intersections of a plane curve by the curves of a linear system. Conversely, if we write

$$X_i = \phi_i(x, y), \qquad F(x, y) = 0,$$

and if the series be not composite, i.e., if the generic curve ϕ through a point of F does not automatically pass through another point of F variable with the first, why then our original curve is birationally transformed to a plane curve, and this birationally to another space curve where the series is cut by the hyperplanes of the space. The problem of linear series may thus be treated either as a problem *in plano*, or else as a problem of series cut by hyperplanes. The whole art consists in the skillful alternation of these two points of view.

At this point the author encounters his first serious difficulty, to explain the meaning of a branch of a curve. He turns it with great skill. First he shows how to transform a curve birationally into a space curve which has no singular points. This may be projected down to a non-singular curve in 3-space, and this again to a plane curve whose only singularities are nodes. The complete vicinity of any point of this curve is given by one, or at most two, power series developments for x and y in terms of an auxiliary parameter t. Hence the complete vicinity of any point of the first curve is given by a finite number of such developments. If there be ν of these for a given point we shall say that there are ν branches there. How do we know that this number ν does not depend upon the method of transformation? To begin with we may show that if two curves are birationally related the exceptional points in the transformation are singular points. Hence if two non-singular space curves are birationally related there are no exceptional points. Hence the number of branches at a singular point is the number of corresponding developments on any non-singular space curve.

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* *Vorlesungen uber Algebraische Geometrie*, Leipzig, 1921, a translation and amplification of his previous lithographed lectures.

I must confess to being struck with admiration at the author's ingenuity in treating the problem of branches in this fashion. A better treatment for his purpose could not be devised. But the treatment, like all things human, has the defects of its qualities. No geometer has reached the heart of the singular-point difficulty, till he has made a thorough study of the power series developments.

Suppose that we have a linear series on a curve, a g_n^r, an r-parameter series of groups of n points. Certainly $n \geqq r$, for there can not be more parameters than points. Is it possible that our series is contained in a $g_n^{r'}$, where $r' > r$ and this in a $g_n^{r''}$, where $r'' > r$, etc.? Yes, this is certainly possible, but we must come to an end some time. Very well, when we have a series which is not contained in another of the same order but larger dimension, it is said to be "complete." How do we know that a given series is not contained in two different complete series? The usual proof is by the classical residuation theorem, the "Restsatz" of Brill and Nöther. But Severi, at this point, has none of the necessary apparatus at his disposal. He therefore turns to arithmetical and analytical considerations. Two groups of the same number of points are said to be "equivalent" if they are groups of the same series. Sums and differences of groups are defined and the laws of addition and subtraction established. If two series have a common group they are both contained in a larger series. Hence there can be but one complete series containing a given series, for if there were two, both would be contained in a third larger. I will hazard the guess that Severi must have regretted the very slight introduction of the zeros and poles of an algebraic function that he needed to prove this theorem, as a breach in the uniformity of his method; it was a very small breach at most.

In Chapter IV we reach the Jacobian group of a series, the group of multiple points, and a proof that this varies continuously with the series. This leads to the strange theorem, which requires simple transcendental considerations, that if we have two one-dimensional series, the difference between the Jacobian group and twice a general group of one, is equivalent to the corresponding difference in the other. There exists thus a single birationally invariant series which includes every such difference. Let the order of this invariant series be $2p-2$. Then p is defined as the "genus" of the curve and easily proved to be a positive integer when not zero. An immediate application is made in finding the group of points of multiplicity $r+1$ of a g_n^r, Brill's formula.

We have at last all the essentials for the theory of linear groups, order, dimension, completeness, deficiency, canonical series, all present, all deduced by very rapid and simple operations. Anyone who is familiar with the subject will be struck by the beauty of it all. How will it impress the beginner? Perhaps he will feel the same way about it. But I cannot escape the feeling that he may find difficulty in understanding the motivation of the different steps. Why does the juggler take a hat from the table? Because he wishes to pull a rabbit out of it? But why a rabbit? Because well-born rabbits come out of hats? If the reader has this feeling at all, it will be intensified in the next chapter, the fifth, which is called the geometry of linear series, "secondo il metodo rapido." It certainly is rapid. Given an algebraic plane curve which is irreducible. Let $p+1$ be the smallest number of generic points such that the complete series including their group has not the dimension 0. Then p is defined as the genus of the curve and is shown to be zero when, and only when, the curve is rational. If the order of a complete generic series be $>p$, then $n-r=p$. If $n-r<p$, the series is said to be "special." These facts being birationally invariant, we may confine ourselves to the consideration of curves with no singularities but nodes. We then introduce the idea of adjoint curves, residual groups, and so the Riemann-Roch theorem and its consequences by the classical processes of Brill and Nöther. It is beautifully done and in a chapter of 25 pages the reader becomes acquainted with all the important theorems about linear series. But if one has learned elsewhere to associate the genus of a curve with a shortage of singular points, this new definition seems to be dragged in by the heels. Still, the result is beautiful.

In Chapter VI, the author turns to the theory of algebraic correspondence between curves and upon one curve. This is the longest and, it seems to me, the most important chapter in the book. There is a vast amount of literature upon the subject, the names of Chasles, Cayley, Brill and Zeuthen come at once to mind. No one can get to the very bottom of this theory without the use of abelian integrals, but the introduction through linear series which Severi exhibited in previous writings and which he gives here in full, is very far ahead of any other with which I am familiar.

The chapter begins with a study of birational transformations of a curve into itself, which leads to the study of projective transformations of a curve of higher space into itself. These are finite in number when the genus is above unity. We thus get the idea of moduli, or birationally invariant numbers connected with a curve. More generally, suppose that we have an algebraic transformation of a plane curve into itself. This may always be written

$$f(x, y) = f(x', y') = 0,$$
$$\phi_1(x, y, x', y') = \phi_2(x, y, x', y') = 0.$$

Castelnuovo determined an upper limit for the number of coincidences of such a correspondence and showed that it is reached when, and only when, there is need of but one auxiliary equation, $\phi_1 = 0$.

Here the groups of points (x', y') corresponding to a variable (x, y) are, with (x, y) itself counted γ times, the groups of a linear series. We call $\gamma < 0$ the *value* of the correspondence. When positive it is an integer. On the other hand, if P correspond to $P_1 \cdots P_\nu$ and Q to $Q_1 \cdots Q_\nu$ and if $(-\gamma)$ be such a positive integer that the groups $P_1+P_2+ \cdots +P_\nu+(-\gamma)Q$ and $Q_1+Q_2+ \cdots +Q_\nu+(-\gamma)P$ are equivalent, we say that the correspondence has the negative integral value γ.

It may be shown by transcendental methods which Severi does not reproduce that on a curve of general moduli there are no correspondences but those of integral or zero value.

The value of the sum of two correspondences is the sum of their values, the value of the product, i.e., of the succession of two correspondences, is the negative of the product of their values. A pencil of curves will cut a correspondence of value unity. Hence we may construct a correspondence of any positive or negative integral value.

If we have a ν to ν' correspondence of value 0, the number of coincidences, which arise from the coalescence of two points on the same branch, is $\nu+\nu'$. Severi* proves this with great care, or rather proves the broader theorem that the sum of the groups corresponding to a given point in such a correspondence and its inverse is equivalent to the group of self-corresponding points. When it comes to correspondences of positive or negative value, we have merely to piece them out to correspondences of value zero by adding known correspondences of opposite value. We thus reach the Chasles-Cayley-Brill correspondence value

$$\nu+\nu'+2p\gamma.$$

In reaching this number, great care must be taken to count multiple coincidences correctly. The rule was found by Zeuthen and is carefully explained by Severi†; he also shows that this number is birationally invariant.

The value of a correspondence leads naturally to the important idea of the linear dependence of correspondences. The chapter closes with the discussion of another characteristic number of a correspondence called its "grade" which depends upon the coincidences between the correspondence and another differing infinitesimally from its inverse. I am willing to take the author's word that this is important, but must confess to finding it a bit obscure.

The great theory of the algebraic correspondence of curves developed in Chapter VI was based on the theory of linear series. The reader would naturally expect further developments in the subsequent chapters. Not at all; the author goes back to his original theme and exhibits two other methods of building up the theory "ab initio." In Chapter VII this is done by the methods of projective geometry in higher space, first brought to perfection by Segre and Castelnuovo, or Castelnuovo and Segre; it would be hard to settle the question of priority. The first thing is to find the complete series containing a given series. The essential point is to show that if two linear series have a common group, they are contained in a series of higher dimension. The plan consists in studying groups on curves in spaces of r and r' dimensions and the ruled variety in $r+r'+1$ dimensions generated by the lines connecting the two, thus getting eventually a more ample system of spaces of $r+r'$ dimensions than that originally given. The next step consists in proving once more the Riemann-Roch theorem, and this is done by means of a formula already established for the number of groups of $r+1$ points common to a g_n^r and a g_m^1. The chapter is short and sketchy.

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* Pp. 221 ff.

† Pp. 224 ff.

In the final chapter, Severi develops the theory of linear series by what he calls the "metodo algebrico." He takes the various steps in the following order:

1. A short account of Cremona transformations.
2. Study of the particular case of quadratic transformations.
3. A discussion of Nöther's idea of clusters of infinitely near singularities and a demonstration that any singular point can be treated as such a cluster.
4. A proof that the first polar of a general point is an adjoint curve whether the singularities are distinct or clustering. This shows that the number of clustering singularities must be finite. The proof that the form of the cluster is independent of the type of transformation used cannot be given satisfactorily without a study of the exponents in the power series developments.
5. A rather laborious proof that the known singularities of a curve impose independent conditions upon another curve of sufficiently high order.
6. Nöther's $AF+B\phi$ theorem.
7. Definition of adjoint curves, and residue theorem. Demonstration that the adjunction conditions are independent.
8. Riemann-Roch theorem as before.

It is part of a reviewer's duty to put the reader on guard against the writer's predilections and prejudices, a candid reviewer should also give warning against his own personal preferences. And the present reviewer must confess to a feeling of regret that the book was not written backwards, with this algebraic method explained at the outset, and the geometric methods added subsequently. This classical procedure which has been followed by Brill and Nöther and Bertini seems to me to penetrate more completely into the heart of the fundamental questions involved, especially when joined to a study of the series developments. Such a study will surely be made in subsequent volumes. The author sees the whole edifice which he plans to build; the reviewer can only see in part, and so he must make the mistake of judging the part separated from the whole. When shall we see the whole? That secret is still in the lap of the gods. The author says in the preface: "Con questo volume inizio la publicazione di un Trattato—se il tempo e le forze mi basteranno. . . . . " Every reader will hope ardently that they do.

J. L. Coolidge

SHORTER NOTICES

Algebra. By Oskar Perron. Berlin and Leipzig, Walter de Gruyter, 1927. Two volumes. Volume I. *Die Grundlagen.* 4 figures; viii+307 pp. Volume II. *Theorie der algebraischen Gleichungen.* 5 figures; viii+243 pp. Göschens Lehrbucherei, I. Gruppe: Reine Mathematik, Band 8, 9.

In view of the large number of texts on higher algebra already on the market, the average mathematician is not apt to hail a new book on this subject with joy or even interest. As one's eye runs hastily down the table of contents and sees a sequence of titles that calls to mind a composite picture of Weber, Serret, Burnside and Panton, et al., one can certainly be excused if one does not at first wax enthusiastic. In fact, the chapter-headings in Volume I are, in order: fundamental notions; polynomial theorem and Taylor's theorem; determinants (including the elements of matrices and of bilinear, quadratic and Hermitian forms); symmetric functions; divisibility of polynomials; existence of roots; and in Volume II the chapter-headings are. numerical solution of equations; reciprocal equations and equations of degree $\geqq 4$; substitution groups; Galois' theory of equations; equations of degree 5. Moreover, the captions of the subdivisions of the eleven chapters are, generally, the classic ones. Thus, first impressions lead one to expect merely one more book to add to the library catalogue and to which a beginning graduate student will look for help when Weber, Serret et al. happen to be in use. But, fortunately, first impressions are unjust in this case. For there are two places in the book where the treatment is sufficiently different from the classic treatment to interest a reader familiar with the usual treatise on the subject.

Probably the outstanding characteristic of the book is the prominence which is given to the notion of field (domain of rationality). This is certainly good pedagogy, for this is one of the most important concepts in mathematics. To be sure, the most central notion of all algebra is that of a linear algebra (hypercomplex numbers) including as special cases fields, groups, matrices and integral equations. But it would be too much to expect the author of a comparatively elementary text to display his various topics as special cases or applications of linear algebras; and we are glad to see that he has clustered the more strictly algebraic parts of his book around the notion of field. In particular, he has presented the subject of the divisibility of polynomials in one or more variables in such a way that there are not those annoying exceptions in the statement of the theorems about the highest common factor of two polynomials in two or more variables, as in Bôcher's text. Also, the emphasis on fields in the first volume admirably prepares the way for the discussion of Galois' theory of equations in the second volume. Although the notion of field is always used in this subject, of necessity, yet the author's presentation is such as to give to the student a number of important theorems about algebraic fields which he does not usually see formulated outside a course in algebraic numbers and should give him a grasp of the elements of this subject such as he does not usually have after taking the usual course in Galois' theory of equations. This aspect of the book is highly to be commended.

On the other hand, when the author has departed to this extent from the precedent set by Weber, Serret et al., it is all the more disappointing to find that he includes the classic material on the numerical solution of equations—Sturm's theorem, the theorem of Fourier and Budan, Descartes' rule of signs, upper limits of the roots, Newton's method, "regula falsi," and the methods of approximation due to Bernoulli and Graffe, although he is wise enough to omit Horner's method. Mathematics of this kind certainly does not belong in the same book with a neat and clear exposition of the divisibility of polynomials in several variables and Galois' theory of equations except in so far as it may be used to illustrate a bit of theory. In fact, such mathematics is not algebra at all, but what one might call "algebraic statistics"—that is, it applies to concrete problems in algebra the methods of statistics rather than the methods of pure algebra.

O. C. HAZLETT

Sur la Théorie des Equations aux Dérivées Partielles du Premier Ordre d'une Seule Fonction Inconnue. By N. N. Saltykow. Paris, Gauthier-Villars, 1925. 172 pp.

This book contains the lectures delivered by the author in the four Belgian Universities during 1923–1924. Much of the material represents results of the author's own researches which have not previously been published in detail or which are now presented in simplified form.

The treatment centers about the well known theorem concerning the equivalence of the two problems: (1) to find a complete integral of a given partial differential equation of the first order in one unknown function; (2) to find the general integral of the equations of its characteristics. The author develops a theory which exhibits the relations between these problems and which gives methods of finding a solution of either one when any incomplete set of integrals of the equations of the characteristics is known. He extends his results, moreover, to simultaneous systems by proofs like those for the single equation. In fact, the similarity is such that one wonders why he does not treat the more general case at the outset.

Since the characteristics can be defined by a linear homogeneous system of partial differential equations, there is an introductory chapter on the theory of such systems. In particular, a new form of the equations of the characteristics, fundamental for the subsequent developments, results from this preliminary discussion

The contributions are both interesting and important. The details of the exposition could perhaps be improved upon. A clear-cut statement of hypothesis and conclusion would help. For example, at the beginning of Chapter II the author announces he is about to prove Jacobi's theorem, but leaves the reader to find out which one. The misprints are numerous but of little consequence.

J. M. THOMAS,
National Research Fellow

The Mechanical Investigations of Leonardo da Vinci. By I. B. Hart. Chicago, Open Court Publishing Company, 1925. 2+240 pp.

This book is a very interesting account of a chapter in the history of mechanics, centering around the picturesque figure of Leonardo da Vinci. Leonardo's fame as the painter of The Last Supper and Mona Lisa has tended to cause an undervaluation of the excellent pioneer work which he did in the field of statics and dynamics. Dr. Hart has here assembled in good form the most important of these results, and has added a translation of the manuscript of Leonardo on the Flight of Birds, the only complete translation into English, so far as I know, of a manuscript of Leonardo.

There are eight chapters: I, pp. 1–12, a general discussion of Leonardo's manuscripts; II, pp. 13–35, mechanical science in the 15th century (largely devoted to the instruments used); III, pp. 36–53, contemporary scientific influences; IV, pp. 54–74, Leonardo's sources; V, pp. 75–98, Leonardo's work in dynamics; VI, pp. 99–142, Leonardo's work in statics; VII, pp. 143–193, Leonardo as a pioneer of aviation; VIII, pp. 194–235, translation of Leonardo's manuscript on the flight of birds. There is a good index.

To give some idea of the scope of Leonardo's work, a few of the results that are brought together in Dr. Hart's book may be mentioned. In dynamics, Leonardo stated the principle of inertia, and that of the equality of action and reaction, at least in special cases. He anticipated Galileo in stating the law for the velocity acquired by a falling body, though he did not succeed in obtaining the correct result for the distance fallen, thinking it, like the velocity, to be proportional to the time. Similar results are stated for bodies rolling down an inclined plane; but here, as also in his attempts at the composition and resolution of forces, he was much handicapped by lack of knowledge of trigonometry, which science was not yet generally known to Europeans, as Regiomontanus's great treatise, though existing in manuscript, was not published till much later (1533).

In statics, Leonardo determines certain centers of gravity, including (perhaps for the first time) that of the pyramid; studies the principles of the lever and pulley, giving nearly all the combinations that modern textbooks discuss; shows how power may be transmitted by various types of gearing; and considers the problem of loaded beams.

The last two chapters are the most important part of the book, since Leonardo's study of the flight of birds forms one of the most significant and valuable of the attempts of men to learn the secret of flight. We find here very careful observations of birds and the various motions of their wings, also the action of the tail as a rudder; and clever reasoning on the center of gravity and center of pressure, and the mechanical principles involved, including the case of soaring flight. The observations and discussions are illustrated by fine drawings.

Thus Leonardo da Vinci is seen to have been a pioneer in the development of aviation; and Dr. Hart's book, which makes these results available to the English reader, is accordingly a valuable addition to the literature of the history of science.

R. B. McClenon

Le Calcul des Probabilités—Son Évolution Mathématique et Philosophique. By L.-Gustave du Pasquier. Paris, Hermann, 1926. 21+304 pp., two tables.

Du Pasquier has written a delightful book in which the classic theory of probability centering around the so-called Theorem of Bernoulli is given in some detail, with an adequate setting in history, logic, and philosophy. Well chosen examples illustrate the use of the probability integral—for which two tables are given in the appendix—but the author is not interested primarily in technique. Nor has he found space for a mathematical treatment of generalized frequency functions, although he mentions the work of Pearson and others, and admits (p. 258) the inadequacy of the Gaussian function for biology. One of his chief purposes is to trace historically the ideas underlying the concept of probability—ideas very hazy at first, clarified to some extent by J. von Kries who insisted upon a "cogent reason" for cases declared to be "equally likely," made more lucid by means of the theory of ensembles used to determine "zones de comportement" (p. 217), and given a finishing touch by R. von Mises* who by the use of an infinite sequence as a "Kollektiv" with its accompanying "Verteilung" eliminates "equally likely cases" as a primary idea.

In Chapter V, six interpretations of probability are unfolded, designated: psychological, practical, logical, empirical, inductive, and interpretation by the zones of comportment. The view is held (pp. 188, 197, Chap. VI) that objective probability or "chance" exists, independent of human knowledge. Furthermore, even complete knowledge does not destroy probability—contrary to the view of Lourié (p. 203) that probability is the science for systematizing ignorance. The examples chosen to support Du Pasquier's contention are taken from the seemingly fortuitous behavior of numbers such as those forming the rth decimal place of the logarithms of consecutive integers. But later (p. 284) the author finds that such examples do not involve the "irregularity" demanded by the Second Postulate of von Mises—so they cannot be admitted to full standing as a "collectif," but must rank as a "syllepte" (p. 286).

The interesting applications in Chapter VII to the kinetic theory of gases, reversible and irreversible phenomena, entropy, etc., have a certain philosophic aroma. We are told (p. 229) that the phenomenon of fluctuation in the density of a gas adds a temperament to the inflexible determinism which rules the material universe.

Chapter VIII is devoted to the exposition of the theory of R. von Mises, setting forth the two postulates which determine a Kollektiv, and explaining the simple operations for deriving cumulative frequency functions from given cumulative frequency functions, "Verteilungen." This is well written, —in the statement of Postulate II, however (p. 266), there appears "ne soient pas nulles" instead of "nicht beide null." The concluding Chapter IX on mathematical probability and experience discusses further the postulates of von Mises and also logical systems.

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* *Grundlagen der Wahrscheinlichkeitsrechnung,* Mathematische Zeitschrift, vol. 5 (1919), pp. 52–99.

The book, throughout, is written in an entertaining style, free from many details that would be uninteresting to the average reader. Although the reviewer was unable to verify the formula in the middle of page 123 and the one at the foot of page 128, the book seemed to be exceptionally free from misprints and infelicities. It will be welcomed by those who are interested in the foundations of probability.

E. L. DODD

Curve Sghembe Speciali Algebriche e Trascendenti. Volume II: *Curve Sferiche, Curve Definite da una Relazione fra Flessione e Torsione, Curve Particolare situate sopra Superficie Assegnate.* By Gino Loria. Bologna, N. Zanichelli, 1925. 255 pp.

The second volume of the treatise on special space curves* treats both algebraic and transcendental curves. Those having tangents belonging to a linear complex base are discussed at length, followed by an outline of those belonging to quadratic and higher complex. Differential properties and methods of proof are particularly featured. Over a fifth of the volume is devoted to spherical curves; it is fairly exhaustive and is well written. Another fifth is given to curves defined by intrinsic equations. The last and longest chapter discusses curves on given surfaces, including helices, lines of curvature, geodesics, and asymptotic lines. The application of the latter to ruled surfaces contained in linear congruences does not take account of a number of important articles.

Extensive references are given, and a list of all the authors quoted in both volumes is given at the end. This feature is a particularly valuable one for bibliographic purposes. The proof reading has been very well done, except that German titles in the footnotes must occasionally suffer.

VIRGIL SNYDER

La Série de Taylor et son Prolongement Analytique. By J. Hadamard and S. Mandelbrojt. Scientia, No. 41. Deuxième édition, revue et mise au courant des progrès récents. Paris, Gauthier-Villars, 1926. 104 pp.

The systematic study of the singularities of analytic functions was begun by Hadamard. In 1901, a very valuable account of his own investigations together with those of other early workers, as Fabry, Leau, LeRoy, Borel and others, was presented by Hadamard in his now classic little book *La Série de Taylor et son Prolongement Analytique* published in the Collection Scientia (No. 12).

This work has now been revised and brought up to date by Hadamard, with the assistance of the brilliant young mathematician Mandelbrojt, who has published in the last few years a number of valuable papers bearing on the subject. In this edition, the authors present in addition to the results of the original edition all recent researches treating of the singularities and the analytic continuation of analytic functions defined by a Taylor's series, and, in the words of the authors, "presenting a real interest."

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* The first volume was reviewed in the Bulletin, vol. 31 (1925), p. 557.

On account of the amount of new material incorporated in this new edition, the work is being issued in two volumes, the first of these two volumes having been published in 1926, while the second volume is in preparation. This first volume contains a revised treatment of the subject matter of the first five chapters making up the first half of the original work, together with the newer researches relating to these topics. The second volume will contain a similar treatment of the other half of the original book. The bibliography has been greatly extended and brought up to date, the original 36 references of the first edition bearing on the first five chapters having now been increased to 91. All mathematicians interested in the study of analytic functions and their singularities will surely welcome this revised edition of a classic in its field.

L. L. SMAIL

La Méthode de Darboux et les Equations $s=f(x, y, z, p, q)$. Mémorial des Sciences Mathématiques, No. 12. By R. Gosse. Paris, Gauthier-Villars, 1926. 52 pp.

The first three chapters of this work are devoted almost exclusively to summarizing results and methods developed more at length in Chapters 4, 6, 7, and 8 of Goursat's *Leçons sur l'Intégration des Equations aux Dérivées Partielles du Second Ordre.* Chapter I concerns the theory of characteristics for the general partial differential equation of the second order in one unknown and two independent variables. It contains a neat proof of the uniqueness of the solution of Cauchy's problem for such an equation. Chapter II deals with invariants of the given equation and with equations of order n in involution with it. In addition to a short exposition of Darboux's method, Chapter III contains some theorems, due to Gau, on the formation of invariants and their reduction to a canonical form.

The last two chapters discuss the problem of determining what equations of the particular type mentioned in the title of the book are integrable by Darboux's method. An assemblage of results contained chiefly in papers by Gau, Goursat, and the author shows that all such equations which do not have an intermediate integral can be reduced by simple transformations to standard types. Some results are also given for two special forms of the equation having an intermediate integral.

The treatment is of necessity very sketchy. For this reason the general mathematical reader will doubtless prefer a standard treatise when seeking information about Darboux's method. The present pamphlet should, therefore, make its appeal to specialists in the field treated in its last two chapters. To them it should be valuable because it contains a synopsis of the present state of the subject, though its usefulness in this respect is apt to be impaired by the absence of a systematic bibliography.

J. M. THOMAS,
National Research Fellow

NOTES

The July, 1927, number of the Transactions of this Society (volume 29, No. 3) contains the following papers: *The contact of a cubic surface with an analytic surface*, by E. P. Lane; *Dynamical space-times which contain a conformal euclidean 3-space*, by H. P. Robertson; *Reduction of the ordinary linear differential equation of the nth order whose coefficients are certain polynomials in a parameter to a system of n first-order equations which are linear in the parameter*, by C. E. Wilder; *On the "third axiom of metric space,"* by V. W. Niemytzki; *Implicit functions and differential equations in general analysis*, by L. M. Graves; *Concerning acyclic continuous curves*, by H. M. Gehman; *On a general formula in the theory of Tchebycheff polynomials and its applications*, by J. Shohat; *A factorization theory for functions* $\sum_{i=1}^{n} a_i e^{\alpha_i x}$, by J. F. Ritt; *Arithmetic of logic*, by E. T. Bell; *Functionals of r-dimensional manifolds admitting continuous groups of point transformations*, by A. D. Michal. The October, 1927, number contains: *The differential equation of the elliptic cylinder*, by J. H. McDonald; *On a class of integral equations with discontinuous kernels*, by R. E. Langer and E. P. Brown; *Some third-order irregular boundary value problems*, by L. E. Ward; *Concerning point sets which can be made connected by the addition of a simple continuous arc*, by G. T. Whyburn; *The Notion of Green's function in the theory of integro-differential equations*, by J. D. Tamarkin; *A general theory of nets on a surface*, by V. G. Grove; *Simply transitive primitive groups*, by W. A. Manning; *The expansion problems associated with regular differential systems of the second order*, by M. H. Stone; Errata, volumes 21, 28, 29.

The July, 1927, number of the American Journal of Mathematics (volume 49, No. 3) contains the following papers: *Linear ordinary self-adjoint differential equations of the second order*, by Anna Pell Wheeler; *The singularities of a function defined by a Dirichlet series*, by D. V. Widder; *On entire function interpolation*, by I. M. Sheffer; *Optics in space of constant non-vanishing curvature*, by J. Pierpont; *On an imprimitive group of order* 5134, by J. R. Musselman; *Rational involutorial transformations in* S_4 *which leave invariant* ∞^4 *quadric varieties*, by H. C. Shaub; *A three-dimensional quartic variety in four-space*, by B. C. Wong; *Compound singularities of the rational plane quartic curve*, by J. H. Neelley; *Pencils of conics in the Galois fields of order* 2^n, by A. D. Campbell; *On generalized lacunae*, by J. J. Gergen; *Properties of certain aggregate functions*, by L. S. Hill; *The theory of group-reduced distributions*, by J. H. Redfield; *Closure of the tangential process on the rational plane cubic*, by F. E. Allen.

At the meeting of the Society to be held in Chicago on April 6–7, Professors E. P. Lane and E. B. Stouffer will be the principal speakers at a symposium entitled Recent developments in projective differential geometry.

The following schedule of lectures has already been arranged for Professor Constantin Carathéodory, of the University of Munich, who will be the first Visiting Lecturer of this Society. (See page 22 of this number of this Bulletin.) The University of Pennsylvania, January 10–13; Ohio State University, January 16–17; The University of Iowa, January 19–20; The University of Chicago, January 23–24; The University of Michigan, January 25–27; Adelbert College, January 30–31; Cornell University, February 1–2. After these lectures, Professor Carathéodory will be in residence at Harvard University until about June 1. Arrangements for additional lectures will be announced later.

The Lobachefsky prize of the Physico-Mathematical Society of Kazan has been awarded to Professor Hermann Weyl, of the Zurich Technical School, for his work as a whole and in particular for his contributions to the problems of space from the point of view of the theory of groups and for his researches on the representation of continuous groups. Earlier awards of this prize were to Lie in 1897, Killing in 1900, and Hilbert in 1903.

Cambridge University has awarded its John Winbolt prize in engineering jointly to C. Hinton, of Trinity College, and W. R. D. Manning, of Sidney Sussex College, for a dissertation on *Transverse oscillation of bridges*.

The Royal Academy of Madrid has announced the following subject for a prize memoir for 1927: *An exposition of the fundamental principles of analysis situs and its most important applications*; competition restricted to the Spanish, Portuguese, and Spanish Americans.

The Reale Accademia dei Lincei has elected Professor Giovanni Georgi, of the University of Cagliari, a correspondent in the section of mechanics, and Professor N. E. Nörlund, of the University of Copenhagen, a foreign member in the section of mathematics.

Professor Rudolf Mehmke has received the honorary degree of doctor of engineering from the Stuttgart Technical School, on the occasion of his seventieth birthday.

The University of Bristol, on the occasion of the opening of its new physics laboratory, has conferred honorary doctorates on Professor Max Born, of Gottingen, Sir William Bragg, of the Royal Institution, London, Professors A. S. Eddington, of Cambridge, Alfred Fowler, of the Imperial College of Science and Technology, and Paul Langevin, of Paris, and Sir Ernest Rutherford, of Cambridge.

The University of Toronto has conferred an honorary doctorate of science on Professor Louise D. Cummings, of Vassar College.

Dr. Paul Alexandroff, of Moscow, and Dr. Heinze Hopf, of Berlin, have been granted International Education Board fellowships for 1927–28, for the purpose of becoming acquainted with the work of American mathematicians in the field of analysis situs. They will study at Princeton and Harvard.

The following have been appointed National Research Fellows in mathematics for 1927–28 (this list includes reappointments): W. L. Ayres, R. W. Barnard, Alonzo Church, C. M. Cramlet, Jesse Douglas, D. A. Flanders, Orrin Frink, G. M. Merriman, T. W. Moore, Hillel Poritsky, H. P. Robertson, C. F. Roos, I. M. Sheffer, M. M. Slotnick, W. M. Whyburn.

Dr. F. Hund, of Göttingen, has been called to an associate professorship of theoretical physics at the University of Rostock.

Professor Robert König, of the University of Münster, has been appointed professor of mathematics at the University of Jena.

Dr. Maximilian Krafft has been appointed associate professor of mathematics at the University of Marburg.

Professor E. Salkowski, of Hannover, has been called to the professorship of descriptive geometry and practical mathematics at the Charlottenburg Technical School.

Dr. Erich Schönhardt has been appointed associate professor of mathematics at the University of Tubingen.

Professor Alois Timpe, of the Berlin Agricultural School, has been appointed professor of mathematics at the Charlottenburg Technical School.

The following have been admitted as private docents: Dr. F. K. Schmidt, in mathematics, at the University of Erlangen; Frau Hilda Pollaczek-Geiringer, in applied mathematics, at the University of Berlin.

Dr. Peter Debye, professor of physics at the Zurich Technical School, has accepted a call to the University of Leipzig, as successor to Professor Otto Wiener.

Dr. W. Saxer has been appointed professor of descriptive geometry at the Zurich Technical School.

Professor Giulio Pittarelli, of the chair of descriptive geometry at the University of Rome, has retired.

Miss Gwenthalyn Jones, of Chicago, has presented Princeton University with $200,000 for the endowment of a professorship of mathematical physics; the chair will be named after her uncle, Thomas D. Jones, of the Princeton class of 1876.

Dr. Ethel L. Anderton has been promoted to an assistant professorship of mathematics at Smith College.

Dr. Tobias Dantzig, professor of mathematics at the University of Maryland, will conduct a course in advanced mathematics for physicists and chemists at the Bureau of Standards during the year 1927–28.

Professor H. C. Gossard, of the University of Wyoming, has been appointed professor of mathematics and dean of men at Nebraska Wesleyan University.

Dr. Harold Hotelling, research associate in the Food Research Institute until the fall of 1927, has been appointed associate professor of mathematics at Stanford University.

Professor F. R. Moulton, director of the department of astronomy at the University of Chicago, has resigned, to become associated in an executive capacity with the Utilities Power and Light Corporation of Chicago.

Dr. B. C. Patterson has been appointed associate professor of mathematics at Hamilton College.

Associate Professor J. B. Reynolds, of Lehigh University, has been promoted to a full professorship of mathematics and theoretical mechanics.

Miss Georgia E. Robinson has been appointed professor of physics at Ozark Wesleyan College.

Dr. G. T. Whyburn has been appointed assistant professor of mathematics at the University of Texas.

The following appointments to instructorships in mathematics are announced.
Amherst College, Mr. B. LeF. Brown;
University of Chicago, Mr. T. F. Cope.

Professor Svante Arrhenius, director of the Nobel Institute, Stockholm, died October 2, 1927, at the age of sixty-eight.

Miss A. L. Van Benschoten, formerly professor of mathematics at Wells College, died September 18, 1927. Professor Van Benschoten had been a member of the American Mathematical Society since 1903.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

Aubry (A.). See Ball (W. W. R.).

Baldus (R.). Nichteuklidische Geometrie. Hyperbolische Geometrie der Ebene. (Sammlung Göschen.) Berlin, de Gruyter, 1927. 152 pp.

Ball (W. W. R.). Récréations mathématiques et problèmes des temps anciens et modernes. 2e édition française. 3e partie, avec additions de MM. A. Margossian, Reinhart, J. Fitzpatrick, et A. Aubry. Paris, Hermann, 1927. 363 pp.

Barzin (M.) et Errera (A.). Sur la logique de M. Brouwer. Bruxelles, 1927. 16 pp.

Becker (O.). Mathematische Existenz. Untersuchungen zur Logik und Ontologie mathematischer Phänomene. Halle, Niemeyer, 1927. 7+369 pp.

Buchanan (H. E.) and Sperry (P.). Plane trigonometry and tables. Richmond, Johnson, 1926. 11+116+112 pp.

Carmichael (R. D.) and Weaver (J. H.). The calculus. Boston, Ginn, 1927. 14+345 pp.

Claren (J.). Neue Methode zur Lösung von kubischen Gleichungen zum Selbststudium. 2te erweiterte Auflage. Leipzig, Verlag Bange, 1927. 20 pp.

Delambre (—.). See Ptolémée.

Descartes (R.). La géométrie. Nouvelle édition. Avec portrait de Descartes d'après Frans Hals. Paris, Hermann, 1927. 91 pp.

Dreetz (W.). See Schülke (A.).

Errera (A.). Exposé historique du problème des quatre couleurs. Bologna, 1927. 24 pp.

——See Barzin (M.).

Euwe (M.). Differentiaalinvarianten van twee covariante-vectorvelden met vier veranderlijken. (Dissertation, Amsterdam.) Groningen, Noordhoff, 1926. 85 pp.

Fine (H. B.). Calculus. New York, Macmillan, 1927. 8+421 pp.

Fitzpatrick (J.). See Ball (W. W. R.).

Gauss (C. F.). Gesammelte Werke. Band XI, Abteilung 1: Varia, Atlas des Erdmagnetismus, Biographisches Register. Berlin, Springer, 1927. 518 pp.

Gauss (C. F.) und Gerling (C. L.). Briefwechsel. Herausgegeben im Auftrage der Gesellschaft zur Beförderung der gesamten Naturwissenschaften zu Marburg von C. Schaefer. Berlin, Otto Elsner Verlagsgesellschaft, 1927. 20+820 pp.

Gerling (C. L.). See Gauss (C. F.).

Halma (—.). See Ptolémée.

HASSE (H.). Hohere Algebra. II: Gleichungen höheren Grades. (Sammlung Göschen.) Berlin, de Gruyter, 1927. 160 pp.

HAZARD (C. T.). See MASON (T. E.).

HOBSON (E. W.). The theory of functions of a real variable and the theory of Fourier's series. 3d edition, revised throughout and enlarged. Volume 1. Cambridge, University Press, 1927. 16+736 pp.

HORN (J.). Gewöhnliche Differentialgleichungen. 2te, völlig umgearbeitete Auflage. (Göschens Lehrbucherei.) Berlin, de Gruyter, 1927. 8+197 pp.

KEASEY (M. A.), KLINE (G. A.), and MCILHATTEN (D. A.). Plane trigonometry, with tables. Philadelphia, Blakiston, 1927. 7+130 pp.

KLINE (G. A.). See KEASEY (M. A.).

KRAITCHIK (M.). Le problème du cavalier. Bruxelles, l'Echiquier, 1927. 5+91 pp.

LITTLE (A. S.). A table of interpolation multipliers. London, Routledge, 1927.

MCILHATTEN (D. A.). See KEASEY (M. A.).

MARGOSSIAN (A.). See BALL (W. W. R.).

MASON (T. E.) and HAZARD (C. T.). Analytic geometry. Boston, Ginn, 1927. 11+224 pp.

PASCAL (E.). Introduzione al calcolo infinitesimale. Napoli, 1926. 4+116 pp.

PASCH (M.). Mathematik am Ursprung. Gesammelte Abhandlungen über Grundfragen der Mathematik. Leipzig, Meiner, 1927. 7+149 pp.

PICARD (E.). Leçons sur quelques types simples d'équations aux dérivées partielles avec des applications à la physique mathématique. Paris, Gauthier-Villars, 1927. 214 pp.

POTRON (—.). Exercices de calcul différentiel et intégral. Volume 2: Solution des exercices. Paris, Hermann, 1927. 4+258 pp.

RUSSELL (B.). The analysis of matter. London, Kegan Paul, and New York, Harcourt, Brace and Company, 1927. 8+408 pp.

PTOLÉMÉE. Composition mathématique. Traduite pour la première fois du grec en français par M. Halma et suivie des notes de M. Delambre. Tome 1 et tome 2. Paris, 1813, 1816. Réimpression fac-similé, Paris, Hermann, 1927. 76+476+48+8+448+40 pp.

REINHART (—.). See BALL (W. W. R.).

SCHAEFER (C.). See GAUSS (C. F.).

SCHÜLKE (A.) und DREETZ (W.). Aufgabensammlung aus der reinen und angewandten Mathematik. Leitfaden der Mathematik. Leipzig, Teubner, 1926–27.

SCHUH (F.). Beknopte hoogere algebra. Tweede stuk. Groningen, Noordhoff, 1926. 287 pp.

SEVERI (F.). Trattato di geometria algebrica. Volume I, parte I. Bologna, Zanichelli, 1926. 358 pp.

SPERRY (P.). See BUCHANAN (H. E.).

VICI (G.). Sulla sezione aurea di un segmento. Milano, Redaelli, 1926. 23 pp.

WEAVER (J. H.). See CARMICHAEL (R. D.).

WIELEITNER (H.). Mathematische Quellenbücher. Band 1: Rechnen und Algebra. Berlin, Salle, 1927. 7+75 pp.

PART II. APPLIED MATHEMATICS.

BOUASSE (H.). Verges et plaques, cloches et carillons. Paris, Delagrave, 1927. 480 pp.

BOUTARIC (A.). La physique moderne et l'électron. Paris, Alcan, 1927. 266 pp.

BRAY (F.). Light. London, Arnold, 1927. 12+284+6 pp.

BURGESS (R. W.). Introduction to the mathematics of statistics. Boston, Houghton Mifflin, 1927. 10+304 pp.

CAVEN (R. M.). Atoms and molecules. Being part I and chapter XII of The foundations of chemical theory. London, Blackie, 1927. 8+141 pp.

ERIKSON (H. A.). Elements of mechanics. New York, McGraw-Hill, 1927. 16+150 pp.

FELDMANN (C.). Die Berechnung elektrischer Leitungsnetze in Theorie und Praxis. Berlin, Springer, 1927. 10+554 pp.

FENDER (W.). See MOULTON (F. R.).

FRANK (P.). See RIEMANN (B.).

GEHRCKE (E.), herausgegeben von. Handbuch der physikalischen Optik. 2ter Band, 2te Hälfte, 1ter Teil. Leipzig, Barth, 1927. 388+6 pp.

GEIGER (H.) und SCHEEL (K.), herausgegeben von. Handbuch der Physik. Band 5: Grundlagen der Mechanik. Mechanik der Punkte und starren Körper. Redigiert von R. Grammel. Band 7: Mechanik der flüssigen und gasformigen Körper. Redigiert von R. Grammel. Berlin, Springer, 1927. 14+623+11+413 pp.

GOCHT (H.). Die Röntgen-Literatur. Stuttgart, Enke, 1927. 12+515 pp.

GRAMMEL (R.). See GEIGER (H.).

GRÖBER (H.). Einfuhrung in die Lehre von der Wärmeubertragung. Berlin, Springer, 1926. 10+200 pp.

GUYOT (J.). See LEMOINE (J.).

HARRISON (H. H.). Elements of telephone transmission. New York, Longmans, 1927.

HARROW (B.). The romance of the atom. New York, Boni and Liveright, 1927. 162 pp.

HEDGES (E. S.) and MYERS (J. L.). The problem of physico-chemical periodicity. London, Arnold, 1926. 95 pp.

HOUSTOUN (R. A.). A treatise on light. 5th edition. London, Longmans, 1927. 489 pp.

KELEN (N.). Die Staumauern. Theorie und wirtschaftliche Bemessung. Berlin, Springer, 1926. 8+294 pp.

LEECHMAN (G. F.). The theory and practice of steering. Glasgow, J. Brown, 1927. 102 pp.

LEMOINE (J.) et GUYOT (J.). Cours de physique. Tome III: Magnétisme, électricité. Paris, Vuibert, 1927. 362 pp.

LEWIS (G. N.). Die Valenz und der Bau der Atome und Moleküle. Übersetzt von G. Wagner und H. Wolff. Braunschweig, Vieweg, 1927.

MACMILLAN (W. D.). Theoretical mechanics. Statics and the dynamics of a particle. New York, McGraw-Hill, 1927. 18+430 pp.

MAYER (M.). Nomographie des Bauingenieurs. (Sammlung Göschen.) Berlin, de Gruyter, 1927.

MERKEL (F.). Die Grundlagen der Warmeübertragung. Dresden, Steinkopf, 1927. 11+234 pp.

MILLIES (A.). Räumliche Vieleckrahmen mit eingespannten Füssen unter besonderer Berücksichtigung der Windbelastung. Berlin, Springer, 1927. 6+96 pp.

MOULTON (F. R.). Einführung in die Himmelsmechanik. 2te, durchgesehene Auflage. Autorisierte deutsche Ausgabe von W. Fender. Leipzig, Teubner, 1927. 13+412 pp.

MÜLLER-BRESLAU (E.). La scienza delle costruzioni. Traduzione italiana di C. Rossi e L. Santarella. Volume III: Sistemi staticamente indeterminati (parte seconda). Milano, Hoepli, 1927. 14+798 pp.

MYERS (J. L.). See HEDGES (E. S.).

NIKURADSE (J.). Untersuchung über die Geschwindigkeitsverteilung in turbulenten Strömungen. Berlin, Verein Deutscher Ingenieure, 1926.

PALMER (L. S.). Wireless principles and practice. New York, Longmans, 1927.

POHL (R. W.). Einführung in die Elektrizitätslehre. Berlin, Springer, 1927. 7+256 pp.

PRASIL (F.). Technische Hydrodynamik. 2te, erweiterte Auflage. Berlin, Springer, 1927.

REICH (F.). Umlenkung eines freien Flüssigkeitsstrahles an einer senkrecht zur Strömungsrichtung stehenden Platte. Berlin, Verein Deutscher Ingenieure, 1926. 152 pp.

REJTÖ (A.). Einige Prinzipien der theoretischen mechanischen Technologie der Metalle. Berlin, Verein Deutscher Ingenieure, 1927. 21+501 pp.

RIEMANN (B.) und WEBER (H.). Die Differential- und Integralgleichungen der Mechanik und Physik. 2ter, physikalischer Teil, herausgegeben von P. Frank. Braunschweig, Vieweg, 1927. 28+863 pp.

ROSSI (C.). See MÜLLER-BRESLAU (E.).

SANTARELLA (L.). See MÜLLER-BRESLAU (E.).

SCHEEL (K.). See GEIGER (H.).

TOLMAN (R. C.). Statistical mechanics with applications to physics and chemistry. (American Chemical Society Monograph Series.) 334 pp.

WAGNER (G.). See LEWIS (G. N.).

WEBER (H.). See RIEMANN (B.).

WICKE (F.). Einführung in die höhere Mathematik unter besonderer Berücksichtigung der Bedürfnisse des Ingenieurs. In zwei Banden. Berlin, Springer, 1927. 10+921 pp.

WOLFF (H.). See LEWIS (G. N.).

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THE THIRTY-FOURTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY

The thirty-fourth Annual Meeting of the American Mathematical Society was held at Nashville, Tenn., Wednesday and Thursday, December 28–29, 1927, in conjunction with the meetings of the American Association for the Advancement of Science and the Mathematical Association of America. The headquarters of the mathematical organizations was at Ward-Belmont College. The meeting of the Society opened on Wednesday morning with a general session at which Professor James Pierpont delivered an address entitled *Mathematical rigor, past and present.* This address has already appeared in the January–February issue of this Bulletin. This was followed by sectional sessions: in the morning (i) Algebra and (ii) Point Sets and Foundations, and in the afternoon (i) Geometry and (ii) Analysis. The Gibbs Lecture was delivered on Wednesday afternoon, after the sectional sessions. Thursday morning was devoted to a general scientific session, followed by the annual business meeting and the election of officers and members of the Council. On Thursday afternoon was held a joint session of the two mathematical organizations and Section A of the American Association for the Advancement of Science.

The joint dinner on Thursday evening in the dining room of Ward-Belmont College was attended by about one hundred seventy-six mathematicians and their friends. Professor Archibald Henderson was toastmaster. After some general remarks on the significance of this meeting, he called for speeches by President Snyder, Professor Slaught, Professor Jewell C. Hughes, Dr. C. F. Roos, Professors Winger, R. L. Moore and Coolidge.

At the business meeting on Thursday morning, a resolution was adopted to put on record the "deep appreciation of the cordial hospitality and efficient services of all those concerned in making this meeting of the Society an exceptional success."

Thirty-four states and Canada were represented among the members present at this first general meeting of the Society in the South. The total number registered was nearly two hundred; the attendance included the following one hundred forty-five members of the Society:

Akers, F. E. Allen, R. B. Allen, Anderton, R. C. Archibald, Ashton, Atchison, R. W. Babcock, Barnhart, W. S. Beckwith, William Betz, Blichfeldt, J. D. Bond, Bowen, Boyd, E. W. Brown, Browne, H. E. Buchanan, Burington, Cairns, Carmichael, I. S. Carroll, Coleman, J. T. Colpitts, Coolidge, A. H. Copeland, L. P. Copeland, Crathorne, Dale, H. T. Davis, Denton, Doak, Dostal, Dresden, W. W. Elliott, G. C. Evans, G. W. Evans, Finkel, L. R. Ford, W. B. Ford, Fort, C. A. Garabedian, Gehman, J. L. Gibson, Hardin, E. R. Hedrick, Archibald Henderson, Herr, Hess, Holder, Hosford, A. M. Howe, J. C. Hughes, Huntington, Hyde, Louis Ingold, Ingraham, Dunham Jackson, C. G. Jaeger, E. H. Jones, B. F. Kimball, Kuhn, E. P. Lane, Larew, Lasley, Latimer, Harry Levy, F. A. Lewis, Luck, Lytle, MacDuffee, Mackie, Maddox, Maizlish, William Marshall, T. E. Mason, May, Messick, Michal, D. C. Miller, G. A. Miller, Mirick, Miser, U. G. Mitchell, Molina, C. N. Moore, R. L. Moore, Richard Morris, Morton, Mossman, Neelley, Olds, Oppenheim, Ore, Osborn, Ott, Palmer, Pierpont, Rainich, Rasor, Rawles, Reinsch, C. N. Reynolds, R. G. D. Richardson, Rider, H. L. Rietz, Risley, H. A. Robinson, Robison, Roos, Safford, Sanders, Sarratt, Saunders, J. H. Scarborough, Shaub, Sheffer, Shirk, Simmons, T. M. Simpson, Slaught, Smail, C. E. Smith, D. M. Smith, I. W. Smith, Virgil Snyder, R. P. Stephens, Stetson, Stouffer, Tappan, Waddell, L. E. Ward, Warren Weaver, W. P. Webber, G. T. Whyburn, Wiener, Wiley, K. P. Williams, A. H. Wilson, Winger, Frederick Wood, F. L. Wren, Wunder, J. M. Young, Yowell.

At the meeting of the Council, the following organization was elected to sustaining membership in the Society:
The Penn Mutual Life Insurance Company, Philadelphia.

The following thirty-five persons were elected to ordinary membership:

Dr. Pedro Manuel Arcaya, minister of interior relations, Venezuela;
Miss Alma Bridgers Bizzell, Louisburg College;
Professor Edward Livingston Carr, Union University;
Dr. William Fitch Cheney, Jr., Tufts College;
Mr. John Alston Clark, Princeton University;
Lieut. George T. Derby, Corps of Engineers, United States Army;
Mr. Joe Lee Dorroh, University of Texas;
Mr. Nat Edmonson, Jr., Rice Institute;
Mr. Joseph M. Feld, Columbia University;

Mr. Alfred Leon Foster, University of Valparaiso;
Mr. Francis Chauncey Hall, Rutgers University;
Professor Lena James Hawks, Ward-Belmont College;
Miss Deborah May Hickey, Rice Institute;
Miss Fannie Hopkins, University of Wisconsin;
Professor Charles Angevine Hutchinson, University of Colorado;
Mr. Harold John Kersten, University of Cincinnati;
Mrs. Mark Kormes, Hunter College;
Miss Ona Kraft, East High School, Cleveland;
Mr. Lloyd Lincoln Lowenstein, Cornell University;
Mr. Francisco Aniceto Lugo, electrical engineer, Maracaibo, Venezuela;
Mr. Edward Roy Cecil Miles, Rice Institute;
Miss Ethel Isabell Moody, Wells College;
Dr. Auguste Adolphe Nouel, Caracas, Venezuela;
Mr. Alexander Oppenheim, University of Chicago;
Mr. Henry Howes Pixley, Rutgers University;
Professor Luis José Fernandes Ribeiro, Pernambuco, Brazil;
Mr. Howard Lester Schug, Cornell University;
Dr. Stefan Serghiesco, Columbia University;
Mr. Leslie Donald Shriver, High School, Charleroi, Pa.;
Professor Charles Eugene Shull, Bridgewater College;
Mr. Ivan S. Sokolnikoff, University of Wisconsin;
Mr. Charles Chapman Torrance, Cornell University;
Mr. Franklin Grandy Williams, Cornell University;
Professor Rose Belle Wood, Greenville Womans College;
Dr. Oscar Zariski, Johns Hopkins University.

The following were elected to membership as nominees of Allyn and Bacon:

Miss Mary Virginia Kenny, Hunter College;
Mr. John Henderson Roberts, University of Texas;

as nominees of the National Life Insurance Company:

Mr. James Mayberry Earl, University of Minnesota;
Mr. Eli Gourin, Columbia University;
Mr. Neal Henry McCoy, University of Iowa;
Mr. Walter Lee Porter, Rice Institute.

The ordinary membership in the Society is now 1758, including 166 nominees of sustaining members and 81 life members. There are also 36 sustaining members. The total attendance of members at all meetings during the past year was 690; the number of papers read was 404. The number of members attending at least one meeting was 452. At the annual election 252 votes were cast.

The reports of the Treasurer and of the auditors (Mr. S. A. Joffe and Professor H. W. Reddick) showed a balance of $3608.67, exclusive of the balances in the Bulletin, Transactions, Colloquium and special funds, and of the life membership reserve. The Society's Endowment Fund now has securities of par value $67,000, yielding an annual income of $3125; sustaining membership fees for the year amounted to $4600. The amount received from sales of the Society's publications was $4831.57.

The Board of Trustees did not meet at Nashville, lacking a quorum. At the adjourned meeting held in New York City, January 2, 1928, they adopted a budget showing estimated expenditures and receipts as $29,839.95 and $28,846.66 respectively.

The Librarian reported that the Library of the Society now contains 7548 volumes.

At the annual election, which closed on Thursday morning, the following officers and other members of the Council were chosen:

Vice-Presidents, Professors H. L. Rietz and J. W. Young.

Secretary, Professor R. G. D. Richardson.

Associate Secretaries, Professors Arnold Dresden (one year) and M. H. Ingraham (two years).

Treasurer, Professor W. B. Fite.

Members of the Editorial Committee of the Bulletin, Professors D. R. Curtiss (two years) and E. R. Hedrick (three years).

Member of the Editorial Committee of the Transactions, Professor H. H. Mitchell.

Members of the Board of Editors of the American Journal of Mathematics, Professors E. W. Chittenden (five years), A. B. Coble (four years), and G. C. Evans (three years).

Members of the Council, Professors W. B. Carver, W. C. Graustein, Louis Ingold, F. D. Murnaghan, and W. L. G. Williams.

The tellers appointed by President Snyder to count the ballots were Professor A. B. Morton and Dr. R. G. Archibald.

The following appointments were reported: to represent the Society at the inauguration of President Burgstahler, of Cornell College, on Friday, November 18, 1927, Professor H. L. Rietz; to represent the Society at the American Mining Congress held in Washington, December 1–3, 1927, Dr. W. D. Lambert; as Committee on Arrangements for the Summer Meeting of 1928, Professors T. C. Esty (chairman), Arnold Dresden, W. A. Hurwitz, H. L. Rietz, A. H. Sprague, and J. W. Young.

At the meeting of the Council, Professor J. L. Coolidge was named to succeed Professor L. P. Eisenhart as one of the three representatives of the Society in the National Research Council. Professors Arnold Dresden and Tomlinson Fort were continued for 1928 as the representatives of the Society in the Council of the American Association for the Advancement of Science. As the committee on the award of the Bôcher Prize, which will take place at the Annual Meeting in 1928, the following were appointed: Professors E. T. Bell, R. D. Carmichael, and Solomon Lefschetz.

The fifth Josiah Willard Gibbs Lecture, entitled *Resonance in the solar system*, was delivered on Wednesday afternoon before an audience of 280 persons, by Professor E. W. Brown, of Yale University. It will appear in full in the May–June issue of this Bulletin. Professors M. B. Porter, Tomlinson Fort, and Solomon Lefschetz were appointed a committee to recommend to the Council a lecturer for 1928.

At the joint session of the Society, the Mathematical Association, and Section A, held on Thursday afternoon, the following papers were read:

I. *The notion of probable error in elementary statistics*, by Professor E. V. Huntington, retiring Vice-President of Section A.

II. *The human significance of mathematics*, by Professor Dunham Jackson, retiring President of the Mathematical Association.

III. *Some philosophic aspects of mathematics*, by Professor Arnold Dresden. (Address delivered at the request of the

American Mathematical Society.) This address will appear in full in an early issue of this Bulletin.

Titles and abstracts of the papers read at the regular sessions of the Society follow below. On Wednesday morning, President Snyder presided over the general session and the Section of Algebra, and Vice-President C. N. Moore over the Section of Point Sets and Foundations. On Wednesday afternoon, President Snyder presided over the Section of Geometry and during the Gibbs Lecture; Professor H. L. Rietz presided over the Section of Analysis. President Snyder also presided at the general session on Thursday morning. At the joint session on Thursday afternoon, Professor Dunham Jackson, chairman of Section A, presided, relieved during the presentation of his own address by Professor W. B. Ford, president of the Mathematical Association. The papers numbered 1–9 were read in the Section of Algebra, Nos. 10–20 in the Section of Point Sets and Foundations, Nos. 21–37 in the Section of Geometry, Nos. 38–53 in the Section of Analysis, and Nos. 54–70 in the general session. The papers of Frank Ayres, W. L. Ayres, Blumenthal, Browne (second paper), Camp, Campbell, Coble, Curtiss, Foster, Franklin, Hazlett, Hollcroft, Kasner, Mathews, Oppenheim, Ranum, Roberts, Roth, Sheffer (third paper), Veblen, Walsh, G. T. Whyburn (third and fourth papers), W. M. Whyburn, and Wilder were read by title. Mr. Holt was introduced by Mr. Burington.

1. Professor D. R. Curtiss: *A class of diophantine equations with bounded positive integral solutions.*

The equations here considered have the form $F(1/x_1, 1/x_2, \cdots, 1/x_n) = k$, where F is a polynomial with positive integral coefficients and k is a positive integer. Every such equation is shown to have only a finite number of positive integral solutions, or no such solution. The proof consists in establishing an upper bound for each x. This proof also applies to certain equations in which the right hand side involves the variables x. An interesting corollary is the proposition that among the equations $x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots + (-1)^n a_n = 0$ all of whose roots are positive integers, there are only a finite number (if any) whose coefficients satisfy a relation $F(a_1/a_n, a_2/a_n, \cdots, a_{n-1}/a_n) = k$, where F and k are as defined above.

2. Professor Olive C. Hazlett: *Note on formal modular invariants.*

In a recent paper in Liouville's Journal, W. L. G. Williams proved a number of theorems to the effect that all polynomials of a certain type in the coefficients of a binary cubic f (with binomial coefficients) are formal modular invariants of f modulo p, where p is a prime. The use of binomial coefficients makes his proofs not applicable when the order of the form is such that any of these binomial coefficients is congruent to zero modulo p. The present paper proves these theorems and a more general theorem which includes all of Williams' theorems as special cases for the general Galois field $GF[p^n]$ in which p is any prime whatsoever and there is no restriction on the order of the binary form.

3. Professor Olive C. Hazlett: *Algebras A defined over an algebra B.*

This paper considers the theory of an algebra A defined over an algebra B, where B is an algebra defined over a field F. After proving that the only case which has any general significance is when B is semi-simple, the paper proves that Wedderburn's fundamental theorem for a linear algebra over a field has a precise analogue here. Moreover, if an algebra A' over F can be regarded as A over B, then the canonical form of A' over F is equivalent to the canonical form of A over B; and thus the central results of Wedderburn's beautiful theory of a linear algebra over a field F can be translated into the analogous results for a linear algebra over B.

4. Professor Oystein Ore: *Arithmetical theory of Galois fields.* First communication.

This paper contains the fundamentals of a new arithmetical theory of Galois fields which derives the ideal properties of the field directly from the defining equation, as in the author's theory of general algebraic fields (Mathematische Annalen, vols. 96-97). It is proved that in Galois fields the complete system of residue classes for powers of prime ideals can be obtained by successive adjunctions of roots of binomial and trinomial canonical congruences. By this method various new results are obtained; in particular it is shown that the Hilbert metacyclic groups for the prime ideals have a very special structure.

5. Mr. L. M. Blumenthal: *On the definiteness of a quadratic form, with an application to the roots of an algebraic equation.*

By mathematical induction and the use of a well known theorem of Sylvester, the author obtains the necessary and sufficient conditions that a real, algebraic, quadratic form be definite. By writing the ternary form $\Sigma_0^2 a_{ij}x_ix_j$ as a quadratic in x_0, it is shown to be definite if and only if its discriminant (a binary quadratic form in x_1, x_2) be positive for all real values of (x_1, x_2) different from $(0, 0)$. These conditions can be expressed in terms of the coefficients of the ternary form, and Sylvester's theorem is employed to obtain the standard form for them. A quadratic form whose discriminant is the same as that of a real algebraic equation is shown to be positive definite if and only if the roots of the equation are real and distinct. Applying the previously deduced conditions that a quadratic form be positive definite, the well known necessary and sufficient conditions for the reality of the roots of the equation are obtained.

6. Professor G. A. Miller: *The number of systems of imprimitivity of transitive substitution groups.*

The main theorem proved in this paper is as follows: The number of systems of imprimitivity of any transitive substitution group is equal to the number of proper subgroups of this group which separately involve as a proper subgroup a fixed one of the subgroups composed of all the substitutions of this transitive group which omit a given letter, and the number of letters in a set of such a system of imprimitivity is equal to the index of this fixed subgroup under the larger subgroup. By means of this theorem the author determines the number of systems of imprimitivity of any transitive group in which the degree of the subgroup composed of all the substitutions which omit a given letter is exactly half the degree of the group. In particular, if in a transitive substitution group G the subgroup G_1 composed of all the substitutions of G which omit a given letter is transitive and omits half the letters of G, then the number of systems of imprimitivity of G is one more than the number of proper subgroups of G_1.

7. Professor C. G. Latimer: *On forms which repeat under multiplication.*

In this paper it is shown that if (A_{ij}) is the adjoint of the general Hermitian matrix (a_{ij}), $(i, j=1, 2, 3)$, then the Hermitian form $\bar{x}_0x_0+\sum_{i,j=1}^{3}A_{ij}\bar{x}_ix_j$ repeats under multiplication. This is seen to be a generalization of Hermite's self-reproductive quadratic form. From the above form we obtain an octenary quadratic form ϕ, which also repeats under multiplication. By proper specialization of the parameters in ϕ, we obtain E. T. Bell's recent generalization (Annals of Mathematics, (2), vol 27, p. 99) of Degen's result on the sum of eight squares.

8. Professor C. G. Latimer: *On the primes in the general cubic Galois field.*

It is known that in every general cubic Galois field a rational prime, not a divisor of the discriminant, is either a prime of the field or is the product of three distinct prime ideals. It is found that a sufficient condition for a prime to be such a product is that it be a cubic residue of the rational integer D, where D^2 is the discriminant of the field F. If F is of class unity, this condition is also necessary. It is shown that a necessary condition that F be of class unity is that $D=9$ or $D=p$, a prime. F. S. Nowlan (this Bulletin, vol. 32, pp. 374–80) has recently treated a problem equivalent to that of determining the primes in two special fields. Our results check with his.

9. Dr. C. G. Jaeger: *A character symbol for primes relative to a cubic field.*

A symbol $\psi(\pi)$ is defined with regard to the primes of a general cubic field analogous to the symbol defined by Dedekind in the study of the class number of a pure cubic field. It is hoped that this symbol will facilitate the study of the zeta function in the general cubic field.

10. Professor H. M. Gehman: *Concerning irreducible continua.*

If a bounded continuum M is an irreducible continuum about a set A, then (1) A' ($=A$ plus its limit points) contains some non-cut points of M; (2) M is an irreducible continuum about those points of A' which are non-cut points of M; (3) A' contains all points of M which are such that there exists a neighborhood about the point of diameter less than any preassigned positive quantity whose removal leaves M connected. As a corollary to (1) follows the well known theorem that every bounded continuum contains at least two non-cut points. The points mentioned in (3) are a special type of non-cut points.

11. Professor H. M. Gehman: *On extending a correspondence in the sense of Antoine.*

A continuous (1-1) correspondence between two plane points sets M and M' can be extended in the sense of Antoine (Journal de Mathématiques, (8), vol. 4 (1921), p. 221), or *A-extended,* to their planes provided there exists a continuous (1-1) correspondence of the planes such that M corresponds to M' under this correspondence. In this paper, continuous curves are classified by types according to the number of simple closed curves, end points, and branch points that they contain, and it is found that there are in all eighteen types for which a correspondence between two curves of the same type (and interior-class) can be *A-extended.* In particular, if M is an acyclic continuous curve, a correspondence between M and M' can be *A-extended* to their planes, if and only if either M contains less than three branch points and the number of end points is finite, or M has three branch points and five end points. Two other theorems are proved which generalize theorems of Antoine on the *A-extension* of a correspondence between sets consisting of collections of continuous curves.

12. Professor R. L. Moore: *Concerning triods in the plane and the junction points of plane continua.*

A *triod* is defined as a continuum containing a point O (its *emanation* point) and consisting of three continua each of which is irreducible between O and some other point and no two of which have in common any point except O. The point O of a continuum M is called a *junction* point of M, if M contains a triod having O as its emanation point and there exists a domain D containing O and such that O is a cut point of the greatest connected subset of $M \cdot D$ that contains O. A bounded continuum M is called *triodic* or *atriodic* according as there does not exist a subcontinuum K of M containing three distinct points and a continuum which contains no one of these points but which contains a point of every connected subset of K that contains two of them. It is shown that, in the plane, (1) no continuum has uncountably many junction points, (2) there does not exist an uncountable set of mutually exclusive bounded triodic continua, (3) if G is an uncountable set of mutually exclusive bounded continuous curves, then all but a countable number of them are either simple continuous arcs or simple closed curves.

13. Professor R. L. Moore: *On the separation of the plane by continua.*

This paper will appear in full in an early issue of this Bulletin.

14. Professor R. L. Moore: *Concerning the sum of a countable number of closed point sets.*

J. R. Kline has recently shown that if G is a countable set of mutually exclusive bounded continua, and, for every positive number ϵ, all but a finite number of them are of diameter less than ϵ, then their sum is not connected. In the present paper it is shown that this theorem remains true if "continua" is replaced by "closed point sets."

15. Professor G. T. Whyburn: *On the structure of plane closed point sets which are accessible from certain subsets of their complements.*

If, in a plane S, R_1, R_2, and R_3 are mutually exclusive connected point sets, then $S-(R_1+R_2+R_3)$ contains not more than two points each of which is accessible from each of these sets. If every point of the closed and bounded point set K is accessible from each of two mutually exclusive connected subsets of $S-K$, then either K is a simple closed curve or there exists an arc which contains K. Let K be an irreducible cutting of a continuous curve M between two points A and B of M, and R_a and R_b the components of $M-K$ containing A and B respectively; then (1) K lies in some single cyclic element of M; (2) if every point of K is accessible from each of the sets R_a and R_b, either K is a simple closed curve or there exists an arc t such that $t \cdot M \equiv K$, and the boundary of no complementary domain of M contains more than two points of K; if, in addition, the boundaries of no two complementary domains of M have a common point, M contains an arc T which contains K. Other consequences of these theorems are given.

16. Professor G. T. Whyburn: *On the disconnection of continua.*

Let M be any plane continuum. (1) If every subcontinuum of M contains an uncountable collection of mutually exclusive point sets $[X]$ each of which cuts M, then every subcontinuum of M is a continuous curve. (2) If, in (1), the point sets $[X]$ are countable, then every subcontinuum of M contains an uncountable collection of mutually exclusive *pairs* of of points each pair of which cuts M. (3) If every subcontinuum of M contains an uncountable collection of mutually exclusive pairs of points each pair of which cuts M, then M is a Menger regular curve. (4) The subcontinuum N of M contains an uncountable collection of mutually exclusive pairs of points each pair of which cuts M if and only if N contains an uncountable set of points each of which is either a cut point of M or accessible from at least two complementary domains of M. (5) There exists a Menger regular curve M containing an arc t which contains a countable set of points H such that no subset of $t-H$ cuts M.

17. Professor G. T. Whyburn: *On irreducible cuttings of continua.*

Let M be any continuous curve; then (1) a cutting K of M is irreducible (see this Bulletin, vol. 33, p. 388) if and only if K is the M-boundary of each component of $M-K$; (2) a cutting K of M between two points A and B of M is an irreducible cutting between these points if and only if K is the M-boundary of each of the two components of $M-K$ containing A and B respectively; (3) every cutting of M between any two closed subsets A and B of M contains an irreducible cutting of M between A and B. If every point of the irreducible cutting K of a plane continuous curve M between two points A and B of M is accessible from each of the components R_a and R_b of $M-K$ containing A and B respectively, then either K contains two points whose sum cuts M or $R_a+R_b=M-K$ and K is an irreducible cutting of M. Every bounded decomposable continuum M which is an irreducible cutting of the plane has the property that however it be decomposed into two continua H and K neither $\equiv M$, $H\cdot K$ is the sum of two mutually exclusive continua.

18. Dr. W. L. Ayres (National Research Fellow): *Continuous curves which are cyclically connected.*

A continuous curve M is cyclically connected if and only if one of the following conditions holds: (1) if K_1 and K_2 are mutually exclusive closed subsets of M each containing more than one point, there exist two mutually exclusive arcs, each of which has a point of K_1 as one end point and a point of K_2 as the other end point and lies in $M-K_1-K_2$ except for these points; (2) if, in (1), K_2 contains just one point P, the two arcs exist as in (1) except that they have just P in common instead of being mutually exclusive; (3) if x, y, and z are distinct points of M, there exist arcs xyz, xzy, yxz of M; (4) (a) the boundary of every complementary domain of M is a simple closed curve or an open curve, (b) no two unbounded complementary domains have a boundary point in common; (5) if M is unbounded, (a) every point of M lies on an open curve of M, (b) the same as (b) of (4).

(Conditions 4 and 5 are for two dimensions only.) A number of conditions are given that every point of an unbounded continuous curve M lie on an open curve of M.

19. Mr. J. H. Roberts: *Concerning continua that contain no domains.*

Menger has shown that if P is a point of a plane continuum M which contains no domain, for every positive number e there exists a simple closed curve enclosing P and of diameter less than e and such that the set of points common to M and this curve is totally disconnected. In the present paper it is shown that for every such curve M there exists a continuous one-to-one transformation of the plane into itself such that in the transformed plane there exists a rectangular coordinate system with the following properties: (a) if l is any line whatever parallel to one of the coordinate axes, then the set of points common to l and M is either vacuous or totally disconnected; (b) if M is a continuous curve, and if R is the interior of any rectangle whose sides lie respectively on the lines $x=a$, $x=b$, $y=c$, and $y=d$, where a, b, c, and d are rational numbers, then the set of all points of M that lie in R is the sum of a finite number of maximal connected subsets and so is the set of all points of M that belong to R'.

20. Professor R. L. Wilder: *Concerning R. L. Moore's axioms Σ_1 for plane analysis situs.*

In his paper *On the foundations of plane analysis situs* (Transactions of this Society, vol. 17 (1916), pp. 131–164), R. L. Moore proposed a system, Σ_1, of eight axioms for plane analysis situs. The present paper establishes that the system Σ_1 may be reduced to a set of seven axioms, by the elimination of axiom 6, which is shown to be a consequence of the other axioms. It is also found, incidentally, that the property stated in axiom 7 may be used to characterize simple closed curve and open curve.

21. Dr. C. F. Roos (National Research Fellow) and Mr. Alexander Oppenheim: *A symmetric method of fitting lines and planes.*

The method usually followed to fit a straight line $y=ax+b$ to observed points (x_i, y_i), $i=1, \cdots, m>2$, is that of choosing a and b so as to minimize $\Sigma(ax_i+b-y_i)^2$. If both x_i and y_i are subject to error, there is no reason to prefer the line so obtained to the line $x=a'y+b'$, different from the first unless the given points are collinear, found by minimizing $\Sigma(a'y_i+b'-x_i)^2$. Using the hypothesis that fitting a straight line is independent of the choice of axes, we find that if $ax+by+c=0$ is the line of fit, the most general function of x_i, y_i, a, b, and c invariant under homogeneous strain, rotation, and translation is some power of ax_i+by_i+c. If ax_i+by_i+c is proportional to the distance from (x_i, y_i) from the corresponding point of the line of fit, we show that $a \cos \alpha+b \sin \alpha=\phi$, where ϕ does not involve a, b, and c, and $\tan \alpha$ is the ratio of the error in y_i to that in x_i, assumed independent of i. We obtain for the line of fit a simple equation well adapted to calculation. The method generalizes to the fitting of planes and hyperplanes, and also offers a method of attack for the fitting of higher plane and space curves.

22. Mr. R. S. Burington and Mr. H. K. Holt: *A classification of plane cubic curves under euclidean transformations.*

Newton showed that every cubic is included in one of four canonical forms which he subdivided by methods partly geometric and partly algebraic into 72 types, with some omissions which have been supplied by later writers. These forms are not canonical forms as we use the term. The transformations which leave one of them unaltered are sometimes more than finite in number. Using a system of invariants and covariants, the authors obtain eight true canonical forms which give all cubic curves with at most a finite number of repetitions.

23. Professor E. T. Browne: *Involutions that belong to a linear class.*

We consider two square matrices A, B of order n which possess the property $A^2=B^2=I$, where I is the unit matrix. If in addition A and B are such that every linear combination of them of the form $C=\xi A+(1-\xi)B$ also possesses the property $C^2=I$, we shall say that A and B determine a linear class of involutions of which C is a member. In this paper a study is made of the properties of pairs of involutions that determine a linear class. The number and distribution of the linearly independent common fixed points and planes of A and B are determined, and necessary and sufficient conditions, both algebraically and geometrically, are given that A and B may determine a linear class. Finally, for $n=3$ and $n=4$ a special study is made of linear classes of harmonic homologies in the plane and of point-plane reflections and line reflections in space. A number of interesting results are obtained.

24. Professor J. H. Neelley: *A note on the rational plane quartic curve with cusps or undulations.*

This paper derives and discusses certain invariants and covariants of the rational plane quartic curve with either cusps or undulations. Also certain degenerate forms are considered.

25. Professor R. M. Winger: *Trinomial curves and monomial groups.*

Maschke (American Journal of Mathematics, vol. 17) has considered monomial groups in the ternary domain, which are derived from two generators of special type, his work was extended by Skinner (ibid., vol. 25). Both authors exclude the interesting groups that leave invariant the trinomial curves $x^n+y^n+z^n=0$. These curves, for rational values of n, have received attention from numerous writers (Loria, *Spezielle Algebraische* *und Transzendente Ebene Kurven*, vol. I, pp. 328 ff.), and particular cases have been studied in connection with their groups, notably by Muth ($n=3$), Berzolari ($n=-2$), Dyck ($n=4$), Snyder ($n=5$), Tappan and Musselman ($n=6$). In a forthcoming paper the author treats in detail the case of the equianharmonic cubic ($n=3$). The present paper generalizes the more important results for arbitrary integral values of n. The group is analyzed and the complete system of invariants, comprising four forms connected by a syzygy, is obtained. The fundamental invariant curves are discussed, in particular, conjugate sets of points, special and general, are constructed. The curves are also studied in relation to a quadratic inversion. Generalization to n-space is indicated.

26. Professor H. A. Simmons: *Strong and weak inequalities involving the ratio of two chords or arcs of a circle; chains of inequalities.*

Let two chords c_1, c_2, $c_1<c_2$, of a semi-circle be drawn, and call their intercepted arcs a_1, a_2 respectively. It is well known that $(c_1/c_2)>(a_1/a_2)$. In the present paper certain areas and volumes that are naturally associated with the chords c_1, c_2 and arcs a_1, a_2 are compared. Special notions of "strong" and "weak" inequalities are introduced early in the paper, and form a large part of the basis of the discussion; they are used in stating theorems and corollaries. Probably the most interesting result of the paper is a chain of trigonometric inequalities appearing in the last section, and developed from the notion of strong inequality.

27. Professor Louis Ingold: *Transformations of the coefficients of linear connection.*

In an earlier paper (Proceedings of the National Academy, vol. 11, pp. 252–256), the author applied a transformation to the coefficients $\Gamma^{\alpha}_{n\lambda}$ of an affine geometry, or a geometry of paths. This transformation was associated with a non-singular matrix a^i_j. In order to define the corresponding transformation in an arbitrary coordinate system, it was found desirable to regard the quantities a^i_j as the components of a mixed tensor. In the present paper the same transformation is applied to the more general non-symmetric coefficients $H^s_{i\lambda}$ and the invariants of the transformation are studied. A series of invariants, given implicitly in the earlier paper, are here formulated explicitly. New invariant properties are studied; for example, it is shown that if, in the original space, there exists a covariant tensor $h_{\sigma_1 \cdots \sigma_k}$ whose covariant derivatives vanish, then the same is true of the transformed space.

28. Professor A. B. Coble: *The invariants of an algebraic curve under Cremona transformation.*

On page 394 of her treatise on Cremona transformations, Miss Hudson remarks that "the outstanding problem of the whole theory, in any number of variables, is still to determine the set of properties which are invariant for Cremona transformations but not for the wider group of Riemann transformations. The present author gives the following version of the matter for a non-hyperelliptic plane algebraic curve f of genus $p \geqq 4$: the Riemann invariants of f are the projective invariants of the normal curve C^{2p-2} in S_{2p-1}, the map of f by its canonical adjoints; the Cremona invariants of f are the simultaneous projective invariants of the normal curve and a *particular* rational surface, Γ, which contains C^{2p-2} as a simple curve. For the excluded cases, hyperelliptic or of genus $p \leqq 3$, special statements must be made. As Γ varies for fixed C^{2p-2}, we obtain the various types of plane curves f, all birationally equivalent, but all distinct under Cremona transformations.

29. Professor M. C. Foster: *On the envelope of lines determined by centers of geodesic curvature.*

This paper is concerned with the envelope relative to the moving tangent plane of the lines which join the centers of geodesic curvature of the curves of an orthogonal system upon a surface. The paper considers the conditions that the point on the envelope coincide with certain special points on the rays of the normal rectilinear congruence formed by these lines.

30. Professor M. C. Foster: *On the loci of centers of geodesic curvature.*

Corresponding to every point on a surface S referred to any orthogonal system are two centers of geodesic curvature, G_1 and G_2, of the curves u=const. and v=const. The loci of G_1 and G_2 are in general two surfaces which are called the G-surfaces for the particular orthogonal system on S. This paper is concerned with these surfaces corresponding to various orthogonal systems.

31. Mr. Frank Ayres: *Plane n-ic curves generated by pairs of involutions on a line.*

H. S. White has shown (American Journal, vol. 48 (1926), that any plane cubic curve can be generated by means of a pair of cubic involutions on a line. In the present paper the following general theorem is proved: Any plane n-ic curve whose equation, $(ax)^n=0$, can be transformed, by means of a collineation of the form $x_j=\alpha_j y_1+\beta_j y_2+\gamma_j y_3, (j=1, 2, 3), \Delta\neq 0$, into one in which y_1 and y_2 are separated, can be generated by means of a pair of n-ic involutions on a line. For such n-ics two points exist such that the first polar of either with respect to the n-ic degenerates into $n-1$ lines through the other. Taking the coordinates of these points as the α_j and β_j in the collineation, and γ_j arbitrary, the required transformation can be effected. Moreover, (α) and (β) are corresponding points on and intersection points of the 1st, $\cdots$, $(n-3)$d Hessian and corresponding Steinerian of the n-ic; the line $(\alpha)(\beta)$ has $(n-\kappa-2)$-point contact with the κth Hessian and $(\kappa-1)$-point contact with the κth Steinerian at (α) and

(β). For a non-degenerate n-ic, the conic polar of each of the $n-2$ further points of intersection of $(\alpha)(\beta)$ and the Hessian has the intersection of the linear polars of (α) and (β) with respect to the n-ic as its double point.

32. Professor A. D. Campbell: *The discriminant of an m-ary quadratic in the Galois fields of order* 2^n.

The prime modulus 2 in the Galois fields of order 2^n makes the ordinary discussion of the discriminant Δ of the m-ary quadratic break down in these fields. In this paper it is shown that, for m even, Δ is the determinant D whose vanishing causes the first polar of every point P with respect to the quadratic to pass through a fixed point P', where the modulus 2 is replaced by zero in D. On the other hand, for m odd, every first polar passes through a fixed point P'; in this case D has 2 as a common factor, and Δ proves to be D with the factor 2 removed and with 2 replaced by zero in the rest of D.

33. Professor A. D. Campbell: *A note on the polar curves of plane algebraic curves in the Galois fields.*

This paper appears in full in the present issue of this Bulletin.

34. Professor T. R. Hollcroft: *On multiple points and curves of surfaces.*

Five problems are solved in this paper, and the following characteristics of algebraic surfaces determined: (1) the maximum number of multiple points of given order on a surface of given order; (2) the maximum order of multiple points of which the surface of given order has a given number; (3) the maximum number of consecutive multiple points of given order on a surface of given order; (4) the maximum multiplicity of a curve of given order on a surface of given order with given multiple curves; (5) the maximum order of a curve of given multiplicity on a surface of given order with given multiple curves.

35. Professor R. M. Mathews: *Dual syzygetic pencils of cubics.*

The nine harmonic polars of an order cubic are dual to the nine flexes and form the base of a syzygetic pencil of class cubics dual to the pencil of order curves on the flexes. Properties of the curves of one pencil can be at once dualized for the other pencil. The present investigation is of figures whose points and lines are the same for both pencils. Among the theorems obtained is the following: Every order cubic of a syzygetic pencil cuts each class cubic of the dual pencil in a G_{18} of points such that the tangents (line elements) at the points form an isomorphic G_{18} on the class curve. The G_{18} is that subgroup of Hesse's G_{216} that leaves each curve invariant.

36. Professor Arthur Ranum: *On spherical, quasi-spherical curves.*

A curve C lying on a sphere S_1 whose tangents touch another sphere S_2, and two curves closely associated with it, are here investigated. If S_1 is not tangent to S_2, C consists, in general, of an infinite number of congruent arches connected by cusps, but the number of arches may be finite and C be a closed curve. If S_1 is tangent to S_2, the point of contact being K, C consists of a single arch and K is an asymptotic point of the curve. The equations of C in general involve elliptic functions.

37. Professor Arthur Ranum: *Spheres osculating a curve and quasi-osculating another curve.*

An intensive study of the "quasi-osculating sphere" of a twisted curve, discovered in 1909 by Hostinsky and the author, leads in this paper to a variety of new curves, and also to a new principle of classification of curves. For instance, if s, $1/\rho$, and $1/\tau$ are the arc length, curvature, and torsion, respectively, and if $\eta = \int ds/\tau$, then a curve that is osculated by the quasi-osculating spheres of another curve is characterized by the intrinsic equation $\rho = a \sec \eta$, and since the equation $\rho = a \cos \eta$ is known to characterize a spherical curve, a close connection is established between the two classes of curves.

38. Professor L. R. Ford: *On the limit points of discontinuous groups.*

Certain theorems concerning the set of limit points of a properly discontinuous group of linear transformations are derived, the proofs being based on the properties of the isometric circle. The results are used to prove that a non-elementary group of non-loxodromic transformations is Fuchsian.

39. Professor G. M. Robison: *On regularity of a certain class of definitions of summability of infinite products.*

Let $P_n = \Pi_{k=1}^{n}(1+a_k)$ be the sequence corresponding to the infinite product $\Pi_{n=1}^{\infty}(1+a_n)$. A new sequence P_n' is defined linearly in terms of a set of constants $(\alpha_{n,k})$ and the elements (a_k) as follows: $P_n' = \Pi_{k=1}^{n}(1+\alpha_{n,k}a_k)$. If $\lim_{n\to\infty} P_n'$ exists and is equal to s, the infinite product $\Pi_{k=1}^{\infty}(1+a_k)$ is said to be summable by the definition to the value s. This paper gives a criterion for the regularity of this type of definition. An example of a definition assigning a value to a divergent infinite product is also given.

40. Professor E. L. Mackie: *The Jacobi condition for a problem of Mayer with variable end points.*

The isoperimetric problem of minimizing the quotient of two simple integrals while keeping a third constant is considered, the three integrands being arbitrary functions of x, y, and y'. The Euler-Lagrange differential equations are deduced as a first necessary condition on the minimizing arc, and a transversality condition is found, by application of the recent results of Bliss in his study of the problem of Mayer. A further necessary condition is then established for this problem by showing that an envelope theorem and a condition analogous to that of Jacobi apply to it as well as to simpler problems in the calculus of variations.

41. Professor P. R. Rider: *Minimizing two types of definite integrals.*

This paper considers the integrals $\int_{x_1}^{x_2} y^n(1+y'^2)^m dx$, $\int_{x_1}^{x_2} y'^n(1+y^2)^m dx$. The equations of the extremals are derived, and also necessary and sufficient conditions for a minimum or a maximum. The results obtained for the second integral are used to extend the conclusions of two earlier papers by the author (this Bulletin, vol. 27, pp. 279–284; Annals of Mathematics, (2), vol. 24, pp. 167–174). The present paper will appear in the Annals of Mathematics.

42. Mr. T. H. Rawles: *The invariant integral and the inverse problem in the calculus of variations.*

In this paper a theory for the inverse problem in the calculus of variations is presented. The Hilbert invariant integral serves as a point of departure. By means of this integral we obtain the general form of the integrand function which is compatible with a prescribed relation between the slopes of the extremal and the normal to the transversal. Conditions are then found which determine when it is possible for a given family of curves to be the extremals of a problem involving this prescribed relation. Finally the particular form of integrand function is developed. The problem is discussed for n variables in the ordinary and in the parametric form.

43. Dr. I. M. Sheffer (National Research Fellow): *Systems of linear differential equations of infinite order with constant coefficients.*

The linear differential equation of infinite order with constant coefficients, $\Sigma_0^\infty a_n y^{(n)}(x) = f(x)$, was solved by Schürer (Leipziger Berichte, vol. 70 (1918), pp. 185–240) for a certain class of analytic functions $f(x)$ and a certain class of constants a_n. The problem of solving n such equations in n unknown functions appears not to have been treated heretofore. It is the purpose of the present paper to effect this generalization. The method used is operational, and permits us to give a complete treatment for the class of functions considered.

44. Dr. I. M. Sheffer: *An operational treatment of the linear differential equation of infinite order with polynomial coefficients of degree one.*

The existence theorems for a linear differential equation of infinite order whose coefficients are polynomials of bounded degree have been considered by von Koch, Perron, Hilb, and others. We consider in this paper the case where the polynomials are of degree not exceeding one. The treatment of such an equation falls into two cases, according as a function $D(t)$, which introduces itself into the problem in a very natural manner, is or is not analytic throughout a certain region. In the former case, the problem can be reduced to the case of an equation with constant coefficients, treated in the preceding paper. If $D(t)$ is not analytic everywhere in the region considered, the discussion is more elaborate. In some cases solutions can be obtained in expansions of certain related functions. In every case, however, it is shown how the original equation can be reduced to a linear differential equation of *finite order*, of Fuchsian type. The treatment is largely operational.

45. Dr. I. M. Sheffer: *On a region of analyticity of a class of analytic functions.*

Let $F(x)=\Sigma_0^\infty F_n x^n$ be analytic about the origin, and let $f(x)=\Sigma_0^\infty F_n x^n/n!$ be the associated entire function. If we are given $f(x)$ with the single property that superior limit $(|f^{(n)}(0)|)^{1/n}=\rho$, the most that can be said of $F(x)$ is that it has $|x|=1/\rho$ as its circle of convergence. It is reasonable to expect that if we know more about $f(x)$ we can determine a larger region of analyticity for $F(x)$. The writer has had occasion to consider the following class of functions $f(x)$: $f(x)=e^{-\xi x}g(x)$, where superior limit $(|g^{(n)}(0)|)^{1/n}=\lambda$, and ξ is any constant (real or complex). For $\lambda \geqq |\xi|$ and for $\lambda<|\xi|$ we obtain regions of analyticity for $F(x)$ that are much more extended than the circle of convergence mentioned above. Borel summability is used to establish these results.

46. Professor C. C. Camp: *Devices for computing the rates together with a statistical study of building and loan associations.*

The author compares methods of computing the rates for building and loan associations in order to secure a maximum of accuracy with a minimum of numerical work. Among the methods examined are the following: interpolation with first, and with second differences; interpolation in seven-figure logarithms from Glover's tables; rule of double false position using logarithms; simple iteration; iteration accelerated by the author's method; Newton's method; and that of binomial expansions. A good random sample of 28 out of 852 associations in Illinois shows a poor correlation between the borrower's and the investor's rates. If we omit the five highest rates for loans, a coefficient of $r=0.775\pm0.056$ is found, and the line of best fit, $y-0.0815=0.69(x-0.0688)$, is obtained. Without this omission the borrower's rate averages 8.87% as against 6.87% for the investor.

47. Professor Philip Franklin: *Sets of functions, orthogonal on an infinite interval, having the power of the continuum.*

The theory of almost periodic functions treats of the expansion of such functions in terms of the sines and cosines which form a non-enumerable set of functions, orthogonal on an infinite interval. We here extend some of the properties of functions orthogonal on a finite interval to general sets of functions of this type. With regard to the existence of sets which are not linear combinations of a finite number of sines and cosines, we show that a complete set of such functions can be constructed from any (enumerable) complete set on a finite interval.

48. Professor Philip Franklin: *Approximation theorems for generalized almost periodic functions.*

The almost periodic functions of H. Bohr are originally defined as continuous functions, having displacement numbers $t(\epsilon)$, such that $D(t) = |f(x+t) - f(x)| \leqq \epsilon$, distributed in a regular fashion. The approximation theorem for these functions identifies them with the class of functions uniformly approximable by trigonometric polynomials. Several generalizations of the notion have been given by Stepanoff, Wiener, Besicovitch, and Weyl, each definition requiring less than continuity, and correspondingly weakening the sense in which $D(t)$ must be small for a displacement number. In this paper we show that for all these generalized almost periodic functions, the approximation theorem has an analogue, the functions of each type being completely equivalent to a class of functions approximable in some sense by trigonometric polynomials.

49. Professor Philip Franklin: *A set of continuous orthogonal functions.*

A set of orthogonal functions was constructed by Haar (Mathematische Annalen, vol. 69 (1910)) which had the property that the development of any continuous function in terms of these functions converged. The functions themselves were not continuous. In the present paper we construct a set of continuous functions which, we show, possesses the same property as Haar's. Our functions are obtained by normalization from broken-line functions.

50. Professor Philip Franklin: *Almost periodic recurrent motions.*

Recurrent motions, a special type of limiting motion associated with the stable trajectories of a steady dynamical system, have been defined and discussed by Birkhoff (Bulletin, Société Mathématique de France, vol. 40 (1912)). If we define an almost periodic motion as one whose coordinates are almost periodic functions (in the sense of H. Bohr) of the time, we note at once that every almost periodic motion is recurrent. For systems satisfying a certain stability condition, we show that conversely every recurrent motion is almost periodic.

51. Dr. W. M. Whyburn (National Research Fellow): *An existence theorem for differential systems.*

In this paper it is shown that there exists a unique set of absolutely continuous functions $y_1(x), \cdots, y_n(x)$, on the interval X: $a \leqq x \leqq b$, such that if $x = c$ is any point of x and $a_1, a_2, \cdots, a_n$ are given constants, then $y_i(c) = a_i$, $i = 1, 2, \cdots, n$, and such that this set of functions satisfies the following system of differential equations *almost everywhere* on X: $dy_i/dx = \sum_{j=1}^{n} A_{ij}(x, y_1, \cdots, y_n) y_j + f_i(x, y_1, \cdots, y_n)$, $i = 1, 2, \cdots, n$, where the A_{ij} and f_i, $(i, j = 1, \cdots, n)$, are functions summable (Lebesgue) of x on X for each fixed $(y_1, \cdots, y_n)$, continuous functions of $(y_1, \cdots, y_n)$ for each fixed x on X, and finally there exist functions $M(x)$ and $L(x)$ that are summable on X such that for all $(y_1, \cdots, y_n)$, $(z_1, \cdots, z_n)$, and every x on X, $|A_{ij}(x, y_1, \cdots, y_n)| \leqq M(x)$, $|f_i(x, y_1, \cdots, y_n)| \leqq M(x)$, and $|A_{ij}(x, y_1, \cdots, y_n) - A_{ij}(x, z_1, \cdots, z_n)| \leqq L(x)\sum_{k=1}^{n} |y_k - z_k|$, $|f_i(x, y_1, \cdots, y_n) - f_i(x, z_1, \cdots, z_n)| \leqq L(x)\Sigma_{k=1}^{n} |y_k - z_k|$, $(i, j = 1, 2, \cdots, n)$. A method of successive approximations is used together with a lemma which shows that the approximating functions are uniformly bounded.

52. Dr. W. M. Whyburn (National Research Fellow): *On the functional relation between the solutions of differential systems and the initial conditions.*

The differential system considered in the author's paper *An existence theorem for differential systems* (see above) is discussed with the initial conditions $y_i(a_i) = b_i$, $i = 1, \cdots, n$, where $b_1, \cdots, b_n$ are constants and $a_1, \cdots, a_n$ are points of the interval of definition, X. A positive constant δ is determined such that if all the points a_i lie on an interval D of length less than or equal to δ, there exists a unique solution of the given differential system which satisfies the initial conditions. The above-mentioned existence theorem is used to extend this solution throughout X. It is shown that the solution is continuous in the $2n+1$ variables $(a_1, \cdots, a_n, b_1, \cdots, b_n, x)$ throughout the manifold defined by $|b_i| \leqq B$, a_i and x on D, $i = 1, \cdots, n$, where B is a constant. Other properties of the solutions of this system are discussed.

53. Professor J. L. Walsh: *On approximation to an arbitrary function of a complex variable by polynomials.*

In the z-plane let two Jordan curves lie exterior to each other except for a single point which they have in common. Let the point set M consist of these curves and their interiors. If the function $f(z)$ is continuous on M and analytic in the interior points of M, then on M this function can be uniformly expanded in a series of polynomials in z. If the bounded point set M consists of a finite number of Jordan arcs and does not divide the plane, then an arbitrary function $f(z)$ continuous on M is developable on M in a uniformly convergent series of polynomials in z.

54. Professors D. J. Struik and Norbert Wiener: *The Heisenberg theory of quanta under general five-dimensional relativity.*

The present paper represents a reconciliation between general relativity and the Heisenberg idea that the Schroedinger wave equation determines a flow of probabilities. By the introduction of the Kaluza five-dimensional theory, the complex probabilities of Hilbert and Heisenberg are made real. The theory assumes a form analogous to that of sound under general relativity and at infinitely high temperatures. It offers the possibility of an interpretation of m, e, and h as dimensional constants. The recent work of Schidlof, identifying as it does the proton and the electron, indicates that M may also be explained as a dimensional constant.

55. Professors C. A. Garabedian: *On thick circular and rectangular plates loaded at the center.*

In the forthcoming number of the Journal de l'Ecole Polytechnique will appear a paper (*Correction de certains résultats sur la flexion d'une plaque circulaire épaisse donnés par de Saint-Venant* . . . .) in which the author shows the necessity of revising certain currently accepted solutions in thick plates under central load. The solutions in question are based upon an erroneous argument first advanced by de Saint-Venant. The cases of interest are those in which the plate is of circular or rectangular contour and clamped or supported at the edge, making in all four problems to be solved anew. The present paper obtains the solutions of these problems, and notes that the correct formulas, as distinguished from those hitherto published, give a consistent stress distribution at the point of application of the concentrated load.

56. Professor G. Y. Rainich: *Rotations in four-space.*

In the six-space the vectors of which represent infinitesimal rotations or antisymmetric tensors of a euclidean four-space, two absolutely perpendicular three-spaces are introduced which are invariant under all rotations in the four-space. Every six-vector is the sum of two three-vectors of opposite type, i.e., of two vectors of the three-spaces mentioned above. The known invariants and simultaneous invariants of six-vectors are simply expressed in terms of the three-vectors. A direction of a plane in four-space can be given by two unit vectors of opposite type, which in turn are determined by the plane. Two simultaneous invariants of three six-vectors which appear to be novel are introduced as triple alternating products of three-vectors. In a special case these two simultaneous invariants characterize the relative position of three planes in four-space. They have been used by the author in his recent work on integrals of tensor fields. The decomposition of an infinitesimal rotation corresponds to the known decomposition of a general rotation in four-space into two elementary rotations.

57. Professor E. R. Hedrick: *Stieltjes integrals in the complex plane.*

The definitions and the properties of Stieltjes integrals in a complex plane are studied, and theorems analogous to those of the traditional theory of functions are stated.

58. Professor C. N. Moore: *On certain criteria for the summability of integrals.*

In this paper there is determined a set of necessary and sufficient conditions that a given infinite integral be Cesàro summable of order n. If the integral is $\int_0^\infty \gamma(x)dx$, the conditions involve the existence of a set of solutions of the equations $\gamma_0(x)=\gamma(x)$, $\gamma_{s-1}(x)=-x\gamma_s'(x)$, $(s=1, 2, \cdots, n+1)$ such that $\int_0^\infty \gamma_i(x)dx$ is summable $(C, n-i)$, $(i=1, 2, \cdots, n+1)$. The results are in the main analogous to some theorems regarding summable series due to Hardy and Littlewood (see Mathematische Zeitschrift, vol. 19 (1923), pp. 71–74), but there are also certain departures from this analogy, as is frequently the case in passing from theorems for divergent series to corresponding theorems for divergent integrals.

59. Professor Dunham Jackson: *On certain problems of approximation in the complex domain.*

A method involving Bernstein's theorem on the derivative of a trigonometric sum or a polynomial, or some form of generalization of it, has been found useful in the treatment of various problems in the approximate representation of real functions. The present paper illustrates the applicability of the method in proving the convergence of certain processes of approximation over a region of the complex plane.

60. Dr. C. F. Roos (National Research Fellow): *On a type of Stieltjes integral equations.*

In the study of linear functionals, especially in connection with Riesz's theorem, it is desirable to know something about the solutions of systems of Stieltjes integral equations of the type $u_h(x) = f_h(x) = \int_a^v K_{hg}(x, s) du_g(s)$, $(h, g = 1, \cdots, m)$, where g is an umbral index. The present paper gives a method which does not make use of an integration by parts. The resolvent kernel is a function which in a certain general sense is reciprocal to $K_{hg}(x, s)$.

61. Professor J. W. Lasley: *Note on the line equation of a plane curve.*

The projective differential geometry of plane curves associates with every curve an ordinary linear homogeneous differential equation of the third order. The Lagrange adjoint of the latter is satisfied by the line coordinates of the curve. The parametric point equations furnish integrating factors for this adjoint. The integrations can be performed, and the general solution of the adjoint equation obtained. There is, however, much latitude in the choice both of dependent and independent variables, and this choice materially affects the differential equations involved. Moreover, one must be careful that the fundamental system of solutions finally obtained leads to the original curve and not merely to some projective transform of it. The present paper outlines a process for effecting this transition from the point to the line equations of the given plane curve.

62. Mr. W. E. Roth: *Solution of the polynomial equation $P(A, X) = 0$ in matrices.* Preliminary communication.

The equation $\mathrm{P}(A, X) \equiv F_0(A)X^p + F_1(A)^{p-1} + \cdots + F_p(A) = 0$, where the $F_i(A)$, $i = 0, 1, \cdots, p$, are polynomials in the given matrix A, is solved in the present paper for such values of X as are expressible as polynomials in A. A criterion for the existence of such solutions is established, and their explicit form is produced.

63. Professor Edward Kasner: *Geometric characterization of the derivative congruence of a general polygenic function.*

In an earlier paper, read at the October meeting of the Society, the author showed that the derivative $r=dw/dz$ of a general polygenic (i.e., non-monogenic) function w is completely represented by a congruence of clocks, a definite clock corresponding to each point z. (See Science, vol. 66 (1927), p. 581, and Proceedings of the National Academy.) If, however, we are given an arbitrary congruence whose clocks are in arbitrary correspondence with the points z, it will not in general be possible to identify this situation with the derivative of any function. Necessary and sufficient conditions are found, first in analytic, then in geometric form. If the congruence and correspondence are defined by a central vector A and a phase vector B, where A and B are represented as complex (in general polygenic) functions, then $(D_x-iD_y)A=(D_x+iD_y)B$, that is $\mathfrak{P}A=\mathfrak{D}B$, in terms of the operators whose geometric meaning is given in the earlier paper. The present paper will be sent to the Comptes Rendus.

64. Professor Edward Kasner: *A new interpretation of the Laplacian operator.*

The Laplacian operator Δ is expressible in the form $\Delta=4\mathfrak{D}\mathfrak{P}=4\mathfrak{P}\mathfrak{D}$, where $\mathfrak{D}$ denotes the mean derivative or central operator, and $\mathfrak{P}$ the phase operator defined in the papers cited above. If w is any complex function, we find that $\Delta w=4\Gamma\mathfrak{D}\Gamma\mathfrak{D}w$, where Γ denotes the operation of taking the conjugate complex value. Hence the Laplacian of any function (real or imaginary) is four times the conjugate of the mean derivative of the conjugate of the mean derivative of that function. This paper will be offered to this Bulletin.

65. Professor Edward Kasner: *Homographic and uniform clocks.*

If a pencil of lines is put into one-to-one or homographic correspondence with the points of a circle, we call this a general or homographic clock. If the angle between two lines varies in proportion to the circular arc determined by the corresponding points, we call the clock uniform. (Necessarily the angular measure of the arc is then twice the angle between the lines.) Uniform clocks may be positive or negative according to whether the sense agrees with the sense of the pencil or disagrees. The clocks in the author's theory of polygenic functions thus appear in the present general theory as negative uniform clocks. For the general case, the centroid is distinct from the center.

66. Professor Oswald Veblen: *Projective tensors and connections.*

The classical theory of differential invariants may be based on the observation that any analytic transformation of coordinates determines a linear transformation in a unique way. It is also true that any analytic transformation determines a unique linear fractional transformation. The coefficients of this linear fractional transformation are the transformation coefficients of the components of a class of invariants, called projective tensors, which have algebraic properties analogous to those of ordinary tensors and seem to provide the formalism needed for a group of problems to which the ordinary tensor analysis is not adequate. This paper contains an introductory account of the theory and will be published in the Proceedings of the National Academy of Sciences.

67. Professor E. T. Browne: *Note on the chacteristic equation of a matrix.*

This paper will appear in full an early issue of this Bulletin.

68. Professor E. P. Lane: *The projective differential geometry of systems of linear homogeneous differential equations of the first order.*

The author considers first the projective geometry, in a linear space of n dimensions, of a system of $n+1$ linear homogeneous differential equations of the first order in one independent and $n+1$ dependent variables. The transformation of dependent variables that is used is determined by the configuration to be studied, which in any case consists of a set of varieties each of which is generated by ∞^1 linear spaces, with the generators in correspondence. The precise number of possible configurations is determined. Certain curves called *intersector curves* on a variety are defined, and the locus of their tangents is investigated. In 5-dimensional space, the intersection of a point configuration with a hyperquadric yields a geometric interpretation in ordinary ruled space. A complete system of invariants is furnished for two ruled surfaces in ordinary space, and a new canonical form of the differential equations therefore is derived. The foundations are laid for the projective differential geometry of a set of varieties generated by ∞^k linear spaces, with the generators in correspondence. The system of equations for such a configuration consists of $k(n+1)$ linear homogeneous partial differential equations of the first order in $n+1$ dependent and k independent variables, the coefficients of which satisfy $\frac{1}{2}k(k-1)(n+1)^2$ conditions of complete integrability.

69. Professor G. T. Whyburn: *Concerning accessibility in the plane.*

In this paper the following theorems are proved. (1) If G is a countable collection of mutually exclusive connected point sets in a plane S, T denotes the sum of all the point sets of the collection G, and K denotes the set of all points X in $S-T$ such that X is accessible from at least three of the sets of G, then K is countable. (2) If every point of a plane continuum M is accessible from at least two complementary domains of M, then M is a Menger regular curve, and if M is bounded, its cut points are countable. (3) In order that the boundary point P of a connected open subset R of a continuous curve (in any number of dimensions) should be regularly accessible from R, it is necessary and sufficient that $R+P$ should be connected im kleinen at P. As a corollary to (1) we have the result that the set of all points of a plane continuum M each of which is accessible from at least three complementary domains of M is countable.

70. Mr. Alexander Oppenheim: *The approximate functional equation for the multiple theta function, and the associated trigonometric sums.*

In this note the approximate functional equation for the multiple theta function is obtained by a process of induction from a slight extension of the corresponding equation for the simple theta function. The latter formula was first given by Hardy and Littlewood in 1914, and various proofs of it have since been published. Application is also made after the manner of Hardy and Littlewood to the study of certain trigonometric sums associated with the powers of a simple theta function.

ARNOLD DRESDEN,
Associate Secretary

A CORRECTION

BY O. D. KELLOGG

Dr. F. Vasilesco has kindly called my attention to an error in my symposium address, *Recent progress with the Dirichlet problem*, which appeared in this Bulletin (vol. 32 (1926), pp. 601–623).

On page 620, in the second paragraph of §8, the property (b) should read "the part of B in any closed region T' has capacity 0"; and the last phrase in this same paragraph should read "then B must have the property (*b*)."

The footnote at this point should refer to a paper by Dr. Vasilesco, *Sur les singularités des fonctions harmoniques*, which will appear in the volume of the Journal de Mathématiques for this year, which is to be dedicated to M. Picard.

HARVARD UNIVERSITY

SEPARATION THEOREMS AND THEIR RELATION TO RECENT DEVELOPMENTS IN ANALYSIS SITUS*

BY J. R. KLINE

1. *Introduction.* The study of those properties of point sets which are invariant under (1-1) continuous transformations form the subject matter of analysis situs. It is the purpose of the present paper to discuss one of the fundamental invariants of this field, that of a point set being separated by various of its subsets.

A set is said to be *connected in the weak sense,* if it is impossible to express it as the sum of two mutually separated sets, that is, two sets that are mutually exclusive and neither of which contains a limit point of the other one. A set M is said to be *connected in the strong sense* if, for every two points P and Q of M, there is in M a closed and connected subset containing both P and Q. If to the set K, consisting of $y=\sin(1/x)$, $[0<x\leqq 1/\pi]$, we add the origin, we obtain a set which is connected in the weak sense but not in the strong sense. A set S is *disconnected in the weak sense* by one of its subsets T, in case $S-T$ is not connected in the strong sense; it is *disconnected in the strong sense* in case $S-T$ is not connected even in the weak sense. Thus every set T which disconnects S in the strong sense also disconnects it in the weak sense but the converse is not true. Thus the set K', composed of the above set K together with the y-axis between $(0, +1)$ and $(0, -1)$ is disconnected in the weak sense by the removal of the origin but is not so disconnected in the strong sense. It is interesting to note that according to a theorem of R. L. Wilder, (1) if one limits himself to separating sets T which are closed, then if S is a continuous curve, T disconnects S in the weak sense only if it disconnects it in

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* An address delivered at the meeting of this Society on May 7, 1927, in New York City, by invitation of the Program Committee.

the strong sense; and (2) continuous curves are the only continua* for which the two types of separation by closed sets are equivalent.†

The theorems of analysis situs are divided roughly into those that are concerned with continuity considerations and those of a combinatorial character. In the present paper we shall attempt to correlate the numerous important and rather widely scattered results that have been obtained in the field which deals with continuity considerations. The theory has been developed most extensively in two dimensions. However, while our material is arranged according as the separating sets and the sets separated are two-dimensional continua of various types, we shall always point out the analogous results in three or more dimensions and attempt to point out some of the difficulties in the cases where the extensions have not been made.

2. *S is a Euclidean Plane and T is a Continuous Curve.*‡ A notable step in the development of modern mathematics was made by Jordan when he expressed the necessity for a proof of the statement that every plane simple closed curve§ divides its plane into an inside and an outside region

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* A set which is both closed and connected will be called a *continuum.*

† See R. L. Wilder, *Concerning continuous curves,* Fundamenta Mathematicae, vol. 7 (1925), Theorem 8, p. 373. This paper will hereafter be referred to as W. F.

‡ A *continuous curve* is the set of points whose coordinates can be represented as $x=f(t)$ and $y=g(t)$, where $f(t)$ and $g(t)$ are continuous, single valued functions of t, $0 \leqq t \leqq 1$. A very useful point set characterization of a continuous curve was given by Hahn, when he proved that a necessary and sufficient condition that a closed and connected point set be a continuous curve is that it be *connected im kleinen* at all its points. A set of points is said to be *connected im kleinen at a point P* if, whenever C is a circle having P as center, there exists a concentric circle $\overline{C}$ such that if Q is a point of the set within $\overline{C}$ then P and Q lie together in some connected subset of the set which lies in C. See Hans Hahn, *Mengentheoretische Characterisierung der stetige Kurve,* Wiener Sitzungsberichte, vol. 123, Part IIa, pp. 2433–2489. For other characterizations of the continuous curve from a point set standpoint see R. L. Moore, *A report on continuous curves from the viewpoint of analysis situs,* this Bulletin, vol. 29 (1923), pp. 289–302.

§ If A and B are distinct points, then a *simple continuous arc* from A to B is defined by Lennes as a closed connected set containing A and B but of which it is the common boundary. Jordan also supplied a proof of this theorem. However, his proof is unsatisfactory in many respects, particularly because he assumes the theorem for the case where the simple closed curve is a simple polygon.* The theorem was first proved completely by Veblen.† Numerous other proofs of the Jordan curve theorem have been given by other mathematicians, among whom we might mention Brouwer, Schoenflies, Alexander, de la Vallée Poussin, Bliss, Ames, Winternitz, and others.‡ The simple closed curve, in addition to dividing its plane into two domains of which it is the common boundary, has the property that each point of the curve is approachable from both domains.§ Schoenflies has proved the following theorem which is commonly known as the converse of the Jordan curve theorem:

Suppose K is a closed bounded set of points lying in a euclidean plane S and $S-K$ is the sum of two sets K_1 and K_2, which are mutually exclusive and are such that

(1) *If P and Q are any two points of K_i $(i=1, 2)$, then there exists an arc from P to Q entirely in K_i;*

(2) *An arc from a point of K_1 to a point of K_2 must contain at least one point of K;*

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containing no proper connected subset that contains both A and B. A subset is a proper subset if its complement in the set is not vacuous. See N. J. Lennes, *Curves in non-metrical plane analysis situs with an application in the calculus of variations*, American Journal, vol. 33 (1911), pp. 287–326. A *simple closed curve* is the sum of two arcs AXB and AYB which have no points in common other than A and B. A simple closed curve may also be defined as the set of all points which can be put into (1-1) continuous correspondence with the points of the circle.

* See C. Jordan, *Cours d'Analyse*, Paris, 2d edition, p. 92.

† See O. Veblen, *Theory of plane curves in non-metrical analysis situs*, Transactions of this Society, vol. 6 (1905), pp. 83–98.

‡ For a complete list of proofs of this theorem see the article by Rosenthal in the Encyklopädie der Mathematischen Wissenschaften, Bd. II^3, Heft 7.

§ A set of points M is said to be *approachable from a domain D*, if whenever P is a point of M and Q is a point of D, then there exists an arc PQ, which lies except for P entirely in D.

(3) *If P is a point of K and Q is a point not in K, then there exists an arc from P to Q, lying except for P entirely in $S-K$.*[*]

R. L. Moore has proved that an open curve, that is, the set of points in (1-1) continuous correspondence with a straight line, divides its plane into two regions, similar to those made by the simple closed curve in the bounded case.[†] A proof of the converse theorem for the unbounded case has been given by the author of the present paper.[‡]

Brouwer has proved the n-dimensional Jordan curve theorem, that is, that if T is a set of points in (1-1) continuous correspondence with the set $x_1^2+x_2^2+x_3^2+\cdots+x_n^2=1$, lying in euclidean space of n dimensions, then T divides S into two regions, of which it is the common boundary.[§] Veblen has proved that n-space is decomposed by a polyhedron. Veblen's proof does not use continuity considerations, in that he bases his proof on axioms I–VIII of his thesis, together with axioms which insure the existence of $n+1$ points not all lying in the same $(n-1)$-space and an n-dimensional closure axiom. No assumption of the archimedean axiom or the axiom of continuity is made.[||] In this connection there is work of an extremely fundamental nature which has been done by J. W. Alexander.[¶]

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[*] See Schoenflies, *Ueber einen grundlagenden Satz der Analysis Situs*, Gottinger Nachrichten, 1902, p. 185.

[†] See R. L. Moore, *On the foundations of plane analysis situs*, Transactions of this Society, vol. 17 (1916), pp. 131–164. An *open curve* is defined, by Moore, from a point set standpoint, as a closed connected set of points M such that if P is any point of the set then $M-P$ is the sum of two mutually separated connected sets.

[‡] See my paper, *The converse of the theorem concerning the division of a plane by an open curve*, Transactions of this Society, vol. 18, pp. 177–184. In a paper presented to the Society at the meeting in Chicago at Christmas 1925, P. M. Swingle has shown that in the presence of the other conditions, condition (1) may be omitted in both Schoenflies' theorem and mine. For an abstract of Swingle's paper see this Bulletin, vol. 32 (1926), p. 110.

[§] See L. E. J. Brouwer, Mathematische Annalen, vol. 71 (1912), p. 314.

[||] See O. Veblen, *On the decomposition of n-space by a polyhedron*, Transactions of this Society, vol. 13 (1912).

[¶] See *A proof and extension of the Jordan-Brouwer separation theorem*, Transactions of this Society, vol. 23 (1922), pp. 333–349.

The converse of the Jordan curve theorem does not hold in three dimensions; this can very easily be seen if one considers the point set obtained from a sphere by deforming it continuously so that diametrically opposite points are brought together along a diameter. The set thus obtained will have the property of dividing three-space into just two domains of which it is the common boundary and such that each point is approachable from both domains; the set cannot be put into (1-1) continuous correspondence with a sphere. It seems probable that, in order to obtain a theorem analogous to the converse theorem of Schoenflies for two dimensions, one will be forced to use a stronger type of approachability. This appears likely because in the proof of Schoenflies' theorem very strong use is made of the fact that if A and B are points of the separating set and AXB and AYB are arcs lying except for A and B entirely in the interior and exterior, respectively, then the sum of AXB and $A{\cdot}YB$ divides the space. This division does of course not hold in three dimensions and it appears probable that one will need an approachability, that will at least have an analogous separation property to that possessed by the arcs in the plane.

Schoenflies next set himself the problem: *What will happen if one immerses a continuous curve in a euclidean space of two dimensions?* Of course it is not necessary that the continuous curve shall divide the plane, for consider merely the case where we have a square plus its interior.* Schoenflies shows that if a continuous curve divides its plane then (1) there are in the complement of the continuous curve M at most a finite number of maximal connected subsets* of diameter greater than a preassigned e†, $[e>0]$, and (2) every point on the boundary of a complementary domain is approachable from all sides from the domain.‡ A condition that is equivalent to Schoenflies' condition (2) is furnished by assuming that the boundary of a complementary domain is *regularly accessible*, that is, if for every point P on the boundary of a complementary domain D and every preassigned $e[e>0]$, there exists a δ_e such that if X is a point of D at a distance from P less than δ_e, then X and P may be joined by an arc in D except for P and of diameter less than e.§ These two properties tell us about points of the continuous curve which are on the boundary of a complementary domain. But it is evident that there are continuous curves in which there are points not on the boundary of any complementary domain. Let $[N]$ be the set of all points of the continuous curve M not on the boundary of any complementary domain. Then (a) if P is a point of $[N]$, Whyburn shows that for every $e>0$, there is in M a simple closed curve of diameter less than e which encloses P||, and (b) $M-[N]$,

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* That a continuous curve should fail to divide its plane, it is necessary and sufficient that every maximal cyclic curve of the set should be a simple closed curve plus its interior. A continuous curve M is said to be a *cyclic* curve if every pair of points belonging to the curve lie on some simple closed curve which is a subset of M. A cyclic curve K is said to be a *maximal cyclic curve* of the continuous curve M, if K is not a proper subset of any other cyclic curve belonging to M. See G. T. Whyburn, this Bulletin, vol. 33, p. 520.

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* A set K is said to be a *maximal connected subset* of a set M if K is a connected subset of M but is not a proper subset of any other connected set in M.

† It is interesting that this is still true if S, the euclidean plane of the hypothesis, is replaced by any plane continuous curve and T is a subcontinuous curve of S. It is not true, however, if S is a continuous curve in three dimensions. See W. L. Ayres, *Concerning continuous curves and correspondences*, Annals of Mathematics, vol. 28 (1927), pp. 396–399.

‡ Suppose P is a point on the boundary of a complementary domain D of the continuous curve M, while A and B are points of this boundary which are distinct from P. Let AXB be an arc lying except for its end points entirely within D while D_1 is any one of the subdomains of D made by AXB which has P on its boundary. Then, if for every point Q of D_1, there is an arc from P to Q lying entirely except for P and Q in D_1, then P is said to be *approachable from all sides*. See Schoenflies, *Die Entwicklung der Lehre von Punktmannigfaltigkeiten*, zweiter Teil, Leipzig, 1908, p. 237, for a proof that these two conditions form a necessary and sufficient set that a subcontinuum of a euclidean plane be a continuous curve.

§ See G. T. Whyburn, *Concerning continua in the plane*, Transactions of this Society, vol. 29 (1927), p. 370. This paper will hereafter be referred to as W. T.

|| See G. T. Whyburn, this Bulletin, vol. 33 (1927), p. 262.

that is, the sum of the boundaries of the complementary domains of M, is strongly connected if and only if $[N]$ does not separate the plane even in the weak sense.* If we have *only* condition (1) of Schoenflies' theorem, then while result (b) is still true, the statement (a) must be modified because we can only say that for every P of $[N]$ and every $d>0$, there exists a continuum (not necessarily a simple closed curve), of diameter less than d separating P from all the points of the plane at a distance from P greater than d, and *M is connected im kleinen at P*. Thus whenever condition (1) of Schoenflies' theorem holds and (2) fails, then all the points at which the continuum fails to be connected im kleinen, lie on the boundary of some complementary domain. The conditions of Schoenflies, while necessary and sufficient that a two-dimensional continuum be a continuous curve, are neither necessary nor sufficient for a continuum in euclidean space of three dimensions. This has been shown by R. L. Moore.†

Let us now consider two points A and B (A and B may or may not be points of the continuous curve M) which have the property that every arc from A to B contains a point of M, different from A and B. Under these circumstances (a) there is a simple closed curve lying in M which separates A from B‡ and (b) while every arc from A to B must contain at least one point of M, it is always possible to join A and B by an arc that does not separate M.§ It follows from (a) that no acyclic continuous curve* divides its plane. Wilder has shown that every subcontinuum of the boundary of a complementary domain of a continuous curve is itself a continuous curve.†

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* See G. T. Whyburn, this Bulletin, vol. 33 (1927), p. 389.

† See R. L. Moore, *Concerning the relation of a continuous curve to its complement in three dimensions*, Proceedings of the National Academy, vol. 8 (1922), pp. 33–38.

‡ This result was obtained for the case where both A and B belong to the complement of M, by R. L. Moore. See *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), p. 159. The case where A and B do not belong to the complement of M was handled by C. M. Cleveland, *Concerning points of a continuous curve that are not accessible from each other*, Proceedings of the National Academy, vol. 13 (1927), pp. 275–276.

§ See R. L. Moore, *Concerning paths that do not separate a continuous curve*, Proceedings of the National Academy, vol. 12 (1926), p. 745.

The papers, which we have mentioned so far, have considered the continuum as given and have asked the question: *What can be said of the complementary domains and their relation to the continuous curve and its subsets?* Then there is also the inverse problem of having a domain given and asking under what conditions the boundary of a single domain shall be a continuous curve or some particular type of continuous curve. Carathéodory first obtained conditions under which the boundary of a single domain will be a simple closed curve.‡ In his treatment, conditions are imposed both upon the boundary and on the relation of the domain to the boundary. In 1918, R. L. Moore found conditions on the domain, *without any reference to the boundary*, these conditions being necessary and sufficient that the boundary of the domain be a simple closed curve. The condition is that the domain be *uniformly connected im kleinen*, that is, for every $\epsilon > 0$, there exists a δ such that if X and Y are two points of the domain at a distance from one another less than δ, then they can be joined by a connected subset of the domain of diameter less than ϵ.§ He afterwards proved that if one requires that the boundary of a simply connected domain be merely a continuous curve instead of specifying that it be the particular type of continuous curve known as a simple closed curve, then the condition of being uniformly connected im kleinen may be replaced by a weaker requirement, namely that of having the S-property,[*] that is, that for every $e\,[e>0]$ the set may be expressed as the sum of a finite number of connected sets of diameter less than e.[†] Conditions equivalent to the assumption of the S-property for the domain have been given by Wilder and Whyburn but in both cases the conditions are of a different character from those of Moore in that they place restrictions on the boundary while Moore's are upon the domain alone. The condition of Wilder is that every connected subset of the boundary be connected in the strong sense[‡] while Whyburn's is that every point of the boundary be accessible from all sides from D, the domain.[§] Whyburn also shows that a necessary and sufficient condition that a continuous curve M be the boundary of a connected domain is that, if J denotes any simple closed curve in M, then (1) $J+I$ or $J+E$, where I and E denote respectively the interior and exterior of J, contain M as a subset and (2) if A and B are any two points of J, then $M-(A+B)$ is not connected.[||]

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* An *acyclic continuous curve* is a continuous curve that contains no simple closed curves. This term was first introduced by H. M. Gehman. Menger and a number of foreign writers call the same sets "*Baumkurven.*"

† See W. F., loc. cit., Theorem 11, p. 361.

‡ See C. Carathéodory, Mathematische Annalen, vol. 73 (1912–1913), p. 366.

§ See R. L. Moore, *A characterization of Jordan regions by properties having no reference to their boundaries*, Proceedings of the National Academy, vol. 4 (1918), pp. 364 370.

There is still a third type of problem that must be considered under the general heading of the separation of the plane by continuous curves. The papers which we have discussed have either (a) taken a given continuous curve and examined its complement or (b) considered a domain as given and hunted for conditions under which the boundary was a continuous curve. The problem in the second case is equivalent to that of being given a domain and asking under what conditions the maximal connected subsets of the complement are continuous curves. In both cases the separating set was either given or was found by taking certain limiting points of sets that were given. Let us now consider the problem where the sets to be separated are given and the dividing sets must be found and can in general not be found by merely taking limit points of sets that are already given.

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[*] See W. Sierpinski, *Sur une condition pour qu'un continu soit une courbe jordanienne*, Fundamenta Mathematicae, vol. 1 (1920), pp. 44–60.

[†] See R. L. Moore, *Concerning connectedness im kleinen and a related property*, Fundamenta Mathematicae, vol. 3 (1922), pp. 232–237.

[‡] See W. F., loc. cit., Theorem 19, p. 375.

[§] See W. T., loc. cit., p. 370.

[||] See W. T., loc. cit., Theorem 6, p. 380.

In 1905, Zoretti showed that if K is a bounded maximal connected subset of a plane closed set M and ϵ is any positive number, then there exists a simple closed curve J enclosing K, passing through no point of M and such that every point of J is at a distance less than ϵ from some point of K.* It is evident that while every point of J is at a distance less than ϵ from some point of K, this is not true of all points within J. For example K might be a simple closed curve of diameter greater than 2ϵ which is in the set M. Moore shows that in the case where the set K of our hypothesis *does not separate its plane*, then in addition to Zoretti's requirements, the simple closed curve J may be so constructed as to have all points within it at a distance less than ϵ from some point of K. Thus with the use of these theorems, if K and H are mutually exclusive maximal connected subsets of the closed set M and neither K nor H separates its plane, then we can get a simple closed curve containing one, excluding the other and passing through no point of M. If the continua K and H satisfy all the previous conditions except that instead of being mutually exclusive they may have a totally disconnected set T in common, then we can construct a simple closed curve that encloses $K-T$ and has $H-T$ in its exterior whenever $K-T$ is connected.† Lubben has carried this work further and has obtained some very interesting results, one of the most important of which is the set of necessary and sufficient conditions for the separation of two sets by a simple closed curve embodied in the following theorem:

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* See L. Zoretti, *Sur les fonctions analytiques uniformes*, Journal de Mathématiques, (6), vol. 1 (1905), pp. 9–11.

† For a proof of these theorems see R. L. Moore, *Concerning the separation of point sets by curves*, Proceedings of the National Academy, vol. 11 (1925), p. 469. The theorems mentioned have been for the bounded case. Moore also considers the case of being given a collection of unbounded continua and constructing open curves which shall separate sets of the collection.

If K and H are closed bounded point sets, whose common part T is a totally disconnected set, while K is a continuum, then a necessary and sufficient condition that there exist a simple closed curve containing a subset of T but containing no point of $K+H-T$ and separating $K-T$ from $H-T$ is the following: $H-T$ (1) *is a subset of a complementary domain of K and* (2) *is not separated by K near T.**

Practically nothing has been done with the corresponding problem in three dimensions. That the problem is an extremely complex one may be seen from an example in the work of Urysohn, in which he exhibits a closed totally disconnected set T having the property that, while for every $\epsilon>0$, the set may be decomposed into a finite number of mutually exclusive parts each of diameter less than ϵ and each of these parts enclosed in a polyhedron not cutting any other polyhedron of the set, *it is not always possible to choose these polyhedra so that each will be of genus zero. Indeed it may happen that the genus must become infinite as ϵ approaches zero.*†

3. *S is a Euclidean Plane and T is a Continuum or Sum of Continua, not necessarily Continuous Curves.* Here we are met with a situation that may be considerably more complicated. We might expect that a certain advantage would be gained by regarding the continuum as the sum of certain

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* If K, H, and T are point sets, then the statement that *$H-KH$ is not separated by K near T* means that if P is a point and C is a circle about P, then there exists a circle C' with the same center such that any pair of points x and y of $H-KH$ within C' can be joined by a connected point set which contains no points of K and is entirely within C. See R. G. Lubben, *The separation of plane point sets by curves*, this Bulletin, vol. 32 (1926), p. 200. For abstracts of other results by Lubben on the same general problem see this Bulletin, vol. 32, pp. 13, 114, and 200.

† See P. Urysohn, *Mémoire sur les multiplicités cantoriennes*, Fundamenta Mathematicae, vol. 7 (1925), p. 121. Examples of a similar nature have been given by Antoine in his thesis, *Sur l'homéomorphie de deux figures et de leurs voisinages.*

properly chosen subcontinua analogous to the case in the Jordan curve theorem, where a number of the proofs depend very strongly on the fact that the simple closed curve is the sum of two arcs which have no points other than their end points in common. However, this separation is not always possible, for we are forced to consider the case where our continuum may be *indecomposable*, that is, where it cannot be expressed as the sum of two distinct proper subcontinua. These continua were first introduced by Brouwer* and have been the subject of a great deal of work by Mazurkiewicz, Kuratowski, Knaster, Urysohn and others. With the aid of these continua examples have been given of continua which divide the plane into any finite number n $[n \geqq 3]$ or a countable infinity of domains, every point of the continuum being a boundary point of each of the domains. In this connection it is interesting to notice that a point set K cannot be the common boundary of three mutually exclusive domains if there are in K two points A and B each of which is accessible from all three of these domains.†

Let us first consider the case where we shall regard our continuum as a single continuum and not as split up into various proper subsets. We will recall the result of R. L. Moore, who showed that whenever two points A and B are separated by the continuous curve T, then T contains a simple closed curve which also separates A from B. The simple closed curve has the property that, while it effects a separation of the plane between A and B, no proper subset of the simple closed curve will separate the plane between A and B. We shall say with Kuratowski that the point set K is an *irreducible separating set* of the plane between A and B, whenever K divides the plane into two or more mutually separated sets such that A and B belong to distinct sets but no proper subset of K effects such a separation between A and B. While K may be an irreducible separating set of the plane between A and B, there may be a different pair of points of the complement of K which can be separated by a proper subset of K, as is illustrated by the following example:

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* See L. E. J. Brouwer, *The impossibility of a linear arrangement of the points of an irreducible continuum*, Proceedings Academy, Amsterdam, vol. 14 (1911), p. 144. If A and B are distinct points, then an *irreducible continuum* from A to B is defined by Zoretti as a closed connected set of points containing A and B but containing no proper closed connected subset containing both A and B. See Zoretti, Annales de l'École Normale, vol. 26 (1909).

† See G. T. Whyburn, this Bulletin, vol. 32 (1926), p. 200.

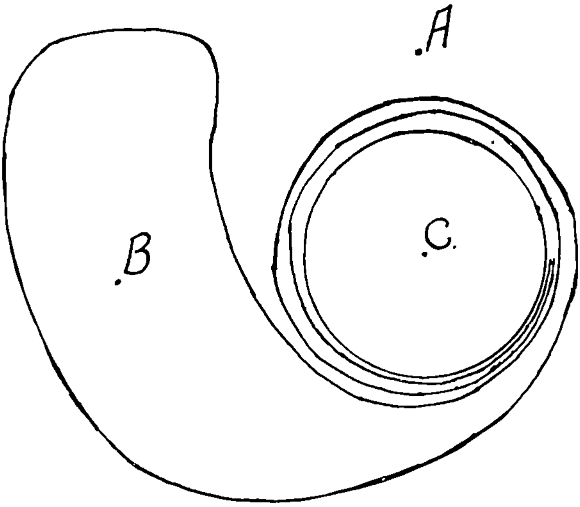

FIGURE 1.

A set is said to be a *completely irreducible separating set* of the plane if the set K divides the plane between some pair of points but no proper subset of K divides the plane between any pair of points. Kuratowski has proved that:

(1) Every continuum separating the plane between A and B contains an irreducible separating set K between A and B and this set K is a continuum.

(2) Every continuum which divides its plane into at most a finite number of complementary domains contains a completely irreducible separating set.*

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* B. Knaster has given an example of a continuum which separates its plane into a countably infinite set of domains but contains no completely irreducible separating set. See *Quelques coupures singulières du plan*, Fundamenta Mathematicae, vol. 7 (1925), pp. 264–289.

(3) Every completely irreducible separating set is either an indecomposable continuum or the sum of two continua each irreducible between the same pair of points. If the plane is divided into more than two domains by the irreducible separating set, then the set is either an indecomposable continuum or the sum of two such continua.*

Suppose we take a continuum, containing more than one point, which has the property that for every subset K of M that is connected, the set $M-K$ is connected in the weak sense. I have proved that every such continuum is a simple closed curve.† If instead of assuming that K is any connected subset of M, we restrict ourselves to the case where K is a closed connected subset of M then

(1) I have shown that, if the plane continuum M is not indecomposable and $M-K$ is connected in the weak sense, then M must divide its plane and in case M is a continuous curve, then M must be a simple closed curve.‡

(2) Kuratowski has shown that if in this case we assume that $M-K$ is connected in the strong sense, then K must be a simple closed curve.§ Kuratowski's result is no longer true if we merely say that $M-K$ is connected in the weak sense. Consider the set composed of $y=\sin(1/x)$ $[0<x\leqq 1/\pi]$, the Y-axis between $(0, +1)$ and $(0, -1)$ and an arc from $(0, -1)$ to $(1/\pi)$ which has no point except its end points in common with the other sets. This set is connected in the weak sense after the removal of the set consisting of the portion of the

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* For a proof of these results of Kuratowski, see *Sur les coupures irréductibles du plan*, Fundamenta Mathematicae, vol. 6 (1924), pp. 130–146. Results of a similar character were announced by G. A. Pfeiffer but never published. For an abstract of Pfeiffer's results, see this Bulletin, vol. 29 (1923), p. 151.

† See my paper *Closed connected sets which remain connected upon the removal of certain connected subsets*, Fundamenta Mathematicae, vol. 5 (1924), pp. 3–11.

‡ See my paper, *Concerning the division of the plane by continua*, Proceedings of the National Academy, vol. 10 (1924), pp. 176–177.

§ C. Kuratowski, *Contribution à l'étude de continus de Jordan*, Fundamenta Mathematicae, vol. 5 (1924), pp. 119–122.

Y-axis between the origin and $(0, -1)$ but not connected in the strong sense.

R. L. Wilder has investigated the properties of sets M which are connected and remain connected upon the removal of any connected subset, when one omits the condition that the set is closed. Such sets he calls *quasi-closed curves*. It is found that (1) if A and B are any two distinct points of M, then M is the sum of two sets M_1 and M_2 each of which contains A and B, is irreducibly connected from A to B and such that $M_1-(A-B)$ and $M_2-(A-B)$ are mutually separated sets, (2) M separates the plane, and (3) if M is connected im kleinen at all its points, then M is a simple closed curve. The last result is particularly interesting as it gives a characterization of a simple closed curve in which the property of being closed is not assumed.*

Using the notion of prime parts of a continuum†, introduced by Hahn, R. L. Moore has further shown that if the plane continuum M has more than one prime part and no one of its prime parts separates the plane, then a necessary and sufficient condition that $S-M$ should be the sum of just two mutually exclusive domains of which M is the common boundary, is that for every subcontinuum K of M, $M-K$ is connected. Under the same hypothesis, in order that $S-M$ shall be the sum of two mutually exclusive domains such that each prime part of M will contain at least one limit point of each domain, it is necessary and sufficient that M be a simple closed curve of prime parts in the sense that it is disconnected by the omission of any two of them which are not identical.* Whyburn has also proved that if a bounded closed set separates the plane in the weak sense but no proper subset does so, then the set must be a simple closed curve.

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* The results mentioned in (1) and (2) were presented at the International Mathematical Congress at Toronto and will appear in the Proceedings of that Congress. The one marked (3) is found in abstract in this Bulletin, vol. 32 (1926).

† If P is a point of a continuum K, then the *prime part* of K, which contains P is defined by Hahn as the set of all points $[X]$ such that for every positive number $e[e>0]$, there exists a finite number of irregular points of K, $X_1, X_2, X_3, \cdots, X_n$, such that X_1 is P, X_n is X and for every i the distance X_i to X_{i+1} is less than e. A point is said to be an *irregular point* of K, if K is not connected im kleinen at the point. See Hans Hahn, *Ueber die irreducible Continua*, Wiener Berichte, vol. 130 (1921).

In this connection I wish also to call attention to an extremely interesting result of Urysohn, who has proved that if K is the common boundary of two connected domains in space of three dimensions and T is a totally disconnected subset of K (that is, a set which contains no connected subsets other than single points) which is closed, then $K-T$ must be connected.†

Let us now consider the case where T is the sum of two or more continua. The first result in this direction is the famous Brouwer-Phragmen theorem, which states that if A and B are any two closed bounded sets, which have no points in common and neither of which disconnects the plane, then their sum cannot separate the plane. Mazurkiewicz in volume 3 of the Fundamenta has proved the unbounded case of the Brouwer-Phragmen theorem.

This fundamental result has been extended as follows.

(a) By a young Polish mathematician, Strasziewicz, who shows that the theorem also holds if we allow the common part to be connected.‡ This result is also contained implicitly in the work of Miss Mullikin.§

(b) By Miss Mullikin, who shows that if $A_1, A_2, A_3, \cdots$ is a countable set of closed sets, no one of which separates the plane, then their sum cannot do so.||

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* For the first of these results see *Concerning the common boundary of two domains*, Fundamenta Mathematicae, vol. 6 (1925), p. 203.

† See P. Urysohn, *Mémoire sur les multiplicités cantoriennes*, Fundamenta Mathematicae, vol. 7 (1925), pp. 103–119.

‡ See S. Strasziewicz, *Ueber eine Verallgemeinerung des Jordan'schen Kurvensatz*, Fundamenta Mathematicae, vol. 4 (1923), pp. 129–135.

§ See Transactions of this Society, vol. 23 (1922).

|| Loc. cit., Theorem 3, pp. 148–155. A new proof of this interesting theorem has been given by Mazurkiewicz in Fundamenta Mathematicae, vol. 6, pp. 37–38. This theorem is closely connected with a theorem of mine *Concerning the complement of a countable infinity of point sets of a certain type*, this Bulletin, vol. 23, pp. 290–292. For extensions of my theorem see the abstract of a paper by R. L. Wilder, this Bulletin, vol. 33 (1927), p. 388.

For the extension of the Brouwer-Phragmen result to n dimensions there is an extremely interesting result by Alexandroff which may be stated as follows.

Suppose that in euclidean space of n dimensions, we have two closed sets F_1 and F_2, neither of which decomposes the space. Then if the common part of these sets is of dimension not greater than $n-3$ the sum F_1+F_2 cannot decompose the space E_n. The common frontier of two connected domains of E_n is a cantorian multiplicity of dimension $n-1$.*†

There is also a theorem closely related to the preceding extensions of the Brouwer-Phragmen theorem, in which we are given some further information about points in the sum. The theorem is one of Gehman's and is as follows. If N is a closed bounded set consisting of a collection of connected sets (E), each of which is a maximal connected subset of N, no one of which separates the plane S, and no more than a finite number are of diameter greater than any preassigned positive number, then N cannot separate the plane and moreover if a point P of a maximal connected subset e of E is accessible from $S-e$, then P is accessible from $S-N$.‡

The theorems of the present section have been negative theorems in the sense that they have told us that when the individual sets have certain properties, then *the sum cannot decompose the plane.* When we look for theorems of a positive type, we are first met by the results of Janiszewski and Miss Mullikin, who working independently obtained the following important results.

(*a*) *If A and B are closed and connected bounded sets, neither of which disconnects the plane, and have a common part which is not connected, then their sum must disconnect the plane.*

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* This concept will be defined in the next section of the present paper.

† See P. Alexandroff, Comptes Rendus, vol. 183, p. 722. A *cantorian multiplicity* is defined by Urysohn as a continuum which remains connected upon the removal of any closed set of dimension not greater than $n-2$.

‡ See H. M. Gehman, *Concerning acyclic continuous curves*, Transactions of this Society, vol. 29 (1927), pp. 558–560.

(*b*) *If the common part of A and B consists of exactly two maximal connected sets, then the plane is divided into exactly two parts by their sum.**

Knaster and Kuratowski have proved that these results remain true, if one only of the continua of the hypothesis is assumed bounded. When both are unbounded, then a necessary and sufficient condition that the sum should cut the plane when neither of the continua alone cut it, is that their common part should contain a maximal connected set which is bounded.† Straszewicz‡ has also proved that if the common part reduces to exactly n maximal connected sets which are maximal, then there are exactly n complementary domains. Still other results in this connection have been given by Mrs. Nikodym.§

Rosenthal has considered a closely related question. He replaces the assumption that neither A nor B divides the plane by the following. Both A and B are continua which are irreducible between the same pair of points a and b and which have no other points in common. His principal result is that the complement of the sum with respect to its plane consists of two principal domains (*Hauptgebiete*), each of which has the entire sum as its boundary, and possibly of a number of secondary domains (*Nebengebiete*), each of which has its frontier wholly in either A or B.‖ W. A. Wilson has generalized these results by showing that this theorem of Rosenthal remains true if the two continua A and B have in common two point sets α and β, where $\alpha\beta=0$, if each of the continua A and B is irreducible between any point a of α and any point b of β, and both A and B are decomposable or one of them is indecomposable and the other is not the union of two indecomposable continua. He also shows that the frontier of any secondary domain is part of either a continuum of condensation, an indecomposable continuum, a pair of indecomposable continua, or the union of an indecomposable continuum and a continuum of condensation.*

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* See S. Janiszewski, *Sur les coupures du plan faites par continus*, Prace Matematyczne-Fisyczne, vol. 36 (1913). This paper is in Polish and the results contained therein were unknown to Miss Mullikin at the time of her work, which was published in the previously quoted paper, pp. 154–162.

† See *Sur les continus non-bornés*, Fundamenta Mathematicae, vol. 5 (1924), p. 36.

‡ See Straszewicz, *Ueber die Zerschneidung der Ebene*, Fundamenta Mathematicae, vol. 7 (1925), pp. 159–188.

§ See S. Nikodym, *Sur les coupures du plan*, Fundamenta Mathematicae, vol. 7 (1925), pp. 14-23.

‖ See A. Rosenthal, *Teilung der Ebene durch irreducible Kontinua*, Sitzungsberichte der Munchener Akademie, 1919, pp. 91–109.

When we are working in three dimensions, the conditions of Janiszewski and Miss Mullikin are neither necessary nor sufficient. That the condition is not necessary may be seen if we consider the sets A and B, where A is the set composed of the great circle g of the sphere K plus one of the hemispheres cut off by g, while B is the other hemisphere plus g. That the conditions are not sufficient may be seen by considering the set composed of two arcs AXB and AYB, which have no point in common other than A and B. In the ninth volume of the Fundamenta Mathematicae, Mazurkiewicz and Strasziewicz have obtained conditions, which are sufficient but not necessary that the sum of two sets, neither of which separates three-space, should separate the same. Let us first introduce several definitions.

To every point t of the interval $t_1 \leqq t \leqq t_2$, we make correspond a point $X(t)$ and suppose that the function $X(t)$ is continuous for the interval under consideration. This defines a continuous curve, which we shall call closed if it is true that $X(t_1)$ and $X(t_2)$ are the same. Let B be a closed point set having no points in common with the closed continuous curve M. Then we shall say that *M is free with respect to B*, if there exists a function of two variables $Y(t, \lambda)$ defined and continuous for $t_1 \leqq t \leqq t_2$ and $0 \leqq \lambda \leqq 1$ where

(*a*) $Y(t, \lambda)$ has nothing in common with B for all values of the variables,

(*b*) $Y(t_1, \lambda) = Y(t_2, \lambda)$,

(*c*) $Y(t, 1) = X(t)$, for all values of t and $Y(t, 0) = x_0$, a constant. In case this function does not exist, then the curve is said to be *interlaced* with B. A closed set in three dimensions is said to be *interlaceable* if there exists a closed curve that is interlaced with B.

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* See W. A. Wilson, this Bulletin, vol. 33 (1927), pp. 733–744.

On the basis of these definitions, the authors prove the following theorems.

THEOREM I. *If A and B are closed sets neither of which divides three-space between the points P and Q and if the common part AB is not interlaceable then their sum does not divide three-space between P and Q.**

THEOREM II. *If A and B are closed sets neither of which cuts three-space, then their sum cuts three-space if their common part is interlaceable while neither of the sets is interlaceable.*†

4. *General Dimension Theory.* Up to the present we have considered our containing set to be a euclidean space of n dimensions, that is, a space satisfying, let us say, Veblen's axioms for geometry with sufficient additional axioms to allow for the existence of $n+1$ points not all in the same $(n-1)$-space, together with an n-dimensional closure axiom. We shall, of course, wish to speak of various sets in this n-dimensional space as having a dimension k, where k may assume all values from zero to n. Various mathematicians have considered this problem of assigning a dimension to subsets of euclidean space of n dimensions. Of course, a theory based on a set of axioms for euclidean geometry is bound to have a special character due to the interplay of the special properties of ordinary space; and it is for this reason that the

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* Alexandroff has recently pointed out that Theorem I is a special case of a theorem due to Alexander. See J. W. Alexander, loc. cit., p. 342. Using Alexander's methods and the notion of the Betti Numbers for a general closed set introduced by himself, Alexandroff has obtained interesting extensions of the Brouwer-Phragmen theorem and of the Janiszewski-Mullikin theorem. See P. Alexandroff, *Zum verallgemeinerten Phragmen-Brouwer'schen Satz*, Fundamenta Mathematicae, vol. 11 (1928), pp. 222–227.

† See S. Mazurkiewicz and S. Straszíewicz, *Sur les coupures de l'espace*, Fundamenta Mathematicae, vol. 9 (1927), pp. 205–211.

founders of the recent developments in dimension theory, Menger and Urysohn, considered themselves immersed in a more general space and considered the problem of assigning a dimension to sets in *general metric space*.

Let us consider a set of elements, E, about which nothing is assumed as to their nature, and a non-negative function $r(x, y)$ satisfying the following conditions:

(1) $r(x, y) = r(y, x)$;

(2) $r(x, y) = 0$, if and only if x and y are identical;

(3) $r(x, y) \leqq r(y, z) + r(z, x)$.

We shall assume that our space in addition to having the above distance function, which makes it a *metric space*, has the property of being *compact*, that is, every infinite set of distinct points has a limit point.

A rather complete survey of the early attempts to define dimension has been given by Menger,* in which he also shows how each of these attempts is unsatisfactory because it assigns a dimension to certain sets which is different from the dimension we would intuitively assign to it; for example the definition of Zoretti of a one-dimensional continuum as an irreducible continuum would assign the dimension one to the surface of a cube which can be a subset of an irreducible continuum in three dimensions. It is rather interesting to note that this problem received a satisfactory solution almost simultaneously at the hands of two young mathematicians, one a Russian, Urysohn, and the other an Austrian, Menger. The theories of these two men are equivalent for *compact metric spaces*.† The theory of Urysohn might be characterized as an *im kleinen* while that of Menger is an *im grossen* theory. We shall first take up the Urysohn theory.‡

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* See K. Menger, *Bericht uber die Dimensionstheorie*, Jahresbericht der Vereinigung, vol. 35 (1926), pp. 113–150.

† For an example of a space satisfying Hausdorff's four axioms for topological space and which contains a set that is two-dimensional according to the definition of Menger, but zero-dimensional by Urysohn's definition, see an article by L. W. Cohen, Comptes Rendus, 1927.

‡ See P. Urysohn, loc. cit.; and also Fundamenta Mathematicae, vol. 8 (1926).

Let P be a point of the set C which lies in a compact metric space and ϵ any positive number. Then we shall say that we have an e-separation of C at point P if we have the following conditions satisfied:

(1) $C=A+B+D$, where A, B, and D are mutually exclusive sets;

(2) A contains P;

(3) $A+B$ is of diameter less than ϵ;

(4) A and D are mutually separated sets.

We shall now define the dimension of a point set as follows:

(1) The dimension of the vacuous set shall be -1;

(2) A set shall be zero-dimensional at a point P if C can be separated at P by a vacuous set, that is, by a set of dimension -1;

(3) The dimension of a point set shall be the greatest positive integer that occurs amongst the dimensions of its various points, (thus we may have sets whose dimension is not finite);

(4) Suppose all the dimensions up to and including the $(n-1)$st have been defined; then we shall say that a set is of dimension n at P if

(a) The set may be e-separated at P by a set of dimension less than n,

(b) The dimension of the set at P is not less than n.

The definition of Menger is as follows: The space is at most n-dimensional if every two closed mutually exclusive sets of the set can be separated by a set which is at most $(n-1)$-dimensional. The definition given by Menger is practically the same as that which Brouwer gave in 1913, but which he carried no farther than showing that every subset of R_n (that is, euclidean space of n dimensions) which contains an open subset is n-dimensional, where an open subset is one whose complement is closed. The above definitions were discovered independently and without any knowledge of the definition of Brouwer. In both systems there are the necessary theorems to show that the dimension as defined leaves sets, to which our intuition would tell us to assign the dimension n, with this dimension. As Menger points out, this definition is already foreshadowed by Euclid when he calls the *line* that whose boundary is points, the *surface* that whose boundary is lines, and the *solid* that whose boundary is surfaces.

To show that the dimension assigned to a point set is in accordance with our intuition and that no contradictions arise in applying it, a long series of theorems must be proved and in a number of cases the argument is very long and involved. The principal theorems follow:

(1) *The dimension of a point set remains invariant under* (1-1) *continuous transformations.*

(2) *If* $F_1, F_2, F_3, \cdots$ *is a countable infinity of closed sets each of dimension* n, *then the sum is of dimension* n.

(3) *Every totally disconnected set which is closed is zero-dimensional.*

(4) *The sum of a set of* n *dimensions and one of* m *dimensions is a set of dimension not more than* $m+n+1$. *With the use of this theorem it is proved that every* n*-dimensional set of a separable space (that is, a space where there is an everywhere dense countable set) is the sum of* $n+1$ *but not fewer zero-dimensional sets.*

(5) *Every closed domain of euclidean space of* n *dimensions is* n*-dimensional.*

This last theorem is proved on the basis of a remarkable theorem due to Lebesgue, which is as follows.

If each point of a domain of n *dimensions belongs to at least one of the closed sets* $F_1, F_2, F_3, \cdots, F_k$, *finite in number, and if the sets are sufficiently small, then there is a point common to at least* $n+1$ *of the sets; on the other hand whatever the domain* G, *and whatever be the diameter of the sets* F_i, *then it is always possible to decompose* G *so that each point is not common to more than* $n+1$ *of the sets.**

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* This theorem was first announced by Lebesgue in 1911 (Mathematische Annalen, vol. 70, pp. 166–168). The complete demonstration of the theorem was found for the first time in the work of Brouwer in Crelle's Journal, vol. 142 (1913). See also Lebesgue, *Sur les correspondences entre les points de deux espaces*, Fundamenta Mathematicae, vol. 2 (1921), p. 257.

Still more remarkable is the result announced by Menger to the effect that every set of k dimensions which is closed is a subset of some continuous curve lying in euclidean space of $2k+1$ dimensions.*

5. *S is a Proper Subcontinuum of the Euclidean Plane and T is a Subset of S.* For the sake of convenience we shall divide the material of this section as follows.

(a) Cut Points of Continua, that is, points of a continuum which separate it.

  (i) Conditions for their Existence.
  (ii) Properties of the Set $[C]$ of all Cut Points.
  (iii) Characterization of Various Types of Continuous Curves by Properties of their Cut Points.

(b) Disconnection by Pairs of Points.

(c) The Menger Curve Theory.

(d) The Set T is a Finite Number (or Countable Infinity) of Continua.

5a. *Cut Points of Continua.* (i) *Conditions for their Existence.* Let us consider three very simple continuous curves, the open curve, that is, the set of points in (1-1) continuous correspondence with the set of points on a straight line, the simple continuous arc from A to B, and the set composed of a square plus its interior. In the first every point is a cut point, in the second every point other than A and B is a cut point, while in the third there is no cut point. When one is examining a continuum as to the existence or non-existence of cut points, it is necessary to examine only the boundaries of the complementary domains of the continuum, for according to a theorem due to G. T. Whyburn, a necessary and sufficient condition that a point P of a continuum M be a cut point of M is that it be a cut point of the boundary of some complementary domain of M.† It one takes two particular points A and B of the continuous curve M, then that there be a point in M which separates A from B in M, it is necessary and sufficient that every two simple continuous arcs in M from A to B shall have a point in common different from A and B.*

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* See K. Menger, *Allgemeine Räume und Cartesische Räume*, Königliche Akademij, Amsterdam, vol. 29 (1927), Nos. 3 and 8.

† See W. T., loc. cit., p. 389. This theorem is an extension of a theorem published in 1925 by R. L. Moore, who showed that in order that a bounded continuous curve K should have no cut point, it is necessary and sufficient

In 1918 Mazurkiewicz showed that (1) every continuous curve M which is bounded has at least two points which do not disconnect M in the weak sense and (2) every continuous curve M which contains a simple closed curve J has the property that J contains at most a countable set of points which disconnect M in the weak sense. It will be noticed that the points of J which disconnect M do not disconnect J.† In these theorems the condition that M be a continuous curve may be replaced by the weaker condition that M be a continuum. This follows from two theorems of R. L. Moore to the effect that (1) every continuum whatsoever contains at least two points neither of which disconnect it in the strong sense and (2) no continuum M (whether it be a continuous curve or not) contains a closed and connected set K (whether K is a simple closed curve or not) which contains an uncountable set of points each of which disconnects M but not K.‡ It is interesting to note that Zarankiewicz has pointed out that the demonstration of Moore's second theorem can be so modified as to apply to the case where M is merely assumed to be connected (bounded or unbounded). The statement of Zarankiewicz's theorem is as follows. If N is any connected

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that the boundary of every complementary domain be a simple closed curve. See Moore, *Concerning the common boundary of two domains*, Fundamenta Mathematicae, vol. 6 (1925), p. 211.

* See W. L. Ayres, *On the separation of points of a continuous curve by arcs and simple closed curves*, this Bulletin, vol. 33 (1927), p. 266.

† See S. Mazurkiewicz, *Un théorème sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 2 (1921), pp. 119–130.

‡ See R. L. Moore, *Concerning the cut points of continuous curves and other closed and connected sets*, Proceedings of the National Academy, vol. 9 (1923), pp. 101–102. This paper will hereafter be referred to as M.C.P.

subset of a connected set C, then the set of points of N which cut N without cutting C is at most countable.*

(ii) *Properties of the Set* $[C]$ *of all Cut Points.* A great deal of interesting work has been done on the set $[C]$ composed of all cut points of a given continuous curve, particularly as to the number of such cut points and various properties possessed by their subsets.

Zarankiewicz has shown that the set $[C]$ is a set F_σ, that is, the sum of a countable infinity of closed sets.† If we consider the subset $[B]$ of all points of $[C]$ such that for every point b of $[B]$, $M-b$ is neither connected or the sum of two connected sets, then $[B]$ is finite or countably infinite.‡ This theorem is proved merely on the assumption that M is connected. In case that M is an acyclic continuous curve, the set $[B]$ becomes the set of *branch points* of M, and hence the theorem contains the result of Menger that the branch points of an acyclic continuous curve are at most countable.§

G. T. Whyburn has proved in his thesis a number of interesting theorems concerning the properties of the subcontinua of $[C]$, the most striking of which are the following.

(1) *If* $[H]$ *denotes the set of end points*‖ *of the continuum* M *while* $[T]$ *is any countable subset of* M, *then every bounded*

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* See C. Zarankiewicz, *Sur les points de division dans les ensembles connexès*, Fundamenta Mathematicae, vol. 9 (1927), Theorem 9, p. 140. This paper will hereafter be referred to as Z.F.M.

† See Z.F.M., loc. cit., Theorem 17, p. 163.

‡ See C. Kuratowski and C. Zarankiewicz, *A theorem on connected sets*, this Bulletin, vol. 33 (1927), pp. 571–575.

§ See K. Menger, *Ueber reguläre Baumkurven*, Mathematische Annalen, vol. 96 (1926), p. 574. A point p is said to be a branch point of an acyclic continuous curve if there exist in the curve three arcs L_1, L_2, and L_3 such that $p=L_1\cdot L_2=L_2\cdot L_3=L_3\cdot L_1$.

‖ If M is a continuous curve, then an *end point of* M is defined by Wilder as a point P such that, whenever a is an arc from P to any point P' of M, then the set $M-(a-P)$ contains no connected subset which contains more than one point and contains P. See W.F., loc. cit., p. 358. As applied to continua in general, Whyburn defines the term *end point* as follows. The point P is said to be an *end point of the continuum* M provided it is true that if N is any subcontinuum of M which contains P, then P is not a limit point of any connected subset of $M-N$.

continuum which is a subset of $[C]+[H]+[T]$ *is an acyclic continuous curve.*[*]

A closely related result is that due to Zarankiewicz in which he shows that in a continuous curve M, if K is any constituent of[†] $[C]$, the set of cut points of M, then $\overline{K}$, that is, the set K plus its limit points, is an acyclic continuous curve.[‡]

(2) *For every positive number* e, $[C]$ *does not contain more than a finite number of continua of diameter greater than* e.

Zarankiewicz shows a slightly more general result in that he proves that you may replace "continua" by constituents and further shows that the constituents of diameter greater than e lie on an acyclic continuous curve which is a subset of M.[§]

(3) *If* $[C]$, $[H]$ *and* $[N]$ *denote, respectively, the cut points, end points and simple closed curves in* M, *then* M *is the sum of these three sets.*

(4) *If* K *is any connected subset of the set* $[C]$, *then in order that* $\overline{K}$ *should be an acyclic continuous curve, it is necessary and sufficient that every point of* $\overline{K}$ *be either an end point or a cut point of* M.

The result of Zarankiewicz mentioned under (1) above shows that for the case where K is a maximal connected subset of $[C]$ which is connected in the strong sense, $\overline{K}$ is always an acyclic continuous curve.

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[*] This result is Theorem 9 of W.T., loc. cit., p. 383. Whyburn's result is not only for the case where M is a continuous curve but for the more general case where M is any plane continuum. C. M. Cleveland announced the special case of Whyburn's result where the subset considered is a maximal connected subset of the cut points $[C]$ of a continuous curve. See this Bulletin, vol. 32 (1926), p. 420.

[†] According to Kuratowski, the *constituent* of a point p in a point set P, is the set of all points of P which lie together with p in a closed connected subset of P.

[‡] See C. Zarankiewicz, *Sur la structure d'un ensemble de points de division dans les continus de Jordan*, Bulletin de l'Académie Polonaise des Sciences et des Lettres, 1926, p. 362. This paper will be referred to as Z.P.B.

[§] See Z.P.B., loc. cit., Theorems 2 and 3.

(5) *If K denotes the set of all cut points of the boundary of a complementary domain D of a continuous curve M, then the sum of D and K is uniformly connected im kleinen.*[*]

(6) *Suppose A and B are two points of a continuous curve M and $[K]$ denotes the set of all points which separate A from B in M. Under these conditions,*

(i) $A+[K]+B$ *is a closed set;*

(ii) *If t is any arc of M from A to B and S is any maximal segment of $t-(A[K]B)$, then M contains a maximal cyclic curve which contains S.*

(7) *If any cut point of M is a point at which some continuum of M is not connected im kleinen, then P is on some simple closed curve in M and the set of all such points is countable.*[†]

(iii) *Characterization of Various Types of Continuous Curves by Properties of their Cut Points.* The first sets to be characterized by their cut points were simple continuous arcs. In 1916, Sierpinski showed that an arc is a bounded set, containing more than two points such that for every point of the set except two, the set is the sum of two closed sets which have only the point P in common.[‡] In the Sierpinski characterization the boundedness assumption is essential. Moore has given another set of conditions in which the boundedness is not assumed but more information is required about the connectedness than was assumed by Sierpinski. Moore's theorem is to the effect that if the continuum M contains two points A and B such that (1) $M-A$ and $M-B$ are connected, (2) $M-P$ is not connected if $A\neq P\neq B$, then M is a simple continuous arc from A to B.

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[*] The results listed above are Theorems 9, 14, 22, 13, and 4, respectively, of W.T.

[†] For results (6) and (7), see this Bulletin, vol. 33 (1927), p. 520.

[‡] See S. Sierpinski, *L'arc simple comme un ensemble de points dans l'espace à m dimensions*, Annali di Matematica, (3), vol. 26, pp. 131–150. As R. L. Moore pointed out, it is implicitly assumed that the two exceptional points do not cut.

If the boundedness is assumed then condition (1) may be omitted.*

An open curve, that is, the set of all points in (1-1) continuous correspondence with the straight line, is characterized as a continuum M which, whenever P is a point of the set, then $M-P$ is the sum of two mutually separated connected sets.† If one considers a continuum consisting entirely of cut points and does not impose the restriction that the sets into which P separates M are connected, then I have shown that M must be (1) a continuous curve, (2) every point P of M is on an open curve of M, and (3) within any circle there are at most a finite number of points from which more than two rays of an open curve depart in M.‡

In order that a bounded continuum be an acyclic continuous curve it is necessary and sufficient that any one of the following conditions be satisfied: (a) Every subcontinuum should contain a non-denumerable set of points each of which disconnects M in the strong sense;§ (b) If K denotes the set of all points of M which do not cut M, then no subset of K disconnects M in the weak sense;‖ (c) The set of all non-cut points shall be totally disconnected in the strong sense.¶

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* See R. L. Moore, *Concerning simple continuous curves*, Transactions of this Society, vol. 21 (1920), p. 335.

† See R. L. Moore, *On the foundations of plane analysis situs*, Transactions of this Society, vol. 17 (1916), p. 159.

‡ See my paper, *Closed connected sets that are disconnected by the removal of a finite number of points*, Proceedings of the National Academy, vol. 9 (1923), pp. 7–12.

§ See M.C.P.

‖ See W. T., loc. cit., Theorem 32, p. 400. This is an extension of a theorem of R. L. Moore, who proved it under the assumption that M is a continuous curve.

¶ A set of points K is said to be *totally disconnected in the strong sense* if it does not cease to be totally disconnected upon the addition of a countable set of points. Thus the set of irrational points on a straight line is totally disconnected but fails to be totally disconnected in the strong sense. See C. Zarankiewicz, Z.F., loc. cit., p. 18.

A continuous curve is said to be *two-way continuous* if for every pair of points in the set there exist at least two arcs joining the two points, neither of which is a subset of the other; it is cyclically connected if every two points of the set lie together on some simple closed curve of the set. Every cyclically connected continuous curve is two-way continuous but not conversely as may be seen if we consider the set composed of two circles which are tangent externally. In order that a continuous curve may be two-way continuous, it is necessary and sufficient that any one of the following be satisfied: (a) every arc contains a non-cut point or (b) the set of all cut points contains no continuum. It is true that the boundary of every complementary domain of a two-way continuous curve is itself two-way continuous.* The fact that a continuous curve has absolutely no cut points is a necessary and sufficient condition for the curve to be cyclically connected.†

5b. *Disconnection by Pairs of Points.* The best example of a continuous curve that is separated by every pair of points is the simple closed curve. A bounded continuum M will be disconnected by the omission of two of its non-cut points, A and B, if and only if there exist two complementary domains such that A and B are both accessible from each of these domains.‡ It is of interest to note that for points which go to make up disconnecting pairs, the following theorem similar in most respects to the corresponding theorem for cut points holds: no continuum M has a subcontinuum K which contains an uncountable infinity of points such that if x and y are any two points of K, then M *but not* K is disconnected by the omission of x and y. The fact that the boundary of a connected domain contains a noncountable infinity of points such that the omission of any pair of them disconnects the boundary is a necessary and sufficient condition that the boundary be a continuous curve.*

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* See G. T. Whyburn, *Two-way continuous curves*, this Bulletin, vol. 32 (1926), pp. 659–663.

† See G. T. Whyburn, *Cyclically connected continuous curves*, Proceedings of the National Academy, vol. 13 (1927), Theorem 1, p. 31.

‡ See G. T. Whyburn, this Bulletin, vol. 33 (1927), p. 388.

Let us consider a continuum M, such that for M there exists a positive integer k such that (1) M is disconnected by the omission of any k of its points but (2) there exists no set of j $[j<k]$ points which disconnects M. Every such continuum is a continuous curve. We have previously discussed the case $k=1$. If $k=2$, then M is a simple closed curve. There are no sets that satisfy these conditions for $k>2$.† In my result the boundedness is not assumed. If one assumes boundedness then in the case where $k=2$, my condition (1) may be omitted;‡ while if we assume that M is connected im kleinen, then in the presence of my conditions (1) and (2) we may dispense with the condition that M is closed and still have the set necessarily a simple closed curve.§

5c. *The Menger Curve Theory.* Menger reserves the name "*curve*" for a bounded closed connected set of points K which has the property that for every point P of K and every $e>0$, there exists a neighborhood of diameter less than e about point P, whose boundary contains no continuum. As the boundary is closed and contains no continua (other than single points) it follows that the boundary must be zero-dimensional‖ and the curve a one-dimensional set if we are dealing with compact metric spaces. In the euclidean plane Menger's curves would consist of those continua which contain the interior of no domain.* Thus Menger's "*curves*" do not include all the sets which are included under the term "*continuous curve*," where, as is well known, a square and its interior may be included.

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* See G. T. Whyburn, *Concerning the disconnection of continua*, Fundamenta Mathematicae, vol. 10 (1927).

† See my paper *Closed connected sets that are disconnected by the removal of a finite number of points*, Proceedings of the National Academy, vol. 9 (1923), pp. 7–12.

‡ See R. L. Moore, *Concerning simple continuous curves*, loc. cit., p. 342.

§ See R. L. Wilder, this Bulletin, vol. 33 (1927), p. 388.

‖ See P. Urysohn, loc. cit., p. 75.

If for a point P of the curve K, there exists for every e $[e>0]$ a neighborhood about P of diameter less than e, whose boundary consists of a finite number of points, then P is called a *regular point* of the curve. If to a point P of the curve K a number n can be attached so that for every e $[e>0]$, there exists a neighborhood about P of diameter less than e whose boundary consists of not more than n points, then P is said to be of *order not greater than* n. A point is *of order exactly* n, if n is the smallest positive integer with the property that for every preassigned e we may find a neighborhood of diameter less than e whose boundary consists of exactly n points. The regular points to each of which we can assign no positive integer as its order (i.e. the points for which it is possible to find a neighborhood consisting of a finite number of points for every preassigned e but where the number n of boundary points becomes infinite as e approaches zero) will be called *points of continually increasing order*. In like manner we can define the points of countable order and the points of the order c of the continuum.

Menger proves that (1) the points of order higher than the first are everywhere dense on K and form a set F_σ, which contains continua in every non-vacuous open subset of the curve, (2) the set of end points of K is either vacuous or zero-dimensional, (3) in order that a continuum should be a connected im kleinen *curve*, it is necessary and sufficient that for every $e[e>0]$, K be the sum of a finite number of continua each of diameter less than e and such that no two have a continuum in common; if any two have at most a countable set in common, then the curve is a continuous curve without any points of order c.*

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* In this connection see the abstract of a paper by R. L. Moore in which he shows that a necessary and sufficient condition that a point set shall contain no domain is that for every two points of the set M there exist a totally disconnected subset of M which separates these two points in M. See this Bulletin, vol. 32 (1926), p. 218.

In a later article Menger shows that every regular curve contains for every point P of order n, n arcs which have P as one end point and are otherwise mutually exclusive; for each point of order continually increasing, there are a countable infinity of arcs ending in P, otherwise mutually exclusive, and such that the diameters of these arcs can be arranged in a sequence converging to zero. The order of a point of a regular curve is thus the greatest integer n such that there can be found in the curve an acyclic continuous curve consisting of n arcs having P and only P in common.†

An interesting complement to the result of Menger just mentioned is a theorem by N. E. Rutt to the effect that if A and B are points of the continuous curve M having the property that it is possible to find n arcs AX_iB ($i=1, 2, 3, \cdots, n$) having no points in common except A and B, then there exist in M a set of points $P_1, P_2, \cdots, P_n$, such that A and B are separated in M by the omission of these points.‡

If K is any curve which lies in a *compact metric space*, then according to Menger there is in *euclidean space* of three dimensions a continuous curve M which contains a subset which is in (1-1) continuous correspondence with K.§ If the curve K is acyclic, then the set can be found in *euclidean space of two dimensions*.‖

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* The above mentioned results are contained in Menger's paper, *Grundzuge einer Theorie der Kurven*, Mathematische Annalen, vol. 95, pp. 277–306.

† See *Zur allgemeine Kurven Theorie*, Fundamenta Mathematicae, vol. 10 (1927), pp. 96–115.

‡ See N. E. Rutt, this Bulletin (abstract), vol. 33 (1927), p. 415.

§ See K. Menger, *Allgemeine Raume und Cartesische Räume*, Königliche Akademij, Amsterdam, vol. 29, pp. 476–482.

‖ See M. Wazewski, *Sur les courbes de Jordan ne renfermant aucune courbe simple fermée*, Annales de la Société Polonaise de Mathématiques, 1924, p. 49. See also H. M. Gehman, *Concerning acyclic continuous curves*, Transactions of this Society, vol. 29 (1927), Theorem 1.

5d. *The Set T is a Finite (or Countably Infinite) Set of Continua.* Most of the characterizations of continuous curves that have been mentioned up to this time, have been based upon the property that certain points or sets of points can be connected in a manner which has been prescribed. However, in 1925, Moore published a characterization, the basic idea of which rests on the ability to separate pairs of points in a certain manner; the result is that a necessary and sufficient condition that a plane continuum be a continuous curve is that, for every two points A and B of the set, there exist a finite number of continua which separate A from B in the set.* According to a theorem due to W. L. Ayres, this disconnection can be made by a set of n arcs, whenever n is the number of maximal connected subsets of $M-(A+B)$ which have A and B both as limit points.†

We will recall Miss Mullikin's result to the effect that if $A_1, A_2, A_3, \cdots$ is a countable infinity of closed sets no one of which disconnects its plane S; then the sum cannot disconnect S. Now, if instead of S, we substitute a proper subcontinuum of the plane and let the A's be subsets of this continuum, then our theorem is no longer true. For example, a circle is not disconnected by the omission of any one of its points but is disconnected by the omission of a countable infinity of them; indeed two suffice to make the separation. Moore, in considering an extension of Miss Mullikin's result, introduced the concept of *semi-continuous collections of continua.*‡

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* See R. L. Moore, *A characterization of a continuous curve*, Fundamenta Mathematicae, vol. 7 (1925), p. 303.

† That the number of such maximal connected subsets is finite follows, with suitable modification, from the previously quoted theorem of Ayres to the effect that if a subcontinuous curve N is subtracted from the continuous curve M, then $M-N$ contains at most a finite number of maximal connected subsets of diameter greater than a preassigned $e>0$.

‡ If A and B are distinct points, then let $r(A,B)$ denote the distance from A to B. If A is a point not in the continuum q, then the distance from A to q, which we shall denote as $r(A,q)$, shall be the lower bound of all numbers $r(A,Y)$ for all points Y of q. If p and q are continua, then the

Moore proves that, if in a plane S, M is a bounded continuum which does not separate S and G is a countable collection of subcontinua of M, no one of which separates M, then if the collection is upper semi-continuous, the sum cannot disconnect M. He further shows that if M is a continuous curve and two particular points A and B are not separated by any one of the upper semi-continuous collection of subcontinua G, then the set of all continua G does not separate A from B in M.* He next considers a plane S every point of which belongs to some continuum of the mutually exclusive set of continua G; then, if (1) the collection of continua is upper semi-continuous and (2) no one of them separates the plane, we may regard each continuum as a point and, defining region suitably, we will find that every axiom of his set for plane analysis situs is satisfied, if the space S is interpreted to mean the collection of continua G. Thus the set of elements G is topologically equivalent to the set of points of a plane S.†

Whyburn takes any plane connected set M and considers the case where M is separated into two mutually separated sets M_1 and M_2, by the omission of n of its connected subsets $A_1, A_2, \cdots, A_n$. Then he shows that (1) $M_i[i=1, 2]+\sum A_i$ is the sum of at most n mutually separated connected sets and furthermore (2) if $M_1+A_1+A_2+\cdots+A_n$ is the sum

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lower distance from p to q, $l(p,q)$, shall be the lower bound [and the upper distance is the upper bound] of the distances $r(X,q)$ for all points X of p. A collection of continua G is said to be *upper semi-continuous* if whenever p is a continuum of the collection G and $p_1, p_2, p_3, \cdots$ is a sequence of continua of G such that the lower distance from p_n to p approaches zero as n becomes infinite, then the upper distance from p_n to p does so also.

* See R. L. Moore, *Concerning upper semi-continuous collections of continua which do not separate the plane*, Proceedings of the National Academy, vol. 10 (1924), pp. 356–360.

† See R. L. Moore, *Concerning upper semi-continuous collections of continua in the plane*, Transactions of this Society, vol. 27 (1925). This concept has also been the basis of extremely interesting work by Alexandroff, Victoria and others.

of k such sets, then $M_2+A_1+A_2 \cdots +A_n$ is the sum of at most $n-k+1$ mutually separated connected sets.*

There is an interesting class of connected sets, that are not closed, known as *biconnex sets*. These sets are the connected sets which cannot be expressed as the sum of two proper subsets which are connected, thus furnishing for connected sets that are not closed an analogue to the indecomposable continua in the closed connected sets. Knaster and Kuratowski gave an extremely interesting example of a biconnex set B, having the further property (b) that there is in B a point P, such that $B-P$ contains no connected subsets other than single points (a connected set which, we might say, is "*exploded*" upon the removal of a single point).† Whether every biconnex set contains such an "*explosion point*" is still unknown;‡ every connected set having such a point is biconnex and no biconnex set can have more than one such point.§ A study of sets H belonging to a connected set K and having the property that $K-H$ contains no connected sets other than single points has been made by R. L. Wilder,‖ who calls them "*dispersion sets.*"

6. *Concluding Remarks*. As is evident from the matter presented in this paper, our knowledge of separation properties of sets immersed in three or more dimensions is comparatively limited. Here is an extremely interesting and important field that is practically untouched. Among the various questions that might be considered, we will mention the following.

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* See G. T. Whyburn, this Bulletin, vol. 33 (1927), p. 388.

† See Knaster and Kuratowski, *Sur les ensembles connexes*, Fundamenta Mathematicae, vol. 2 (1921), p. 241.

‡ See Fundamenta Mathematicae, vol. 3, p. 321.

§ See my note, *A theorem concerning connected sets*, Fundamenta Mathematicae, vol. 3, pp. 238–239.

‖ See *On dispersion sets of connected sets*, Fundamenta Mathematicae, vol. 6, pp. 214–228; see also B. Knaster, *Sur un problème de M. R. L. Wilder*, in the same journal, vol. 7, pp. 191–198.

(a) The problem of finding approachability conditions which are necessary and sufficient that the separating set be a simple closed surface, i.e., the set of points that can be put into (1-1) continuous correspondence with the surface of a sphere.

(b) The problem of securing necessary and sufficient conditions that the sum of two closed connected sets, neither of which separates three-space, may have a sum that separates three-space (the conditions of Mazurkiewicz and Strasziewicz mentioned above are sufficient but not necessary).

(c) The question of when a continuous (1-1) correspondence T between two simple closed surfaces in three dimensions, may be extended to a (1-1) continuous correspondence $\overline{T}$ of their surrounding three-spaces such that T and $\overline{T}$ are identical for points of the surfaces given.*

(d) The problem of finding conditions analogous to those of Schoenflies that are necessary and sufficient that a continuum in three dimensions be a continuous curve. The conditions of Schoenflies might be characterized as *external conditions* in that they give the relation of the continuum to the surrounding space. The connectivity im kleinen condition of Hahn, the S property of Sierpinski, the requirement of Wilder that every connected subset of an open subset be arcwise connected, and Moore's separation condition are all of them *internal conditions* and hold in any number of dimensions. The problem of finding external conditions is an open one.†

We do not wish, however, to create the impression that the problems of two dimensions are by any means completely solved. An interesting set of problems among those still unsolved is concerned with the question as to whether every continuum such that every subcontinuum, containing more than one point, is homeomorphic with the whole set must be a simple continuous arc and whether there are bounded homogeneous plane continua which are not simple closed curves.* A solution of the various problems still unsolved in two and three dimensions will open up tremendous possibilities for future work in this important and fascinating branch of mathematics.

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* That this extension is not always possible has been shown by J. W. Alexander by means of an interesting example in the Proceedings of the National Academy, vol. 10 (1924).

† For a discussion of the difficulties encountered see the article of R. L. Moore, *Concerning the relation of a continuous curve to its complement in three dimensions*, loc. cit.

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* That such continua, if they exist, cannot be continuous curves follows from the results of Mazurkiewicz, *Sur les continus homogènes*, Fundamenta Mathematica, vol. 5 (1924), pp. 137–146. A set M is said to be *homogeneous* if for every two points x and y of M there exists a (1-1) continuous transformation of M into itself which turns x into y.

EXPANSION IN SERIES OF NON-INVERTED FACTORIALS*

BY W. J. TRJITZINSKY

An expansion of the form

$$f(z) = \sum \frac{b_n}{(z+1)(z+2)\cdots(z+n)}$$

can be obtained from the consideration of Cauchy's formula

$$2\pi i f(z) = \int_C \frac{f(t)dt}{z-t},$$

if $f(z) = 0$ at infinity, together with the result†

$$(1)\qquad \frac{n!}{(z+1)(z+2)\cdots(z+n+1)} = \int_0^1 u^n(1-u)^z du,$$

where $(1-u)^z$ denotes the branch reducing to unity for $u=0$. The above relations can also be used for deriving an expansion in series of non-inverted factorials. By (1) we have

$$\frac{1}{z-t} = \int_0^1 (1-u)^{z-t-1} du.$$

Consider

$$(1-u)^{z-t-1} = (1-u)^z(1-u)^{-t-1}.$$

Since

$$(1-u)^z = 1$$

when $u=0$, we may write

$$(2)\quad (1-u)^z = 1 - \frac{z}{1!}u + \frac{z(z-1)}{2!}u^2 - \cdots + \frac{(-1)^n}{n!}z(z-1)\cdots(z-n+1)u^n + \cdots.$$

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* Presented to the Society, September 9, 1927.

† Whittaker and Watson, *Modern Analysis*, 3d edition, Cambridge University Press, 1920, p. 144.

The binomial expansion (2) will be uniformly convergent for $0 \leqq u \leqq 1$ when $R(z) > 0$.* Also let us introduce the condition

$$R(-t-1) > 0,$$

that is,

$$R(t) < -1.$$

Then the expansion

$$(3) \qquad (1-u)^{z-t-1} = (1-u)^{-t-1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} u^n (1-u)^{-t-1} z(z-1) \cdots (z-n+1),$$

which holds for $0 \leqq u \leqq 1$, can be integrated termwise, so that we may write

$$(4) \qquad \frac{1}{z-t} = \int_0^1 (1-u)^{z-t-1} du = \int_0^1 (1-u)^{-t-1} du + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \cdot \left[\int_0^1 u^n (1-u)^{-t-1} du \right] \cdot z(z-1) \cdots (z-n+1).$$

But

$$\int_0^1 u^n (1-u)^{-t-1} du = \frac{-n!}{t(1-t)(2-t) \cdots (n-t)},$$

and hence we have

$$(5) \qquad \frac{1}{z-t} = -\frac{1}{t} - \sum_{n=1}^{\infty} \frac{(-1)^n z(z-1)(z-2) \cdots (z-n+1)}{t(1-t)(2-t) \cdots (n-t)},$$

where $R(z) > 0$, $R(t) < -1$.

Let $R_n(u)$ denote the remainder after $(n+1)$ terms of the series (2) multiplied by $(1-u)^{-t-1}$. Since (2) is uniformly convergent, given ϵ, n_0 can be found so that for $n \geqq n_0$ and all u, $0 \leqq u \leqq 1$, we have

$$| R_n(u) | < \epsilon.$$

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* $R(z)$ denotes the real part of z.

If we let $R_n'(t)$ denote the remainder after $(n+1)$ terms of the series (5), we may observe that

$$R_n'(t) = \int_0^1 R_n(u)du,$$

and hence

$$|R_n'(t)| \leqq \int_0^1 |R_n(u)|\, du < \epsilon,$$

where ϵ is independent of t; consequently, (5) is a uniformly convergent series in t.

Let $f(z)$ be a function analytic on and outside a closed contour C situated to the left of $R(z) = -1$, and vanishing at infinity; then

$$\begin{aligned} -2\pi i f(z) &= -\int_C \frac{f(t)dt}{z-t} \\ &= \int_C f(t)\left[\frac{1}{t} + \sum_{n=1}^{\infty}(-1)^n \frac{z(z-1)\cdots(z-n+1)}{t(1-t)\cdots(n-t)}\right]dt \end{aligned}$$

when $R(z) > 0$. Since integration termwise is justifiable, we have

$$(6)\qquad -2\pi i f(z) = \int_C \frac{f(t)dt}{t} + \sum_{n=1}^{\infty}(-1)^n \cdot\left(\int_C \frac{f(t)dt}{t(1-t)\cdots(n-t)}\right) z(z-1)\cdots(z-n+1).$$

Hence we may state the following theorem.

THEOREM I. *Let $f(z)$ be a function analytic on and outside of a closed contour C situated to the left of $R(z) = -1$, and vanishing at infinity; then for all z with $R(z) > 0$*

$$(7)\qquad f(z) = b_0 + b_1 z + b_2 z(z-1) + \cdots + b_n z(z-1)\cdots(z-n+1) + \cdots,$$

where

$$(8)\qquad b_n = \frac{(-1)^{n+1}}{2\pi i}\int_C \frac{f(t)dt}{t(1-t)\cdots(n-t)}.$$

If we take $(1-u)^{z-t-1}$ as

$$(1-u)^{z+k}\cdot(1-u)^{-t-1-k},$$

where k may be complex, repeat the steps by means of which Theorem I was deduced, and replace z by $z+k$ and t by $t+k$, we find the following generalized theorem.

THEOREM II. *Let $f(z)$ be a function analytic on and outside of a closed contour C situated to the left of $R(z)=-R(1+k)$, and vanishing at infinity, then for all z with $R(z)>-R(k)$, we have*

$$(9)\quad f(z) = b_0 + b_1(z+k) + b_2(z+k)(z+k-1) + \cdots + b_n(z+k)(z+k-1)\cdots(z+k-n+1) + \cdots,$$

where

$$(10)\quad b_n = \frac{(-1)^{n+1}}{2\pi i}\int_C \frac{f(t)dt}{(t+k)(1-t-k)(2-t-k)\cdots(n-t-k)}.$$

Let U be max. $|f(t)|$ on C and l the length of C; then considering the expansion defined by Theorem I, it is observed that $t=t_1+it_2$ has $-t_1>1$, since $R(t)<-1$ so that $|n-t|\geqq n-t_1>n+1$, and hence

$$\frac{1}{|t(1-t)\cdots(n-t)|}<\frac{1}{(n+1)!}.$$

Consequently

$$(11)\qquad |b_n|<\frac{h}{(n+1)!}.\qquad \left(h=\frac{Ul}{2\pi}\right).$$

THE UNIVERSITY OF VALPARAISO

NOTE ON A CONVERGENCE PROOF

BY DUNHAM JACKSON

Some years ago I published a particularly simple proof of the convergence of the Fejér mean of the Fourier series for an arbitrary continuous function.* I did not notice until some time later that the same proof had already been given by Haar† in his thesis. The present note constitutes a renewed attempt to contribute something to the theory of the method in question, by applying it to a problem which is not treated by Haar, in the passage cited at any rate. The substance of the note consists in the proof of the following theorem:‡

Let $f(x)$ *be an arbitrary continuous function of period* 2π. *With each positive integral value of* n, *let an integer* m_n *be associated, subject merely to the condition that* $m_n \geqq n$, *and let*

$$\tau_n(x) = \frac{1}{nm_n} \sum_{i=1}^{m_n} f(t_i) \frac{\sin^2 \frac{1}{2}n(t_i - x)}{\sin^2 \frac{1}{2}(t_i - x)}, \tag{1}$$

where $t_i = 2i\pi/m_n$. *Then* $\tau_n(x)$ *converges uniformly toward* $f(x)$ *as* n *becomes infinite.*

The reasoning is given in full, so that it can be understood without reference to Haar's paper. By a well known identity,

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* *Note on a method of proof in the theory of Fourier's series*, this Bulletin, vol. 27 (1920–21), pp. 108–110.

† A. Haar, *Zur Theorie der orthogonalen Funktionensysteme*, Dissertation, Göttingen, 1909; p. 29; reprinted in Mathematische Annalen, vol. 69 (1910), pp. 331–371; pp. 353–354.

‡ For the case $m_n = n$, see D. Jackson, *A formula of trigonometric interpolation*, Rendiconti del Circolo Matematico di Palermo, vol. 37 (1914), pp. 371–375; S. Bernstein, *Sur la convergence absolue des séries trigonométriques*, Comptes Rendus, vol. 158 (1914), pp. 1661–1663; L. Fejér, *Über Interpolation*, Gottinger Nachrichten (1916), pp. 66–91; pp. 87–91. For a corresponding generalization of the ordinary formula of trigonometric interpolation, see D. Jackson, *Some notes on trigonometric interpolation*, American Mathematical Monthly, vol. 34 (1927), pp. 401–405. For the underlying idea of the present treatment, see also Hahn, *Über das Interpolationsproblem*, Mathematische Zeitschrift, vol. 1 (1918), pp. 115–142.

$$
\begin{aligned}
\frac{\sin^2 \frac{1}{2}n(t_i - x)}{\sin^2 \frac{1}{2}(t_i - x)} &= n + 2(n-1)\cos(t_i - x) \\
&\quad + 2(n-2)\cos 2(t_i - x) + \cdots + 2\cos(n-1)(t_i - x) \\
&= n + 2\sum_{k=1}^{n-1}(n-k)(\cos kt_i \cos kx + \sin kt_i \sin kx),
\end{aligned} \tag{2}
$$

so that $\tau_n(x)$ is a trigonometric sum of order $n-1$ in x. Since $n-1<m_n$, and since

$$\sum_{i=1}^{m_n} \cos kt_i = \sum_{i=1}^{m_n} \sin kt_i = 0$$

for $0<k<m_n$, it is seen that

$$\frac{1}{nm_n}\sum_{i=1}^{m_n}\frac{\sin^2 \frac{1}{2}n(t_i - x)}{\sin^2 \frac{1}{2}(t_i - x)} = 1. \tag{3}$$

This may also be expressed by saying that if $f(x)\equiv 1$, the corresponding $\tau_n(x)$ is identically equal to 1 for all values of n.

To take another very special case, let $f(x)$ be of the form $\cos px$, where p is a given positive integer, and let the form of the corresponding $\tau_n(x)$ be determined with the aid of (2). The expression $\sum_i \cos pt_i \sin kt_i$ is equal to zero for all values of k. The question ultimately at issue being one of convergence for $n=\infty$, it is sufficient to consider values of $n>2p$. Then $p<m_n/2$, and $\sum_i \cos^2 pt_i = m_n/2$. Under the hypotheses, $p+n-1$ may or may not be less than m_n. If $p+n-1<m_n$, $\sum_i \cos pt_i \cos kt_i = 0$ for all the values of k (including $k=0$) that come into consideration, except $k=p$, and $\tau_n(x)$ reduces to a single term:

$$\tau_n(x) = \frac{n-p}{n}\cos px. \tag{4}$$

If $p+n-1\geqq m_n$, there is one other term, resulting from the fact that $\sum_i \cos pt_i \cos(m_n - p)t_i = m_n/2$, and

$$\tau_n(x) = \frac{n-p}{n}\cos px + \frac{n-m_n+p}{n}\cos(m_n - p)x. \tag{5}$$

But $n-m_n \leqq 0$, and $(n-m_n+p)/n \leqq p/n$, which approaches zero as n becomes infinite. So, whichever of the expressions (4), (5) may be in force from time to time as n takes on successive values, it is clear that

$$\lim_{n=\infty} \tau_n(x) = \cos px,$$

uniformly for all values of x. There is a corresponding proof if $f(x) \equiv \sin px$.

On the other hand, the $\tau_n(x)$ corresponding to the sum of any finite number of functions is the sum of the τ's constructed for the various functions separately, and converges if each of the latter τ's is convergent. *So* $\tau_n(x)$ *converges uniformly toward* $f(x)$, *whenever* $f(x)$ *itself is identically a trigonometric sum.*

In transition, it is to be noted from (1) and (3) that $|\tau_n(x)| \leqq M$, if M is the maximum of $|f(x)|$.

Finally, let $f(x)$ be an arbitrary continuous function of period 2π. Let ϵ be an arbitrary positive quantity. By Weierstrass's theorem there exists a trigonometric sum $T(x)$ such that

$$(6) \qquad |f(x) - T(x)| \leqq \epsilon/3$$

for all values of x. If $\tau_n(x)$ is defined by (1), and if $\tau_{n1}(x)$ is similarly formed with $T(t_i)$ in place of $f(t_i)$, it follows from the preceding paragraph, applied to the difference $T(x)-f(x)$, that

$$(7) \qquad |\tau_{n1}(x) - \tau_n(x)| \leqq \epsilon/3$$

for all values of n and x. And by the italics at the end of the second paragraph preceding,

$$(8) \qquad |T(x) - \tau_{n1}(x)| \leqq \epsilon/3$$

if n is sufficiently large. For such values of n, by combination of (6), (7), and (8), $|f(x) - \tau_n(x)| \leqq \epsilon$, which is equivalent to the conclusion of the theorem.

THE UNIVERSITY OF MINNESOTA

AN ELEMENTARY PROPERTY OF BOUNDED DOMAINS*

BY W. L. AYRES†

It is the purpose of this note to prove the following property of a bounded domain‡ lying in a two-dimensional euclidean space.

THEOREM. *If P is a point of a plane and bounded domain, there exists a triangle such that it contains P in its interior, has its vertices on the boundary of the domain, and every other point of the triangle and its interior lie in the domain.*

PROOF. Let D be the domain and B denote the boundary of D. Let r denote the lower limit of all numbers $d(P, x)$, where x is any point of the boundary B and $d(P, x)$ denotes the distance from P to x. Let C be the circle with center at P and radius r. Since B is a closed set, the circle C contains a point X of B. Let E_1 and E_2 be points whose distance from P is $\frac{1}{2}r$ and which lie on the line through P perpendicular to the line XP. Let $E_iF_i(i=1, 2)$ be a ray through E_i parallel to the ray XP.§ If y is any point of the ray XP such that $d(P, y) \geqq r$, let C_y be the circle with center at P and

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† National Research Fellow in Mathematics.

‡ A connected set of points D is said to be a *domain* if for every point P of D there is a circle with center at P, such that D contains every point in the interior of the circle. The *boundary* of a domain is the set of all limit points of the domain which do not belong to the domain. The boundary of a bounded domain is a closed and bounded point set, and if H is any connected point set containing a point of the domain and a point not belonging to the domain, then H contains a point of the boundary of the domain.

§ If l is a line and A and B are points of l, the *ray* AB (but not the ray BA) is the maximal connected subset of $l-A$ which contains B. If AB is a ray and C is a point not on the line AB then if l denotes the line through C parallel to the line AB the *ray through* C *parallel to the ray* AB is the maximal connected subset of $l-C$ which lies on the B-side of the line AC.

radius $d(P, y)$ and let y_i be the intersection of C_y and the ray E_iF_i. Let $[y]$ be the set of all points y of the ray XP such that $d(P, y) \geqq r$ and the arc y_1yy_2 of C_y contains a point of the boundary B. Using the fact that the boundary B is closed and the ray XP contains at least one point of B, it is not difficult to show that the set $[y]$ is non-vacuous and closed. Since $[y]$ is closed, there is a first point A of $[y]$ on the ray XP.* The arc A_1AA_2 of C_A contains a point Y of the boundary B and every point of the segment XY belongs to the domain D. There are two cases to consider according as $Y=A$ or $Y \neq A$.

Case I. If $Y \neq A$, let Q be the intersection of the line XY and the line through P parallel to the line AY. Let $[z]$ be the set of all points z of the ray QP such that either (a) the segment Xz contains a point of the boundary B, or (b) the segment Yz contains a point of B, or (c) the point z belongs to B. It follows easily that the set $[z]$ is non-vacuous, closed and contains no point of the interval QP. Let G be the first point of the set $[z]$ on the ray QP. Then either G is a point of the boundary B or the segment XG or YG contains a point of the boundary B.

Since G does not lie on the interval QP, the point P is in the interior of the triangle XYG, and every point in the interior of this triangle lies in the domain D. If the segment XG contains a point of B, let Z be the first point of B on the ray XG. We may show that there is a first point of B on the ray XG as follows. Since the ray XG makes an acute angle with the ray XP, the ray XG contains a point H of the circle C. Now the points of B on the ray XG form a set which is closed except possibly for the point X. But X is not a limit point of the set of points of B which lie on the ray XG, since no point of the segment XH belongs to B. Hence the points of B which lie on the ray XG form a closed set and there is a first point Z of B on the segment XG in the order X to G.

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* If K is a set of points lying on a ray AB, a point p of K is said to be the *first point of* K on the ray AB if for every point q of K distinct from p the point p lies on the segment Aq.

Let l be the line through X parallel to the line AY and let K denote the point of intersection of l and the circle C which is distinct from X (the line l is not tangent to the circle C since $A \neq Y$). The segment YK contains the point P and there is no point of B on or inside the triangle XYK except possibly its vertices. Hence the angle XYZ is greater than the angle XYK and so P lies in the angle XYZ. Similarly the angle YXZ is greater than the angle YXA and so P lies in the angle YXZ. Then P lies in the interior of the triangle XYZ. The segment XZ belongs to the domain D, since the segment XH belongs to the domain D and Z is the first point of B on the ray XG. The segment YZ and the interior of the triangle XYZ belong to the domain D, since they are subsets of the interior of the triangle XYG. Therefore the triangle XYZ satisfies all the conditions of our theorem and is the desired triangle.

If no point of the segment XG belongs to the boundary B, but the segment YG contains at least one point of B, then if Z is chosen as the first point of B on the ray YG, the same proof shows that the triangle XYZ is the desired triangle. If neither the segment XG nor the segment YG contains a point of B, the triangle XYG satisfies all the conditions of our theorem and is the required triangle.

Case II. If $A = Y$, let l be the line through P perpendicular to the line XY and let G_1 and G_2 be points of l such that P is between G_1 and G_2. On the ray PG_1, let $[z]$ denote the set of all points z such that z belongs to the boundary B or the segment Xz contains a point of B or the segment Yz contains a point of B. It is easily seen that $[z]$ is non-vacuous and closed. Let H be the first point of the set $[z]$ on the ray PG_1. The set of points of B which lie on the ray XH is closed except possibly for the point X. Since the angle YXH is acute, there is a segment XK which lies entirely in the interior of the circle C and thus entirely in the domain D. Then the set of points of B on the ray XH is closed. If the segment XH contains a point of B, let Z be the first point of B on the segment XH in the order X to H. If the segment XH contains no point of B but the segment YH contains a point of B, let Z be the first point of B on the segment YH in the order Y to H. There is a first point since the set of points of B which lie on the ray YH is closed. If neither the segment XH nor the segment YH contains a point of the boundary B, let H be the point Z. In any case the triangle XYZ has its vertices on the boundary B and every other point of the triangle and its interior belongs to the domain D, the segment XY contains the point P, and the angles XYZ and YXZ are acute.

In exactly the same manner, we determine a point W of the boundary B on the G_2-side of the line XY, such that the triangle XYW has the same properties as the triangle XYZ, where Z is replaced by W.

Since the angles XYZ, YXZ, XYW and YXW are all acute, every point of the segment ZW lies in the interior of the quadrilateral $XZYW$. If the segment ZW does not contain the point P, the point P lies in the interior of the triangle XZW or YZW. In this case the triangle XWZ or the triangle YWZ (whichever contains P in its interior) is the desired triangle. If the segment WZ contains the point P, we have a quadrilateral $XZYW$ having its vertices on the boundary B and every other point of the quadrilateral and its interior belonging to the domain D, and such that P is the intersection of the diagonals of $XZYW$. Either the rays XZ and WY have no point in common or the rays ZX and YW have no point in common. Let us suppose that the rays XZ and WY have no point in common. If the lines XZ and WY are parallel, let PQ be the ray through P parallel to the ray XZ. If the lines XZ and WY intersect in a point M, let Q be a point of the line PM such that P is between Q and M. On the ray PQ, let $[v]$ be the set of all points v such that v is a point of B or the segment Xv contains a point of B or the segment Wv contains a point of B. As above, we may show that the set $[v]$ is non-vacuous and closed. Let N be the first point of the set $[v]$ on the ray PQ. If the segment XN contains a point of B, let V be the first point of the boundary B on the ray XN. There is a first point V since the set of points of B which lie on the ray XN is a closed set. Since V is outside the quadrilateral $XZYW$, the angle XWV is greater than the angle XWZ and the angle WXV is greater than the angle WXY. Hence the point P lies in the interior of the triangle WXV and this is the desired triangle. If the segment XN contains no point of B but the segment YN contains a point of B, let V be the first point of B on the ray XN, and again the triangle XWV is the desired triangle. If neither the segment XN nor the segment YN contains a point of B, the point N belongs to B and the triangle XWN is the desired triangle. The case in which the rays ZX and YW have no point in common is exactly the same as the above.

Thus, in any possibility, we have established the existence of a triangle having the properties required in our theorem.

It is of interest to notice that while for any point of any bounded domain there is a triangle having the properties required in our theorem, there exist bounded domains, and in fact bounded domains with simple closed curves as boundaries, such that there is *no simple polygon of more than three sides* having its vertices on the boundary of the domain and so that every other point of the polygon and its interior lie in the domain. An example of such a domain is the bounded domain whose boundary is a three-cusped hypocycloid.

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EXTENDED POLYGONAL NUMBERS*

BY L. E. DICKSON

1. *Introduction and Summary.* The xth extended polygonal number of order $m+2$ is $e(x)=\frac{1}{2}m(x^2+x)-x$, while $e(-x)$ is the xth polygonal number of order $m+2$. We take $m>2$ and thereby exclude the classic cases of triangular numbers and squares. If k is an integer $\geqq 0$, the values of

$$(1)\qquad e(x-k)=(x-k)\left[\tfrac{1}{2}m(x-k+1)-1\right]$$

for integers $x\geqq 0$ are the first k polygonal numbers and all the extended polygonal numbers.

We shall prove that every integer $p\geqq 0$ is a sum of E numbers 0 or 1 and four values of function (1) for integers $x\geqq 0$, where $E=m-2$ if $k=0$; $E=m-3$ if $k=1$; while if $k=2$, $E=m-6$ when $m\geqq 8$, $E=2$ when $m=7$, $E=1$ when $m=5$ or 6, $E=0$ if $m\leqq 4$; and finally if $k\geqq 3$, $E=m-6$ when $m\geqq 7$, $E=1$ when $m=6$, and $E=0$ when $m\leqq 5$. In no case will a smaller E serve for every p.

These results have a very simple interpretation when $E=0$, so that every integer $p\geqq 0$ is a sum of four values of (1). When $k=1$, $m=3$, this is equivalent to the fact that every positive integer $24p+4$ is the sum of the squares of four numbers of the form $6x-5$ with $x\geqq 0$. When $k=2$, $m=4$, and when $k=3$, $m=5$, the equivalent facts are

$$8p+4=\sum(4x-7)^2,\quad 40p+36=\sum(10x-27)^2,$$

each summed for four integers $x\geqq 0$. When $k=1$, $m=4$, the equivalent fact is that, for every integer $p\geqq 0$, one of $8p\pm 4$ is a sum of the squares of four numbers $4x-3$ with $x\geqq 0$. Each of our theorems has a similar interpretation concerning four squares, besides the obvious one for $E+4$ squares. The single improvement on the last fact is furnished by the last part of Theorem 3, which is equivalent to

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* Presented to the Society, September 9, 1927.

$$56A + 125 = \sum_5 (14x - 23)^2,$$

for five integers $x \geqq 0$ and every $A \geqq 0$. By Theorem 5,

$$3p + 4 = \sum_4 (3x - 8)^2 \quad \text{if} \quad p \neq 8n + 4.$$

Assistance was provided by the Carnegie Institution for preparing the tables and verifying the theorems for the necessary initial cases. The tables in the three papers are of constant use in a memoir to appear in the American Journal of Mathematics, which treats the like problem for all quadratic functions.

2. *General Formulas.* A quadratic function has an integral value for every integer $x \geqq 0$ if and only if it has the form $\frac{1}{2}mx^2 + \frac{1}{2}nx + c$, where m, n, c are integers such that $m+n$ is even. Comparison with (1) gives

$$(2) \quad n = m - 2 - 2mk, \qquad c = \tfrac{1}{2}m(k^2 - k) + k.$$

For these values, formulas (4), (6), (8), (13), and (15) of the writer's paper in this Bulletin (vol. 33 (1927), pp. 713–720) give

$$(3) \quad A = mw + 4c + r - b, \qquad w = \tfrac{1}{2}(a + b) - kb,$$

$$(4) \quad U = 24mA + m^2(9 - 12k - 12k^2) - 24mk - 60m + 36,$$

$$(5) \quad V = 2mA - 2mE + (m - 2)^2, \quad P = 2m(d - k) - m - 2,$$

$$(6) \quad F = (2V + W)^2 - VU > 0, \quad 8W = U - P^2.$$

The (reduced) minor conditions are

$$(7) \qquad A \geqq 4c + 4E, \qquad A \geqq 4c + \tfrac{2}{3}m$$

if $n \geqq 0$, but the same and

$$(8) \qquad 3A \geqq 12c + 2m - 2n - n^2/m, \qquad (\text{if } n < 0).$$

When $k \geqq 2$, the latter follows from (7_2). In fact, the sum of its last two terms is negative, since $-n$ is positive and $2m+n$ is negative.

3. *The Case* $k=0$. We shall prove that every integer

$A \geqq 0$ is a sum of $E=m-2$ numbers 0 or 1 and four extended polygonal numbers $0, m-1, 3m-2, \cdots$. Note that $E(m-2)=m-2$. When r takes the values $0, 1, \cdots, m-2$, and b the odd values β and $\beta-2$, evidently $b-r$ takes the values of $\beta, \beta-1, \cdots, \beta-m$, which include a complete set of residues modulo m. We choose $b-r$ congruent to $4c-A$. Then (3) determines an integer w and hence an odd integer a. There will be two consecutive odd values β, $\beta-2$ of b if the difference between the limits for b exceeds 4. Hence we take $d=4$. Then

$$U = 24mA + 9m^2 - 60m + 36, \qquad V = 2mA - m^2 + 4,$$
$$P = 7m - 2, \qquad W = 3mA - 5m^2 - 4m + 4,$$
$$F = m^2A^2 - 92m^3A + 64m^2A + 58m^4 - 4m^3 - 152m^2 + 144m > 0,$$

which evidently holds if $A \geqq 92m-64$, $m \geqq 3$. Conditions (7) hold if $A \geqq 4m$.

In Table III*, the entries involving the same multiple of m and the intervening numbers will be said to form a *block*. We suppress $15m-5, \cdots, 69m-13, 14, \cdots$, and all such entries down to the last entry of any block which differs by 3 or more from the next entry of that block. If we subtract the largest entry of any abridged block from the least entry of the next abridged block, we always obtain a difference $\leqq m-1$.

TABLE III. SUMS OF FOUR EXTENDED POLYGONAL NUMBERS

0, $m-1$, $2m-2$, $3m-2$–3, $4m-3$–4, $5m-4$, $6m-3$–5, $7m-4$–5, $8m-5$–6, $9m-5$–6, $10m-4$, 6, 7, $11m-5$, 7, $12m-6$–8, $13m-6$–8, $14m-7$, 8, $15m-5$, 8, 9, $16m-6$–9, $17m-7$–9, $18m-7$–10, $19m-8$–10, $20m-8$–10, $21m-6$, 8, 9, 11, $22m-7$, 9–11, $23m-8$, 10, 11, $24m-8$–12, $25m-9$, 11, 12, $26m-10$–12, $27m-9$–12, $28m-7$, 10–13, $29m-8$, 11–13, $30m-9$–13, $31m-9$–13, $32m-10$–14, $33m-11$–14, $34m-10$–14, $35m-11$–13, $36m-8$, 11–15, $37m-9$, 12–15,

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* Tables No. I and No. II occur in the previous papers by the author, this Bulletin, vol. 33, pp. 713-720, and vol. 34, pp. 63-72.

$38m-10$, 11, 13–15, $39m-10$–13, 15, $40m-11$, 13–16, $41m-12$–16, $42m-11$, 12, 14–16, $43m-12$, 13, 15, 16, $44m-13$–16, $45m-9$, 13–17, $46m-10$, 12, 14–17, $47m-11$, 13, 15–17, $48m-11$, 12, 14–17, $49m-12$–17, $50m-13$–18, $51m-12$, 13, 15–18, $52m-13$–18, $53m-14$–17, $54m-14$, 15, 17, 18, $55m-10$, 13, 15–19, $56m-11$, 14, 16–19, $57m-12$, 14–17, 19, $58m-12$, 13, 15–19, $59m-13$, 16–19, $60m-14$, 16–20, $61m-13$–17, 19, 20, $62m-14$–20, $63m-15$–20, $64m-15$–20, $65m-14$, 16, 17, 19, 20, $66m-11$, 15, 17–21, $67m-12$, 16–21, $68m-13$, 16–21, $69m-13$, 14, 17, 19–21, $70m-14$, 15, 18–21, $71m-15$–21, $72m-14$–22, $73m-15$–17, 19–22, $74m-16$–19, 21, 22, $75m-16$–22, $76m-15$–22, $77m-16$, 17, 19–22, $78m-12$, 17–23, $79m-13$, 17–21, 23, $80m-14$, 18–23, $81m-14$–17, 19–23, $82m-15$, 17–23, $83m-16$–21, 23, $84m-15$, 16, 18–24, $85m-16$, 17, 19–24, $86m-17$, 19–24, $87m-17$–21, 23, 24, $88m-16$, 18, 20–24, $89m-17$, 19–24, $90m-18$, 19, 21–25, $91m-13$, 18–21, 23–25, $92m-14$, 19–25.

THEOREM 1*. *If* $k=0$, *then* $E=m-2$.

4. *The Case* $k=1$. The following is a complete list to $8m-4$ of sums by four of 1 and extended polygonal numbers:

$$
(9)\qquad
\begin{array}{l}
0-4,\ m-1,\ m,\ m+1,\ m+2,\ 2m-2-1,\ 2m,\\
3m-3-1,\ 3m,\ 3m+1,\\
4m-4-1,\ 5m-4-3,\ 6m-5-1,\ 6m,\\
7m-5-2,\ 8m-6-4.
\end{array}
$$

Hence $E(6m-6)=m-3$. We next prove

THEOREM 2. *If* $k=1$, *then* $E=m-3$.

First, let $m\geqq 4$. For $b=\beta$, $\beta-2$ and $r=0, 1, \cdots, m-3$, then $b-r$ takes the values β, $\beta-1, \cdots, \beta-(m-1)$, which form a complete set of residues modulo m. Hence $d=4$. Then

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* In the current number of the Proceedings of the American Philosophical Society, the writer gives another proof, analogous to that by Cauchy for ordinary polygonal numbers.

$$U = 24mA - 15m^2 - 84m + 36, \quad V = 2mA - m^2 + 2m + 4$$
$$P = 5m - 2, \quad W = 3mA - 5m^2 - 8m + 4,$$
$$F = m^2A^2 + m^2A(64 - 44m) + 34m^4 + 2m^3 + 112m^2 + 168m.$$

Evidently $F>0$ if $A \geqq 44m-64$. The minor conditions hold if $A \geqq 4m$. By (9), $E(A) \leqq m-3$ for $A \leqq 8m-4$. We annex $15m-7=1+3m-2+2(6m-3)$ to Table III and abridge it as in §3. Then the gaps are $\leqq m-2$ from $8m-4$ to $44m-16$. This proves Theorem 2 when $m \geqq 4$.

Second, let $m=3$. Then β, $\beta-2$, $\beta-4$ form a complete set of residues modulo 3. Hence $d=6$,

$$U = 72A - 351, \quad V = 6A + 1, \quad W = 9A - 122,$$
$$F = A^2 - 334A + 1639 > 0 \quad \text{if} \quad A \geqq 6.$$

The minor conditions hold if $A \geqq 7$. But the numbers $<3m$ in (9) are 0-8. Hence Theorem 2 holds if $m=3$.

5. *The Case* $k=2$. Then $E(m-2)=m-6$.

First, let $m \geqq 8$. We shall prove that $E=m-6$. For $b=\beta$, $\beta-2$, $\beta-4$, $\beta-6$, $0 \leqq r \leqq m-6$, evidently $b-r$ takes the values β, $\beta-1, \cdots, \beta-m$, whence $d=8$. Then

$$U = 24mA - 63m^2 - 108m + 36, \quad V = 2mA - m^2 + 8m + 4,$$
$$P = 11m - 2, \quad W = 3mA - 23m^2 - 8m + 4,$$
$$F = m^2A^2 - 200m^3A + 136m^2A + 562m^4 - 4m^3 + 616m^2 + 336m.$$

Thus $F>0$ if $A \geqq 200m-136$ and in fact if $A=198m-136+t$, $t \geqq 0$, since then $F=t^2m^2+tm^2(196m-136)+166m^4+268m^3+616m^2+336m$.

From Table IV we suppress $4m+8$, 5, and all such entries in any block down to the last entry which differs by 3 or more from the next entry of that block. We shall prove that $E(A) \leqq m-6$ if A lies between consecutive blocks. This will follow if proved when $A+1$ is the least entry $tm-s$ of an abridged block. Then A is the sum of $m-6$ or $m-7$ and a number n in the abridged table if $tm-s$ is the sum of $m-5$ or $m-6$ and n. In other words, if $-s$ is the term free of m in the least entry of an abridged block, then $-(s-5)$ or $-(s-6)$ is that of some entry of the preceding abridged block. An inspection of Table IV shows that this holds with $-(s-5)$ except for

$$t = 10, 38, 56, 71, 78, 82, 91, 108, 131, 138, 142, 159, 169, 172, 176, 191. \tag{10}$$

It holds with $-(s-6)$ for all cases in (10) except $t=10$. But for $m \geqq 8$, $10m-8$ is the sum of $m-8$ and $9m$, while the gaps of 3 from $9m$ to $9m\pm 3$ are now permissible since $E \geqq 2$.

TABLE IV. SUMS BY FOUR OF 1, $m+2$, AND EXTENDED POLYGONAL NUMBERS

0–4, $m+5$, 4, 3, 2, 1, 0, -1, $2m+6$, 5, 4, 3, 2, 1. 0, -1, -2, $3m+7$, 6, 4, 3, 1, 0, $-1-3$, $4m+8$, 5, 2, 1, 0, $-1-4$, $5m+3$, 2, 0, -1, -3, -4, $6m+4$, 1, 0, $-1-5$, $7m+1$, 0, $-1-5$, $8m+2$, 1, 0, $-1-6$, $9m+3$, 0, $-3-6$, $10m-1$–7, $11m+0$, $-1-5$, -7, $12m+1$, 0, $-2-8$, $13m+2$, -1, $-3-8$, $14m-2-8$, $15m-2-5$, $-7-9$, $16m-1-9$, $17m+0$, -1, $-3-9$, $18m+1$, -2, -3, $-5-10$, $19m-4$, -5, $-7-10$, $20m-3$, $-6-10$, $21m-3-9$, -11, $22m-2-11$, $23m-1$, -2, -4, -5, $-7-11$, $24m+0$, -3, $-6-12$, $25m-5-9$, -11, -12, $26m-4$, -6, -7, $-9-12$, $27m-5$, $-7-12$, $28m-4-13$, $29m-3-9$, $-11-13$, $30m-2$, -3, -5, -6, $-8-13$, $31m-1$, 4[*], 7–13, $32m-6$–14, $33m-5$, 6, 8, 9, 11–14, $34m-8$-14, $35m-7$–13, $36m-5$–15, $37m-4$–9, 11–15, $38m-3$, 4, 6, 7, 9–11, 13–15, $39m-2$, 5, 8–13, 15, $40m-7$, 8, 10–16, $41m-6$, 9, 11–16, $42m-9$–16, $43m-8$–13, 15, 16, $44m-7$–16, $45m-6$–9, 11–17, $46m-5$–17, $47m-4$, 5, 7–13, 15–17, $48m-3$, 6, 8–17, $49m-8$, 9, 11–17, $50m-7$, 10–18, $51m-9$–13, 15–18, $52m-9$–18, $53m-8$, 9, 11–17, $54m-13$–15, 17, 18, $55m-7$–13, 15–19, $56m-6$–19, $57m-5$, 6, 8, 9, 11–17, 19, $58m-4$, 7, 10–19, $59m-9$, 10, 12, 13, 15–19, $60m-8$, 11–20, $61m-11$–17, 19, 20, $62m-10$–20, $63m-9$, 12, 15–20, $64m-13$–20, $65m-12$–17, 19, 20, $66m-8$–21, $67m-7$–13, 15–21, $68m-6$, 7, 9–21, $69m-5$, 8, 11–14, 16–21, $70m-10$, 11, 13–15, 17–21,

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[*] From here on, we omit minus signs in continuations.

$71m-$9, 12, 13, 15–21, $72m-$11–22, $73m-$11–17, 19–22, $74m-$10, 12–19, 21, 22, $75m-$12, 15–22, $76m-$13–22, $77m-$12–17, 19–22, $78m-$9–12, 14–23, $79m-$8–13, 15–21, 23, $80m-$7, 8, 10, 11, 13–23, $81m-$6, 9, 12–17, 19–23, $82m-$11–23, $83m-$10, 12, 13, 15–21, 23, $84m-$13–24, $85m-$12–17, 19–24, $86m-$11, 14, 17–24, $87m-$15–21, 23, 24, $88m-$14–24, $89m-$13, 14, 16, 17, 19–24, $90m-$12, 15–19, 21–25, $91m-$10–13, 15–21, 23–25, $92m-$9–25, $93m-$8, 9, 11, 12, 14–17, 19–25, $94m-$7, 10, 13–25, $95m-$12, 13, 15–21, 23–25, $96m-$11, 14, 17–25, $97m-$14–17, 19–26, $98m-$13–23, 25, 26, $99m-$12, 15–21, 23–26, $100m-$15–26, $101m-$14–17, 19–25, $102m-$14, 15, 17–26, $103m-$13, 16–21, 23–26, $104m-$15, 17–26, $105m-$11–14, 17, 20–25, 27, $106m-$10–19, 21–27, $107m-$9, 10, 12, 13, 15–21, 23–27, $108m-$8, 11, 14–18, 20–27, $109m-$13, 14, 16, 17, 19–25, 27, $110m-$12, 15, 18–27, $111m-$15–21, 23–27, $112m-$14–28, $113m-$13, 16, 17, 19–25, 27, 28, $114m-$15, 18–23, 25–28, $115m-$16–18, 20, 21, 23–28, $116m-$15, 16, 18, 19, 21–28, $117m-$14, 17, 19–25, 27, 28, $118m-$18–28, $119m-$18–21, 23–28, $120m-$12–15, 17–29, $121m-$11–17, 19–25, 27–29, $122m-$10, 11, 13–19, 21–29, $123m-$9, 12, 15–21, 23–29, $124m-$14, 15, 17–19, 21–29, $125m-$13, 16, 17, 19–25, 27–29, $126m-$16–27, 29, $127m-$15–21, 23–29, $128m-$14, 16–30, $129m-$17, 19–25, 27–30, $130m-$17–23, 25–30, $131m-$16, 17, 19–21, 24–29, $132m-$15, 18, 20–30, $133m-$19–25, 27–30, $134m-$18–30, $135m-$17–21, 23–29, $136m-$13–18, 20–31, $137m-$12–17, 19–25, 27–31, $138m-$11, 12, 14, 15, 17, 18, 21–31, $139m-$10, 13, 16–20, 23–29, 31, $140m-$15, 16, 18, 19, 22, 23, 25–31, $141m-$14, 17, 19–25, 27–31, $142m-$17–27, 29–31, $143m-$16–21, 23–29, 31, $144m-$15, 18, 21–32, $145m-$20–25, 27–32, $146m-$18–23, 25–32, $147m-$17, 18, 20, 21, 23–29, 31, 32, $148m-$16, 19–32, $149m-$19–25, 27–32, $150m-$18, 20–32, $151m-$19–21, 23–29, 31, 32, $152m-$18, 19, 21–32, $153m-$14–17, 20, 21, 23–25, 27–33, $154m-$13–18, 22–33, $155m-$12, 13, 15, 16, 18, 19, 21, 23–29, 31–33, $156m-$11, 14, 17–33, $157m-$16, 17, 19–25, 27–33, $158m-$15, 18–27, 29–31, 33, $159m-$18–21, 23–29, 31–33,$160m-$17–33,

$161m-16$, 19, 21, 22, 24, 25, 27–33, $162m-20$–23, 25–34, $163m-19$–21, 23–29, 31–34, $164m-18$, 19, 21–34, $165m-17$, 20–24, 27–33, $166m-19$, 21–34, $167m-20$, 21, 23–29, 31–34, $168m-20$–27, 29–34, $169m-19$, 20, 22–25, 27–33, $170m-18$, 21–34, $171m-15$–18, 21, 23–29, 31–35, $172m-14$–19, 22–35, $173m-13$, 14, 16, 17; 19–22, 24, 25, 27–33, 35, $174m-12$, 15, 18–27, 29–31, 33–35, $175m-17$, 18, 20, 21, 23–29, 31–35, $176m-16$, 19, 22–35, $177m-19$–22, 24, 25, 27–33, 35, $178m-18$–23, 25–35, $179m-17$, 20, 23–29, 31–35, $180m-22$–36, $181m-20$–25, 27–33, 35, 36, $182m-19$–36, $183m-18$, 20, 21, 23–29, 31–36, $184m-22$–36, $185m-22$, 24, 25, 27–33, 35, 36, $186m-21$–36, $187m-20$, 21, 23–29, 31–36, $188m-19$, 22, 23, 25–36, $189m-23$–25, 27–33, 35, 36, $190m-16$–19, 22, 23, 25–27, 29–31, 33–37, $191m-15$–21, 24–29, 31–37, $192m-14$, 15, 17, 18, 20–37, $193m-13$, 16, 19–22, 24, 25, 27–33, 35–37, $194m-18$–23, 26–35, 37, $195m-17$, 20, 23, 25–29, 31–37, $196m-20$–37, $197m-19$–25, 27–33, 35–37, $198m-18$, 21, 23–37.

If $m=7$, Table IV lacks $61=9m-2$ and $62=9m-1$, since the maximum in the preceding block is $8m+2=58$ and the minimum in the subsequent block is $10m-7=63$. If $m=6$, Table IV lacks $28=4m+4=5m-2=6m-8$. If $m=5$, it lacks $23=3m+8=4m+3=5m-2=6m-7$. Hence E cannot have smaller values than those in

THEOREM 3. *If* $k=2$, $E=m-6$ *for* $m\geqq 8$, $E=2$ *for* $m=7$, $E=1$ *for* $m=5$ *or* 6, $E=0$ *for* $m=4$. *When*[*] $m=7$, $E(A)\leqq 1$ *if* $A\neq 62$.

Since $E\leqq m-2$ for every m, conditions (7) hold if $A\geqq 8m$. This completes the proof of Theorem 3 when $m\geqq 8$. For $m=5$, 6, or 7, the values of $b-r$ for $b=\beta$, $\beta-2$, $\beta-4$ include a complete set of residues modulo m, whence $d=6$. We have the same U as before and

$$P = 7m - 2, \qquad W = 3mA - 14m^2 - 10m + 4.$$

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[*] Hence every integer $\geqq 0$ is a sum of five values of $e(x-2)$ for $m=7$.

If $m=7$, $U=168A-3807$, $V=14A-3$, $W=21A-752$,

$$F = 49A^2 - 7\cdot 2926A + 7\cdot 80449 > 0 \quad \text{if} \quad A \geqq 389.$$

The discussion made when $m \geqq 8$ applies when $m=7$ except for $10m-8$. The latter now equals $9m-1$ and gives no trouble.

If $m=6$, $U=36(4A-80)$, $V=12A+4$, $W=18A-560$,

$$F/36 = (7A-92)^2 - (4A-80)(12A+4) = A^2 - 344A + 8784,$$

and $F>0$ if $A \geqq 317$. When $A<317=55m-13$, we have only the first two cases in (10). But $38m-16=37m-10$ is in the preceding block, while $10m-8=9m-2$ exceeds by unity the greatest entry in its preceding abridged block.

If $m=5$,

$$U = 120A - 2079,\ V = 10A - 1,\ W = 15A - 396,$$

$$F/25 = A^2 - 278A + 6253 > 0 \quad \text{if} \quad A \geqq 254.$$

But $E(A) \leqq 1$ for $A<254$.

Finally, if $m=4$, we shall prove that $E=0$. Here (1) is $2x^2-7x+6$, whence $A=2a-7b+24$.

First, let A be odd. Take $b \equiv A-2 \pmod 4$. Then a and b are odd and the general method applies. Here $d=4$, $P=10$,

$$U = 4(24A-351), \quad V = 4(2A+1), \quad W = 12A-188,$$

$$F/16 = (7A-45)^2 - (2A+1)(24A-351) = (A-24)^2 + 1800.$$

Second, let $A=2S$. Take $b=2B$, $B \equiv S-2 \pmod 4$. Then $a=S+7B-12$, $a \equiv 2 \pmod 4$. Hence we may apply Lemma 3 of the writer's paper in this Bulletin for January (vol. 34, pp. 63–72). Its conditions

$$4B^2 + 2B + 1 > 3a, \quad B^2 < a$$

become on elimination of a

$$8B > 19 + u^{1/2},\ 2B < 7 + v^{1/2},\ u = 48S - 231,\ v = 4S + 1.$$

The difference between these two limits for B shall exceed 4.

Hence $4v^{1/2}-u^{1/2}>23$. The left member is $\geqq 0$. Squaring twice, we get

$$S^2 - 432S + 2220 > 0, \quad S \geqq 427, \quad A \geqq 854.$$

It was verified that $E(A)=0$ for $A<854$.

6. THEOREM 4. *For* $k\geqq 3, E=m-6$ *if* $m\geqq 7, E=1$ *if* $m=6$, $E=0$ *if* $m=5$.

Since 0 and 1 are the only polygonal and extended polygonal numbers $<m-2$, $E(m-2)=m-6$ if $m\geqq 6$. If $m\geqq 8$, our theorem now follows Theorem 3.

When $m=7$, it remains to prove that $E=1$ if $k=3$. As at the beginning of §5, $d=8$. Then $P=61$ and

$$U = 168A - 7503, \quad V = 14A + 11, \quad W = 21A - 1403,$$

$$F/49 = A^2 - 656A + 40606 > 0 \text{ if } A \geqq 587 = 86m - 15.$$

Conditions (7) hold if $A\geqq 101$. When $A<587$, the discussion at the beginning of §5 shows that $E(A)\leqq 1$ except for

$$10m - 8 = 9m - 1 = 2(3m - 2) + 3m + 3.$$

Let $m=5$, $k=3$. We shall prove that $E=0$. Here $d=10$, $P=63$,

$$U = 120A - 3999, \quad V = 10A + 9, \quad W = 15A - 996,$$

$$F/25 = A^2 - 1182A + 39699 > 0 \text{ if } A \geqq 1148.$$

Conditions (7) hold if $A\geqq 76$. It was verified that $E(A)=0$ for $A<1148$.

Finally, let $m=6$. By (1),

$$1 + 3e(x - k) = (3x - 3k + 1)^2.$$

Hence the following equations are equivalent:

$$(11) \quad A = \sum_4 e(x - k), \quad 3A + 4 = \sum_4 (3x - 3k + 1)^2.$$

When the former holds we write $E_k(A)=0$, attaching the subscript k to the earlier E. Then also $E_g(A)=0$ if $g>k$. Hence $E_g(A)>0$ implies $E_k(A)>0$ for every $k\leqq g$. Let k be any given integer $\geqq 0$. To prove that $E>0$, it evidently suffices to exhibit one positive integer A for which $E_k(A) > 0$. For a sufficiently large integer n,

$$g = \tfrac{1}{3}(5 \cdot 4^n - 2) \tag{12}$$

is an integer $\geqq k$. Take $3A+4=4^{2n} \cdot 46$. Then $E_g(A)>0$ by the following lemma.

LEMMA 1. *There do not exist four integers* $x \geqq 0$ *satisfying*

$$4^{2n} \cdot 46 = \sum_4 (3x + 3 - 5\ 4^n)^2. \tag{13}$$

For $n=0$, $46 = \sum(3x-2)^2$ is not solvable. In fact, the summands <46 are $(-2)^2$, 1^2, 4^2. If $2 \cdot 23$ is a sum by four of 1, 4, 16, the value 1 must be used twice, while the maximum sum by two of 4 and 16 is $32 < 46-2$. For $n \geqq 1$, Lemma 1 follows by induction from $n-1$ to n. Let (13) hold. Since the sum of the four squares is a multiple of 8, each square is even. Thus x is odd, $x=2y+1$, where y is an integer $\geqq 0$. Cancellation of 4 gives

$$4^{2n-1} \cdot 46 = \sum(3y + 3 - 10\ 4^{n-1})^2.$$

The four squares are again all even, whence $y=2z+1$, where z is an integer $\geqq 0$. Cancellation of 4 now gives a result like (13) with n replaced by $n-1$. Hence the induction is complete.

By Theorem 3, $E_2(B) \leqq 1$ for every $B \geqq 0$. This completes the proof that $E=1$ for every $k \geqq 2$.

It is an interesting fact that g in (12) is the greatest integer for which $E_g(A)>0$ when $3A+4=4^{2n}\ 46$. This is a consequence of $E_{g+1}(A)=0$. The latter follows from

LEMMA 2. *There exist four integers* $x \geqq 0$ *satisfying*

$$4^{2n} \cdot 46 = \sum(3x - 5 \cdot 4^n)^2. \tag{14}$$

This is true for $n=0$ since 46 is the sum of 25, 4, 1, 16, which are the squares of the values of $3x-5$ for $x=0, 1, 2, 3$. For $n \geqq 1$, we proceed by induction from $n-1$ to n. Hence assume (14) with n replaced by $n-1$. Multiply the assumed equation by 4^2 and write $\xi = 4x$; we get (14) with x replaced by ξ.

Instead of Lemmas 1 and 2, we may employ

LEMMA 3. *If n is an odd integer* $\geqq 3$ *and* $C=4^{n-3}\cdot 88$, *then*

$$C \neq \sum(3x+3-2^n)^2,\quad C=\sum(3x-2^n)^2, \tag{15}$$

each summed for four integers $x \geqq 0$. *In other words, if* $t=(2^n-2)/3$ *and* $3A+4=C$, *then* $E_t(A)>0$, $E_{t+1}(A)=0$.

The proof is by induction from $n-2$ to n and is like that for Lemmas 1 and 2 except for the initial value $n=3$. For $n=3$, the squares in (15) must be even, whence $x=2y+1$ in (15_1) and $x=2y$ in (15_2), and the relations become

$$22 \neq \sum(3y-1)^2,\quad 22=\sum(3y-4)^2.$$

These follow since 22 is not a sum by four of 1, 4, 25, $\cdots$, while $22=(-4)^2+2(-1)^2+2^2$.

For $n=1$, $g=6$; for $n=5$, $t=10$. Hence

$$E_i(244)>0 \text{ if } i<7,\quad E_7(244)=0,$$
$$E_j(468)>0 \text{ if } j<11,\quad E_{11}(468)=0.$$

When $k=3$ or 4, the only numbers $A \leqq 1000$ for which $E_k(A) \neq 0$ are 244, 468, 500, 676, 852, 980. This holds also for $k=5$ or 6 except for 676, which is the sum of $e(7)=161$, $e(11)=385$, and the double of $p(5)=65$. For $k=7$, 8, 9, 10, the A's are 468, 852, 980, since 500 is the sum of $e(5)=85$, $e(9)=261$, $p(3)=21$, $p(7)=133$. Next,

$$852=2e(5)+2p(11),\quad 980=e(5)+e(13)+p(3)+p(11),$$

since $p(11)=341$, $e(13)=533$. Hence $E_{11}(B)=0$ if $B \leqq 1000$.

7. THEOREM 5. *If* $m=6$, $k=3$, $A \not\equiv 4 \pmod 8$, *then* $E(A)=0$.

Here $A=3a-16b+84$. If A is odd, $d=6$, $P=28$,

$$U=9\cdot 16(A-39),\quad V=12A+16,\quad W=18A-800,$$
$$F/36=(7A-128)^2-4(A-39)V=(A+8)^2+64\cdot 294.$$

Conditions (7) hold if $A \geqq 88$. The numbers before $16m$ in Table IV include all $\leqq 88$ except

$$28 = 1 + 1 + 5 + 21, \quad 52 = 2\cdot 5 + 2\cdot 21,$$
$$60 = 1 + 5 + 21 + 33, \quad 84 = 4\cdot 21.$$

Next, let $A \equiv 2 \pmod 4$. Take $b = 2B$. Then a is an integer if $B \equiv A \pmod 3$, and $a \equiv 2 \pmod 4$. Then $B^2 < a$ if

$$3B < 16 + v^{1/2}, \quad v = 3A + 4.$$

Lemma 3 in this Bulletin for January (vol. 34, pp. 63–72) applies if

$$4B > 15 + u^{1/2}, \quad u = 4A - 115 > 0, \quad A > 29.$$

The difference between the two limits for B exceeds 3 since $(A - 77)^2 + 6936 > 0$.

Finally, let $A = 8p$. By §1, $24p+4$ is the sum of the squares of four integers $6x-5$ with $x \geqq 0$. Write $y = 2x+1$. Thus $3A+4$ is the sum of the squares of four integers $3y-8$ with $y \geqq 0$. By (11) with $k = 3$, this implies $E(A) = 0$.

8. THEOREM 6*. *For every* $k \geqq 0$, *there exist integers* $A > 0$ *such that* $E_k(A) > 0$ *for the octagonal function* $p_8(x-k)$.

In (11) we replace $x-k$ by $h-x$ and conclude that

$$(16) \quad A = \sum_4 p(x - h), \quad 3A + 4 = \sum_4 (3x - 3h - 1)^2$$

are equivalent equations. Choose an integer $n \geqq 0$ such that $g \geqq \frac{1}{2}k$, for g defined by (12). Then $6q+4 = 4^{2n}\cdot 46$ determines an integer $q > 0$. Take $A = 8q+4$. We shall prove that $E_{2g}(A) > 0$, whence $E_k(A) > 0$. Suppose that $E_{2g}(A) = 0$, so that (16_2) holds for $h = 2g$. But $3A+4 = 24q+16$. Hence the four squares are all even and $x = 2y+1$. Cancellation of 4 gives

$$4^{2n}\cdot 46 = \sum (3y - 3g + 1)^2 = \sum (3y + 3 - 5\cdot 4^n)^2,$$

in contradiction with Lemma 1. We may vary this proof by taking $2g - 1 \geqq k$, or by taking $6q+4$ to be C of Lemma 3 and choosing n so that $2t \geqq k$.

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* This completes the proof of Theorem 5 of the author's paper in this Bulletin for January (vol. 34, pp. 63–72).

THE PROBLEM OF DEPRECIATION IN THE CALCULUS OF VARIATIONS*

BY C. F. ROOS†

1. *Introduction.* In a recent article Hotelling has shown that the older treatments of depreciation involve a number of serious errors of reasoning and has formulated the problem in such a way that many of these errors are overcome.‡ His view-point is that the owner of a machine will do everything in his power to maximize the present value of the sum of the anticipated rentals which the machine will yield from the present time t_1 to some future time t_2 plus the present value of the salvage value of the machine at the time t_2 when it is salvaged. Although the depreciation problem appears to be a Lagrange problem in the calculus of variations,§ Hotelling has chosen to consider it as a problem in the theory of ordinary maxima of functions.

If, in the light of recent developments in the new dynamical economics,|| the depreciation problem is formulated as a Lagrange problem with variable end-points, the resulting problem is sufficiently general to include as special cases all of the existing depreciation theories, i e., such theories as the straight line, the compound interest, the sinking fund, the unit-cost-plus, and the theory due to Hotelling. In

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† National Research Fellow in Mathematics.

‡ H. Hotelling, *A general mathematical theory of depreciation*, Journal of the American Statistical Association, September, 1925.

§ Bliss, *Lectures on the Calculus of Variations*, University of Chicago, Summer, 1925, mimeographed by O. E. Brown, Northwestern University, Evanston, Ill.

|| Roos, *Dynamical economics*, Proceedings of the National Academy, vol. 13, No. 3 (March, 1927); Roos, *A dynamical theory of economics*, Journal of Political Economy, October, 1927. See also, Roos, *A mathematical theory of competition*, American Journal of Mathematics, vol. 57, No. 3, July, 1925, and G. C. Evans, *Dynamics of monopoly*, American Mathematical Monthly, vol. 31 (1924), No. 2.

as much as existing depreciation theories assume a constant rate of production, the importance of a calculus of variations treatment which allows the rate of production to vary, is immediately evident.

2. *The Value of a Machine.* In order to obtain an expression for the operating expense or cost of production for a machine or other property, mathematical economists, following the example set by Walras, define certain quantities called coefficients of production. These coefficients are defined as the quantities of the services of the factors of production, i.e. services of land, services of persons and services of capital, that enter into the manufacture of a unit of a given commodity.* For the static case Walras assumes these quantities to be constant. For the dynamic case an obvious extension would be to suppose the coefficients to be functions of the time, but this is not enough. There seems to be justification for writing the coefficients of production for a commodity C as functions of the rate of production of C, the selling price of C and the first derivatives of these quantities with respect to time.† In this paper we will, therefore, suppose the coefficients of production to be functions of the rate of production of C, the price of C, and the first derivatives of these quantities with respect to the time.

If there is one producer manufacturing an amount $u(t)$ of C in unit time and if, furthermore, the selling price of one unit of C is $p(t)$, the coefficients of production are functions

$$f_\alpha(u, u', p, p', t), \qquad (\alpha = 1, \cdots, m),$$

where m is the number of services required to manufacture one unit of C and primes denote derivatives with respect to time. If we denote the prices of the m services required to produce one unit of C by $p_\alpha(t)$, then by the definition of the coefficients of production the cost of producing $u(t)$ units of C is

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* V. Pareto, *Manuel d'Economie Politique*, pp. 607.

† Roos, *Dynamical economics*, loc. cit.

$$(1)\quad \psi(u,u',p,p',p_1,\cdots,p_m,t) = \sum_{\alpha=1}^{m} u(t)f_\alpha(u,u',p,p',t)p_\alpha(t).$$

This gives the cost of production before depreciation has been taken into account.

The profit or rent which will be obtained from a machine in unit time will, therefore, be

$$R(t) = p(t)u(t) - \psi(u,u',p,p',p_1,\cdots,p_m,t).$$

Now, the value of a machine to its operator at a time t_1 is the sum of the anticipated rentals which it will yield from the time t_1 to a time ω at which it is to be salvaged each multiplied by a discount factor to allow for interest plus the salvage value also discounted. In the most general case the interest will vary with the time. We can, therefore, for the general case write the value of a machine to its operator at a time t_1 as

$$(2)\quad V = \int_{t_1}^{\omega} [pu - \psi(u,u',p,p',p_1,\cdots,p_m,t)]e^{-\int_{t_1}^{t}\delta(\nu)d\nu}dt + Se^{-\int_{t_1}^{\omega}\delta(\nu)d\nu},$$

where $\delta(\nu)$ is the force of interest, which is defined as the rate of increase of an invested sum s divided by s, and S is the salvage value of the machine at the time ω.* This salvage value S is the cost price K of the machine at the time t_1 minus the depreciation in the market value of the machine after it has been operated for the period of time t_1 to ω. The depreciation in the market value of the machine is in general a functional of the rate of production, of the price of the article produced and of the time derivatives of these quantities. We may, therefore, write

$$(3)\quad S = K - \int_{t_1}^{t_2} D(u,u',p,p',t)e^{-\int_{t_2}^{t}\delta(\nu)d\nu}\,dt$$

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* Hotelling, *A general mathematical theory of depreciation*, loc. cit., p. 342.

where $D(u, u', p, p', t)$ is the rate of depreciation. If we substitute the value of S defined by (3) in equation (2), we obtain on transposing the second term of the right-hand member

$$\left[V - Ke^{-\int_{t_1}^{t_2}\delta(\nu)d\nu}\right] = \int_{t_1}^{t_2}[pu - \psi(u,u',p,p',p_1, \cdots ,p_m,t) - D(u,u',p,p',t)]e^{-\int_{t_1}^{t}\delta(\nu)d\nu}dt.$$

The quantity represented by the second term of the left hand member is the value at t_1 of a sum K necessary to replace the machine at the time t_2. In order to simplify notation in the work which follows let us write this last expression in the form

$$(4)\quad I = \int_{t_1}^{t_2}[pu - Q(u,u',p,p',p_1, \cdots ,p_m,t)]e^{-\int_{t_1}^{t}\delta(\nu)d\nu}dt,$$

where

$$Q(u, u', p, p', p_1, \cdots , p_m, t) = \psi(u, u', p, p', p_1, \cdots , p_m, t) + D(u,u', p, p', t).$$

3. *Equations of Demand and Supply.* For the commodity produced by the machine there will be a function ξ defining the rate of demand. As I have already pointed out, this rate of demand in its most general form will be a functional of the type

$$\xi = g(u,u',p,p',t) + \int_{t_1}^{t}H(u,u',p,p',t,s)ds$$

where as before primes indicate derivatives with respect to time and s is a parameter of integration.*

If as many units are sold as are produced, the demand per unit time will be $u(t)$ so that the demand equation will be

$$(5)\qquad \phi(u,u',p,p',t) = \int_{t_1}^{t}H(u,u',p,p',t,s)ds$$

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* Roos, *A dynamical theory of economics*, loc. cit.

where as notation $\phi(u, u', p, p', t) = u - g(u, u', p, p', t)$. When the rate of demand depends only slightly upon the history of prices and rates of production, we are justified in writing $H \equiv 0$. For this case we obtain a first order differential equation of demand

$$\phi(u, u', p, p', t) = 0 \qquad (t_1 \leqq t \leqq t_2). \tag{6}$$

In as much as this last form of the demand equation is sufficiently general to include as special cases all forms of the demand equation now used in statistical investigations, it will be sufficiently general for the purposes of this paper.

4. *Necessary Conditions for a Solution.* A likely assumption to make regarding the operation of the machine is that the operator will endeavor to maximize the expression (4), which is the difference between the value of the machine at the time t_1 and the discounted cost price. This assumption is not equivalent to the assumption that the operator will endeavour to maximize V, for t_2 is a variable end-value, but it has the advantage of simplicity.* We will find it convenient to introduce the notation $y_1(t) = u(t)$ and $y_2(t) = p(t)$ in this paragraph in stating the problem of depreciation and in writing the conditions necessary for a solution. Under the hypothesis that (4) is to be maximized, we may state the problem of depreciation as that of choosing the rate of production $y_1(t)$ and the price $y_2(t)$ satisfying a differential equation of demand

$$\phi(y_1, y_1', y_2, y_2', t) = 0 \qquad (t_1 \leqq t \leqq t_2),$$

and having end-points which satisfy end-equations

$$\theta_\mu[t_1, y_1(t_1), y_2(t_1), t_2, y_1(t_2), y_2(t_2)] = 0, \quad (\mu = 1, \cdots, n \leqq 6),$$

so that they maximize the integral

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* Hotelling assumes that V is to be maximized, but assumes $u(t)$ and $p(t)$ to be known functions; see *A general mathematical theory of depreciation*, loc. cit., p. 343. In a paper entitled *A general problem of maximizing an integral with discontinuous integrand* to be offered to the Transactions of this Society, I have given the mathematical analysis necessary to solve the problem under the assumption that V is to be maximized.

$$I = \int_{t_1}^{t_2} [y_1 y_2 - Q(y_1, y_1', y_2, y_2', p_1, \cdots, p_m, t)] e^{-\int_{t_1}^{t} \delta(v) dv} dt,$$

with $p_1, \cdots, p_m$ considered as known functions of t.

In order to obtain a solution we assume further that

(1) the functions $y_i(t)$, $(i=1, 2)$, defining the maximizing arc E_{12} are continuous on the interval $t_1 t_2$, and this interval can be sub-divided into a finite number of parts on each of which the functions have continuous derivatives,

(2) in a neighborhood R of the values t, y_1, y_1', y_2, y_2' on the maximizing arc E_{12} the functions Q and ϕ have continuous derivatives up to and including those of the second order, and

(3) the functions θ_μ have continuous derivatives up to and including those of the second order near the end-values $(t_1, y_1(t_1), y_2(t_1), t_2, y_1(t_2), y_2(t_2))$, and at these values the $6n$-dimensional matrix

$$\| \ \theta_{\mu t_1} \quad \theta_{\mu y_i(t_1)} \quad \theta_{\mu t_2} \quad \theta_{\mu y_i(t_2)} \ \| \text{ has rank } n.$$

Under these hypotheses, the analysis for the problem of Lagrange with variable end-values as arguments of the integrand applies to the problem of depreciation. I have shown in another paper* that the solution must be such that

$$\text{(a)} \quad F_{y_i'} = \int_{t_1}^{t} F_{y_i} dt + C_i, \qquad (i = 1, 2), \text{ along the arc } E_{12},$$

where

$$F(y_1, \cdots, y_2', t, t_1) = \lambda_0 f(y_1, \cdots, y_2', t, t_1)$$
$$+ \lambda_1(t) \phi(y_1, \cdots, y_2', t),$$

where $f(y_1, y_1', y_2, y_2', t, t_1)$ is the integrand of I, where the C_i are arbitrary, and where the λ are Lagrange multipliers;

(b) At the end-points all determinants of order $n+1$ of the matrix

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* Roos, *A general problem of minimizing an integral with discontinuous integrand*, to be offered to the Transactions of this Society.

$$\left\| \begin{matrix} H(t_1) - y_{k'}(t_1) H_{y_{k'}}(t_1) & H_{y_\iota'}(t_1) & H(t_2) - y_{k'}(t_2) H_{y_{k'}}(t_2) & H_{y_\iota'}(t_2) \\ \theta_{\mu t_1} & \theta_{\mu y_\iota(t_1)} & \theta_{\mu t_2} & \theta_{\mu y_\iota(t_2)} \end{matrix} \right\|$$

vanish, where as notation

$$H(t_\sigma) = (\sigma - 2)F(t_1) + (\sigma - 1)F(t_2)$$

$$+ \int_{t_1}^{t_2} [F_{t_\sigma} + F_{y_k(t_\sigma)} y_{k'}(t_\sigma)] dx \qquad (\sigma = 1, 2),$$

$$H_{y_\iota'}(t_\sigma) = (\sigma - 2)F_{y_\iota'}(t_1) + (\sigma - 1)F_{y_\iota'}(t_2) + \int_{t_1}^{t_2} F_{y_\iota(t_\sigma)} dt,$$

k is used as an umbral index with range 1, 2, and subscripts y_ι, y_ι', and t_σ denote differentiation.

(c) In addition to the preceding conditions (a) and (b) certain second order conditions of the calculus of variations must be satisfied by an arc E_{12} which does actually maximize I. It is true, however, that if an arc E_{12}, which maximizes I does actually exist, it is the one defined by the differential equations $\phi(y_1, y_1', y_2, y_2', t) = 0$ and the Euler-Lagrange equations in the Du Bois-Reymond form (a) and having end-points satisfying the end-equations $\theta_\mu(t_1, y_1(t_1), y_2(t_1), t_2, y_1(t_2), y_2(t_2)) = 0$, $(\mu = 1, \cdots, n)$, and the transversality conditions (b). Rather than consider the additional conditions of the calculus of variations, let us assume that there does actually exist a maximizing arc E_{12} for the particular problem under consideration and that t, $u(t)$, $p(t)$ in the space t, u, p give the desired solution.

We desire next to obtain an expression for depreciation, which has been defined as the rate of decrease of value.* To do this let us suppose that a machine is operated at a rate $u(t)$ which maximizes I, so that in the expression (4) the functions $u(t)$ and $p(t)$ and the end-values t_1 and t_2 are those defining the maximizing arc. Then the expression (4) will define the maximum value of the machine at the time t_1, and the value of the machine at any other time t $(t_1 \leqq t \leqq t_2)$ can be obtained by replacing t_1 in (4) by t. We obtain

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* H. Hotelling, *A general mathematical theory of depreciation*, loc. cit.

$$V(t) = \int_t^{t_2} [pu - Q(u,u',p,p',\tau)] e^{-\int_t^{\tau} \delta(\nu)d\nu} \, d\tau + K e^{-\int_t^{t_2} \delta(\nu)d\nu} . \tag{7}$$

If we let $\Delta(t) = -dV(t)/dt$ be the depreciation of the machine to its operator, we can write by differentiating (7) with respect to t:

$$\Delta(t) = - dI(t)/dt - \delta(t) K e^{-\int_t^{t_2} \delta(\nu)d\nu} . \tag{8}$$

5. *Relation to Existing Depreciation Theories.* It is most interesting and instructive to see what further hypotheses must be made to obtain the various depreciation theories now commonly used. If, in equation (2), we suppose the salvage value S to be a point function of the time $t_2 = n$ at which the machine is to be salvaged instead of a functional of the rate of production and price as done in this paper, then on changing the parameter of integration from t to τ equation (2) becomes for $t_2 = n$ and $t_1 = t$

$$V(t) = \int_t^{n} [pu - \psi(u,u',p,p',\tau)] e^{-\int_t^{\tau} (\delta\nu)d\nu} \, d\tau + S(n) e^{-\int_t^{n} \delta(\nu)d\nu} . \tag{9}$$

If the cost of production function $\psi(u, u', p, p', \tau)$ is a point function $\psi(\tau)$ instead of the more general function ψ of (9), this formula reduces to that given by Hotelling.[*]

Now, by (9), $V(n) = S(n)$, and hence the total depreciation of the machine for the period $\tau = t$ to $\tau = n$ is evidently $V(t) - S(n)$. If we desire only simplicity, we may assume that the depreciation per unit time is equal to the average depreciation $[V(t) - S(n)]/n$ and obtain the well known straight line formula for depreciation.

Again, if p, u, and ψ of (9) are constant for a certain number of years of the machine's life and then change abruptly at $t_2 = n$ in such a way that it is evident that the end of the useful life of the machine has come, and if the force of interest $\delta(t)$ is a constant equal to δ, we can perform the integration in (9) and write

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[*] See citation (1), p. 343.

$$V(t) = [pu - \psi]\frac{1 - e^{-\delta(n-t)}}{\delta} + Se^{-\delta(n-t)}. \tag{10}$$

Now, when $\delta(t) = \delta$, a constant, the discount factor $e^{-\delta}$ is equal to $(1+i)^{-1}$, where i is the rate of interest in the ordinary sense. We obtain, therefore,

$$V(t) = [pu - \psi]\frac{1 - (1 + i)^{-(n-t)}}{\delta} + S(1 + i)^{-(n-t)}, \tag{11}$$

and, at $t = 0$, this becomes

$$V(0) = [pu - \psi]\frac{1 - (1 + i)^{-n}}{\delta} + S(1 + i)^{-n}. \tag{12}$$

If we eliminate $[pu - \psi]/\delta$ from the equations (11) and (12), we obtain

$$\begin{vmatrix} V(t) - S(1 + i)^{t-n} & 1 - (1 + i)^{t-n} \\ V(0) - S(1 + i)^{-n} & 1 - (1 + i)^{-n} \end{vmatrix} = 0.$$

By adding $-S$ times the second column to the first column, then subtracting the second row from the first row and expanding, we obtain at once

$$V(0) - V(t) = [V(0) - S(n)]\frac{(1 + i)^{t} - 1}{(1 + i)^{n} - 1}.$$

When we introduce the customary notation

$$s_{\overline{t|}} = \frac{(1 + i)^{t} - 1}{i},$$

this formula becomes

$$V(0) - V(t) = [V(0) - S(n)]\frac{s_{\overline{t|}}}{s_{\overline{n|}}}, \tag{13}$$

which is the well known formula for the accumulation of depreciation allowances at the end of the tth year under the sinking-fund method.*

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* Rietz, Crathorne and Rietz, *Mathematics of Finance*, pp. 112–121.

If now the formula for $V(t+1)$ obtained from (13) by replacing t by $t+1$ be subtracted from (13), we obtain

$$V(t) - V(t+1) = [V(0) - S(n)]\frac{(1+i)^t}{s_{\overline{n}|}}. \tag{14}$$

This formula states that the depreciation for any year is equal to the depreciation charge for the first year at compound interest at the rate of i per annum. This is, therefore, the so-called "compound interest method" of providing for depreciation.

If we perform the indicated differentiation in (8), we obtain on solving for p

$$p = \frac{Q + \delta K e^{-\delta(n-t)} - dV(t)/dt}{u}, \tag{15}$$

which is equivalent to the formula by which J. S. Taylor defines *unit cost plus*.*

From the general theory of this paper we have thus derived the popular "sinking fund" or "equal annual payment" formula for depreciation (13) and the "compound interest formula" (14) by assuming that the rate of production u, the price p and the cost of production ψ are constant throughout the useful life of the machine. We have also seen that the unit cost plus formula can be obtained directly from the formula for depreciation as given in this paper when the price p and the rate of production u are known functions of the time.

6. *Some Generalizations.* If instead of the hypothesis of §3, we adopt Hotelling's hypothesis that the operator of a machine will do everything in his power to maximize the present value $V(t_1)$ of his machine, the problem of depreciation for a fixed t_1 is a Mayer problem of a general type considered by Bliss.† When t_1 is variable this prob-

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* J. S. Taylor, *A statistical theory of depreciation based on unit cost*, Journal of the American Statistical Association, December, 1923.

† Bliss, *The problem of Mayer with variable end points*, Transactions of this Society, vol. 19 (1918).

lem can be made over into the Bliss problem by introducing a new variable z satisfying the differential equation $z = 0$ and the end condition $z(t_1) = t_1$. It is not, however, necessary to complicate matters by introducing another variable to obtain the form given by Bliss, for, if we take the term in K in (4) under the integral sign, we obtain

$$V(t_1) = \int_{t_1}^{t_2} \left[(pu - Q) e^{-\int_t^\tau \delta(\nu)\delta\nu} + \frac{K e^{-\int_{t_1}^{t_2} \delta(\nu) d\nu}}{t_2 - t_1} \right] d\tau,$$

and this is a special case of the general problem referred to in §3.* For the conditions (a) and (b) of §3 the function $f(u, u', p, p', t, t_1)$ of that section becomes a function $f(u, u', p, p', t, t_1, t_2)$ identically equal to the above integrand.

I have already discussed a related problem for the case of several competing machines and I believe that the analysis given there can be remodeled after the methods of this paper to fit the depreciation problem for several competing machines.†

There is another important possible direction of extension of this paper. As I have pointed out in §2, there are instances when the more general functional equation of demand as given by (5) should replace the differential equation of demand (6). The resulting problem is a problem in the maxima of functionals which would be well worth special investigation.‡

THE UNIVERSITY OF CHICAGO

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* Roos, *General problem of minimizing an integral with discontinuous integrand*, loc. cit. See also E. H. Clarke, *On the minimum of the sum of a definite integral and a function of a point*, Doctoral Dissertation, University of Chicago, and Miss Sinclair, Annals of Mathematics, (2), vol. 10, pp. 55–80.

† Roos, *Generalized Lagrange problems in the calculus of variations*, Transactions of this Society, not yet published.

‡ For a related problem in the maxima of functionals, see Hahn, *Über die Lagrange'sche Multiplicatorenmethode*, Sitzungsberichte der Akademie Wien, vol. 131 (1922), pp. 531–550.

MOULTON ON EXTERIOR BALLISTICS

New Methods in Exterior Ballistics. By F. R. Moulton. Chicago, University Press, 1926. vi+257 pp.

This book is the outgrowth of Professor Moulton's work for the Ordnance Department of the U. S. Army during the World War and, to some extent at least, the result of his teaching the subject since that time.

The importance of the book is not questioned and an effort will be made properly to evaluate it in this review, with the definite understanding that statements made are merely opinions. But I wish to point out two aspects of work on ballistics in this country, pertaining to this book and its contents, that are somewhat disappointing.

Toward the close of the World War, and even during the period of hostilities, every person who made any pretense of being interested in contributions to mathematics longed to secure a book on ballistics from which he could compute a trajectory; not by the old antiquated methods which, he had heard, had been entirely discarded, but by the methods that had been devised by American mathematicians who had patriotically devoted their talents to the solution of these important problems. Except for a very few who were in one way or another officially connected with the Army, those who desired such a book at that time or even now have been doomed to disappointment. After two books have been published on the subject, one by R. S. (then Captain) Hoar in 1922, and the book under review by Professor Moulton, the student and the professor of mathematics have at their disposal no book from which a trajectory can be actually computed. Both books are devoid of tables, and without tables, a trajectory can no more be computed by the methods they have devised than a spherical triangle could be solved without the proper tables. From the standpoint of interesting persons in the subject of ballistics this is to be regretted. Such tables are regarded as confidential information by the U. S. Army. Regarding the propriety of this attitude the reviewer has no comment to make and no opinion to express. But it is proper, I think, to warn the reader that this withholding of the tables from not only public examination (to which there may be objections) but from the perusal and use of mathematicians, physicists, and engineers might give to such tables an importance that the facts do not justify.

The second disappointing feature, in the opinion of the reviewer, is of a more serious character. There has been a failure on the part of some properly to appraise the work done in this country during the War by our mathematical men who were interested in ballistics. The contributions that they made to the subject were advertised in most complimentary terms at meetings of the national mathematical organizations and elsewhere. No one was disposed to question them; no one was in a position to question them. It came to be usual to say certain things about these researches in ballistics. The failure adequately and fairly to appreciate the exact worth of this work is embodied in the following quotation from another recent review of Professor Moulton's book: "The fundamental differential equations of motion of a projectile in flight have been known since the days of Euler. Up to the World War the progress in ballistics had consisted in devising approximate algebraic expressions whereby to wrench these equations into soluble form." "These approximations had been improved and improved, but still could not keep pace with the development of modern artillery. So finally Professor F. R. Moulton of the University of Chicago, while serving as a Major in the U. S. Army during the World War, cut the Gŏrdian knot by going back to the Eulerian equations and solving them in their original and exact form by numerical integration. Thus he laid the cornerstone for an entirely new science of ballistics." Moreover, the same reviewer in his following paragraph compares Professor Moulton's scientific achievement to that of Morse and Bell, and states that he should be given equal credit.

There is little doubt that the above quotation is an expression of confirmed opinion on the part of its author. Also it must be said that it embodies the attitude that has become all too prevalent regarding this work and which has not been corrected. Since I have said that this has not been corrected, let me give what I consider a more accurate appraisal of the same work: it is understood that this is my opinion and nothing more. (1) Solutions of these equations by numerical integration are just as truly approximations as solutions obtained in different ways by such men as Cranz, Richmond, Vahlen, Dufrenois, and others, all of whose methods were either developed or recast from other previously devised methods during the same period. Research in ballistics was not confined to this country. Many problems were investigated and at least partially solved abroad before they were undertaken on this side of the water.

(2) Numerical integration like any other method of successive approximation, if it is to become accurate, must be corrected from time to time. For instance, the square root or any rational root of an integer may be obtained by successive approximations because the number itself serves as a check upon the error made. But unless there is some means by which the errors of approximation are corrected, there is no guarantee that accuracy will be obtained, no matter how carefully the work may be done. This is exactly the case in the case of integrating the differential equations of the motion of a projectile. The only check is a tabulated function G which is built upon observed data. Although these data are the most reliable data obtainable, they are subject to errors of various kinds. The particular table referred to, according to all the evidence that can be obtained (and it is not easy to obtain) is built upon Gavre firings, firings at Aberdeen, upon interpolation, and upon some extrapolation. The point to be made here is that a trajectory computed upon this table as a basis could be no more accurate than the table itself. And even more important than this, if one had more reliable tabulated data, he might be able to obtain the results desired without so much trouble.

(3) It is admitted by all that the differential equations of the motion of a projectile in flight are insoluble by purely mathematical methods. The

best that can be done is to approximate such solutions. Thus the problem, from the standpoint of the practical ballistician at least, is one of engineering mathematics. If these premises are correct, then what is to be gained by making the mathematical method of approximation more difficult than is neccessary? If the reviewer has acquired the point of view of other ballisticians, such as Cranz, Dufrenois, and Vahlen, it is briefly summarized in this: Let us aim to get the physical data or tabulated data as accurately as possible, and the method of approximation as simple as possible. Surely this is the practical point of view, and it must be admitted to be preeminently the case when one takes into account the kind of educational training usually possessed by army officers of any nation. Their training must be broad and practical in the extreme, and they want mathematics presented in as simple and usable form as possible.

(4) The argument often presented to show the great advantage of methods of numerical integration in solving problems in ballistics over any other method of approximation is something like this: Other methods of approximation are practical up to a certain point but by methods of numerical integration results can be made as accurate as we please (by making intervals sufficiently small). This is far from correct. This statement involves several assumptions of a fundamental character. The method of numerical integration as outlined and perfected by Professor Moulton and his associates can be used to compute trajectories *as accurately as we please* only upon the following conditions:

(a) That the G table is absolutely correct.

(b) That the H table is absolutely correct.

(c) That the characteristic motion of a projectile is completely defined by its ballistic coefficient.

None of these is absolutely correct. The H table is highly satisfactory; the G table, although probably the best available, is subject to large errors; the ballistic coefficient only approximately characterizes the motion of a projectile.

In Chapter I the problem is outlined and the translatory motion of the projectile is described; the rotation of the projectile is purposely omitted from this discussion for the sake of simplicity and because rotation is the subject that lies at the basis of the last, and, in my opinion, the most important chapter of the book. It is particularly important to point out the kinds of things that might effect the motion of a projectile and especially the degree of accuracy that is needed. This is admirably done in Chapter II. The method of numerical integration is given in Chapter III, presented as clearly and as attractively as it could be done without the use of tables. The subject of differential variation as presented in Chapter IV will be interesting to the mathematician, but is not elementary, and in my opinion, is not given in a sufficiently elementary and practical manner. The adjoint system of differential equations is introduced as if it were such a familiar portion of mathematics that this could be done without explanation. Chapter V is theoretical in the extreme. It is devoted to a theoretical substantiation of the validity of the process of numerical integration, based upon the work of Picard. This is most attractively presented from the standpoint of logical sequence, but, as Professor Moulton suggests, this chapter should be omitted by those who are reading the book for practical purposes only.

In Chapter V, which occupies eighty-five of the two hundred fifty-seven pages of the book, Professor Moulton has set forth the mathematical basis upon which modern advanced ballistics reposes. This study of the rotating projectile is of fundamental importance in ballistics and especially in projectile design. Projectiles are constructed in such a way that both heads and tails may be varied by screwing such portions of different designs upon the body of the projectile. Moreover by screwing a cylinder to different positions within the body of the projectile, its moments of inertia may be varied. These projectiles are then fired through cardboard or beaverboard and their periods of precession, yaw, etc. may be studied. Although this has been added to since the War, it is the best presentation of the subject in our language, or in any language, that I have seen. The study of projectile design, and in general the problems of the rotating projectile, may be studied through the mathematics of this chapter as a tool, provided that experimentation can be conducted as it should be conducted. In this sense Professor Moulton has put the subject of ballistics upon such a basis that no essential change will have to be made in the mathematical method of attacking such problems for some time to come.

There have been three outstanding products of the work of our ballisticians and mathematicians during the War. The translation of the A. L. V. F. Tables; the book published by R. S. Hoar in 1922; and the book under review by Professor Moulton. In my opinion the most practical product is the translation of the A. L. V. F. Tables; they are just what the ballistician wants and needs. I doubt seriously that we could construct better or more useful tables. Placing ballistic research upon a sound mathematical basis is also a matter of great importance. Credit for this must be given to Professor Moulton and his associates. Also, they must have credit for giving an impetus to mathematical research in the Army and Navy that has had excellent results. This book records the mathematical methods of attacking problems in modern ballistics as developed in this country during the period of the World War. The same problems have been attacked and solved in slightly different ways by ballisticians in other countries during the same period. Greater accuracy in such work from the practical standpoint will depend upon more accurate physical data rather than upon new methods of mathematical attack. Although Professor Moulton's book is by far the best book on modern ballistics published in this country, it is for the use of the mathematician rather than for the practical ballistician. In the reviewer's opinion, it is not Professor Moulton's outstanding contribution to mathematical science. We still need a practical book on ballistics, provided with adequate tables, for the use of the officer, the teacher, and the ballistician.

J. E. Rowe

PÓLYA AND SZEGÖ'S PROBLEMS IN ANALYSIS

Aufgaben und Lehrsätze aus der Analysis. By G. Pólya and G. Szegö. Berlin, Julius Springer, 1925. vol. 1, xvi+338 pp; vol. 2, x+407.

There are but few books which could be compared with this one as to the richness and charm of material, and amount of suggestions which an attentive reader is able to get out of it. The purpose of the book, and hence the material and its treatment, is quite different from those of numerous collections of problems already existing. The mere development of technique is of secondary importance with the authors: their main efforts are directed rather toward cultivating good habits in the mathematical thinking of their readers. The reader is constantly urged to give his attention not only to *what* he is being questioned about, but also to *how* and *where* he is questioned. Accordingly the most essential feature of the book, and one to which the authors gave much care, is the relative order of the problems. Isolated problems and examples comprise but a small part of the book: usually the reader has to deal with sets of problems, each of which is devoted to an independent and more or less substantial notion or question.

The material treated in the book (see the detailed list of contents below), is taken from the modern parts of the classical theory of functions. This choice seems to be the wisest, because it not only agrees with the personal taste of the authors, but it also is fitted for the purpose better than any other part of analysis. it gives an adequate idea of the modern development of the science without being so abstract as to scare a beginner, even an advanced one. As sources, the authors use to a great extent recent memoirs. Many a problem has been communicated to them by different mathematicians as well as those personally found, and appears in print for the first time. There is no doubt, therefore, that the book may be of great value to specialists as well as to beginners.

The first volume is devoted to the fundamental notions, while the second volume treats of questions of a more specialized type.

Volume I contains three sections:

Sec. I (Chapters 1–4) Infinite Series and Sequences.
Sec. II (Chapters 1–5) Integral Calculus.
Sec. III (Chapters 1–6) Generalities on Functions of a Complex Variable.

Of these. Chapters I, 3 (the Structure of Sequences and Series of Real Terms); II, 1 (The Definite Integral as a Limit of Sums of Rectangles); II, 3 (Inequalities); II, 4 (Different Kinds of Distribution, i. e. Multiples of an irrational number, distribution of digits in a logarithmic table, etc.); II, 5 (Functions of Large Numbers); III, 3 (Geometrisches über den Funktionsverlauf); III, 4 (Cauchy's Integral and the Principle of the Argument); III, 6 (The Principle of the Maximum) are of extraordinary interest and value.

Volume II contains six parts:
Sec. IV (Chapters 1–3) Functions of a Complex Variable, special part.
Sec. V (Chapters 1–3) Position of the Zeros.
Sec. VI Polynomials and Trigonometric Polynomials.
Sec. VII Determinants and Quadratic Forms.
Sec. VIII (Chapters 1–5) Theory of Numbers.
Sec. IX Geometric Problems.

This volume is still more interesting although harder than the first.

The readers will find considerable help in the solutions which are separated from the problems and collected in the second part of each volume. The solutions are given in a very condensed form, and require very careful reading. They are often followed by suggestions of possible generalizations and contain bibliographical references.

The total number of problems in the first volume is 747, and in the second volume 877.

It is the authors' teaching experience that each chapter can be worked through in an advanced class in one semester (2 hours per week); this certainly requires that the students be excellently prepared.

Both volumes are beautifully printed and like most of the monographs of this series by J. Springer (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen) seem to contain almost no misprints.

Of course nobody would expect that a book so rich in content and original ideas would be entirely free from a few minor defects. Let us indicate some of them which occurred to us at the first reading.

Problems I, 75, 76 should be placed before I, 72–74. Problem I, 113 should be placed after I, 139. The person who does not know the method of solving Problem II, 202, could hardly work out the solution of II, 58.

Solutions of some problems could be simpler and more natural (this point is very important, because we are interested not only in the facts as such, but also in the methods of obtaining these facts): this is the case with Problems I, 73–74, 82, 174.

Some references are not quite exact, for instance, the solution of Problem II, 199 is ascribed to P. Czillag, while actually it was published (in more general form) by C. N. Haskins in his well known paper *On the measurable bounds and the distribution of functional values of summable functions*, Transactions of this Society, vol. 17 (1916), p. 181–194.

Some topics, in view of their importance and relative simplicity, should be presented in greater detail than has been done in the *Aufgaben*. Such topics are, in our opinion: dominant series; summability of divergent series.

To conclude our review, let us modify one of the metaphors used by the authors in the Introduction: We feel as though we were in a large and prosperous city; our guides are successful in their intention to show us how to travel from any place to any other place in the shortest and surest way; their wide knowledge and fine taste enable them, meanwhile, to point out the places of unexpected and unusual beauty, and we are thankful for this.

J. D. TAMARKIN

RIETZ ON STATISTICS

Mathematical Statistics. By Henry Lewis Rietz. Chicago, Open Court, 1927. vii+181 pages.

This is Number Three of the Carus Mathematical Monographs of the Mathematical Association of America.* As such, it is designed to give an exposition "comprehensible not only to teachers and students specializing in mathematics, but also to scientific workers in other fields." The author defines his main purpose as "shifting the emphasis and point of view in the study of statistics in the direction of the consideration of the underlying theory" and introducing "some of the recent advances in mathematical statistics to a wider range of readers." It must readily appear to the careful reader of the book that Professor Rietz has handsomely achieved his own stated purpose as well as the general purpose of the Carus series. The book does render available, as does no other volume written in English which is known to the reviewer, the essentials of an introductory survey of the underlying mathematical theory which must receive increasing attention from specialists in statistics, if the widespread use of statistical method is to be systematically helpful. Moreover, Professor Rietz has developed this theory so skillfully that the "workers in other fields," provided only that they have a passing familiarity with the grammar of mathematics, can secure a satisfactory understanding of the points involved.

As a general basis for developing the concepts of statistics, the author prefers to regard probability as the limit (if it exists) of the relative frequency, and it is likely that most practicing statisticians will agree with him.† Quite regardless of the considerable helpfulness of the concept of *a priori* probability, there can be little doubt that the notion of probability generally arises in practical problems of statistics as an adjunct in describing—one almost says "explaining"—observed frequencies. The idea of *description* is again prominent in the author's remark concerning the utility of fitted theoretical frequency curves: "Indeed, the use of the theoretical curve is likely to be justified in a large way only when it facilitates the study of the properties of the class of distributions of which the given one is a random sample by enabling us to make use of the properties of a mathematical function $F(x)$ in establishing certain theoretical norms for the description of a class of actual distributions."

The author's definition of probability appears constantly in the background, particularly in Chapter II, entitled *Relative frequencies in simple*

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* The first and second volumes of the series are: *Calculus of Variations*, by G. A. Bliss, 1925; and *Analytic Functions of a Complex Variable*, by D. R. Curtiss, 1926.

† His definition, to which he frankly says there are "some objections," is: "If the relative frequency of success approaches a limit when the trial is repeated indefinitely under the same set of circumstances, this limit is called the probability of success in one trial."

sampling, where he develops the probability integral and the Poisson exponential function. He says, for example: "The theorem of Bernoulli deals with the fundamental problem of the approach of the relative frequency m/s of success in s trials to the underlying constant probability p as s increases"; and again: "The De Moivre-Laplace theorem deals with the probability that the number of successes m in a set of s trials will fall within a certain conveniently assigned discrepancy d from the mathematical expectation sp." In connection with this latter citation, it may be remarked that the author seeks to make use of recent research in the history (as well as theory) of statistics. Thus, he names the theorem implying the "normal error curve" jointly for De Moivre and Laplace, instead of Gauss. The difficulties of substituting for the name of Gauss in the present extensive literature will be considerable, and not the least of these will be that of determining a name for the "Gaussian" curve. The word "normal" has such misleading implications that the substitution of the name "normal frequency function" does not seem entirely fitting.

Chapter III, *Frequency functions of one variable*, gives an exposition of the so-called Gaussian curve and of the Pearson and Gram-Charlier systems of curves. It is here that we find with especial frequency those strikingly helpful clarifying statements which feature all parts of the book. Thus, in referring to the "Gaussian" curve: "The difficulty is not one of deriving this function but rather one of establishing a high degree of probability that the hypotheses underlying the derivation are realized in relation to practical problems of statistics." Again: "The idea of obtaining a suitable basis for frequency curves in the probabilities given by terms of a hypergeometric series is the main principle which supports the Pearson curves as probability or frequency curves, rather than as mere graduation curves." And: "That is, the excess E is equal to the coefficient by which to multiply the ordinate at the centroid of the normal curve to get the increment to this ordinate as calculated by retaining the terms in $\phi^{(3)}(x)$ and $\phi^{(4)}(x)$ of the Type A series."

Correlation is the subject of Chapter IV, and the author presents the development both by use of the regression concept and by reference to the frequency surface.* In spite of the brevity of treatment, the topic of ordinary correlation is covered effectively; and the same may be said of multiple correlation.

The treatment of partial correlation deserves particular comment, especially in reference to the matter of definition. Professor Rietz suggests the possibility of regarding the partial correlation coefficient "as a sort of average value of the correlation coefficients of x_1 and x_2 in subdivisions of a population" in each of which the other variables have assigned values, or "as the correlation coefficient between the deviations of x_1 and x_2 from the corresponding predicted values given by their linear regression equations" on the other variables. Both of these notions are helpful, but the first is especially useful in pointing out the nature of the error involved in the frequent definition of the partial correlation coefficient as that between two variables "when the other variables are held constant." It can not too strongly be emphasized that *no* variable is held *constant* in the partial correlation measurement. This process of measurement seeks merely to eliminate from the indicated correlation between two variables the part which arises from their mutual linear dependence upon certain other specified variables. The development in the book is especially happy in that it shows the only way in which constant or "assigned" values enter the concept of partial correlation.*

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* A typographical error appears in the section heading on page 78, and is cited because its prominent position may render it especially misleading. "Regressive" should read "regression."

The remaining chapters, on *Random sampling fluctuations*, *The Lexis theory*, and *The development of the Gram-Charlier series*, are of somewhat less general interest than the earlier portions of the book. They are, however, important to the general reader; and the exposition of the probable error (standard error, rather than probable, is considered in the text) derivations and their significance is especially worthwhile.

The reviewer has not sought to verify in detail the symbolic portions of the text; but, if there be any mistakes, they are not such as to impede careful reading and full understanding of the discussion.

A difficult task has been handled by the author with admirable skill; and statisticians generally will be grateful for a volume which is available for courses and reference, for mathematicians and "other workers."

W. L. CRUM

PRINCIPIA: VOLUMES II AND III

Principia Mathematica. By Alfred North Whitehead and Bertrand Russell. Volumes II and III. Second Edition. Cambridge, University Press, 1927. xxxi+742 pp., and viii+491 pp.

The second and third volumes of the second edition of Whitehead and Russell's *Principia Mathematica* do not differ from the corresponding volumes of the first edition. An account of the changes which the authors think desirable is contained in the introduction and appendices to the first volume of the second edition, of which a review has previously appeared,† but the text of all three volumes has been left unchanged.

The second and third volumes of the *Principia Mathematica* are devoted to building up, on the basis of the system of logic developed in the first volume, the theories of cardinal numbers, relations and relation-numbers, series, well-ordered series and ordinal numbers, and finally of the continuum and of real numbers. The task of developing these theories on the basis of the theorems and processes of logic only, as well as that, undertaken in the first volume, of investigating logic itself by mathematical methods, are both of the highest importance to our understanding of the foundations of mathematics. The magnitude of the undertaking thoroughly justifies the formidable appearance of the three volumes, which contain what is, so far, the most nearly successful attempt to accomplish these tasks. In spite of serious difficulties which remain unsolved, the work has established its claim to be ranked as a notable achievement.

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* In this connection, the reviewer regrets the parenthetical use (p. 101) of the words "held constant," although he is sure that no careful reader will be misled by them after going through the preceding discussion in the text.

† B. A. Bernstein, this Bulletin, vol. 32 (1926), pp. 711–713.

Volume II begins with an account of cardinal numbers, based on the definition, given originally by Frege, that the cardinal number of a class α is the class of all classes similar to α. This definition has a pragmatic justification, in that it leads to cardinal numbers which have the properties we require. But it seems worth while to notice the possibility of another definition* more in accord with our intuitive idea of number, namely that the cardinal number of a class α is the abstraction from α with respect to the propositional function *x is similar to y*. In order to make this definition, we require the following postulate†: If ϕ is any propositional function of two variables which is transitive and symmetric, and if A is any term such that $\phi(A,x)$ is true for some x, then A_ϕ, the abstraction from A with respect to ϕ, exists and has the property that $A_\phi = B_\phi$ if and only if $\phi(A,B)$ holds. The obvious objection is that this is an unnecessary additional postulate which ought to be avoided, but this objection, we believe, can be removed by a consideration of the treatment of classes in the first volume of the *Principia Mathematica*.

According to this treatment the existence of classes is not assumed and the word class itself is not defined, but a determinate meaning is given to propositions about classes. On this basis, classes appear as shadowy things without actual existence. In a certain sense this is the correct view, for classes are, essentially, not a part of reality but fictions devised for their usefulness as instruments in understanding reality. But the same statement is true of propositions and propositional functions and, in fact, all the terms of logic and of mathematics. A postulate asserting the existence of a logical or mathematical term ought, indeed, to be understood merely as making this term, by fiat, a part of a certain abstract structure which is being built. The treatment of classes given by Whitehead and Russell seems not well balanced in that it ascribes a greater degree of reality to propositions and propositional functions than to classes. A postulate asserting the existence of classes probably ought to be included in their treatment.

If, however, we accept the postulate about abstractions proposed above, it is not necessary to postulate the existence of classes, because they can be introduced as follows. A class is determined by a propositional function, but classes differ from propositional functions in that two equivalent but distinct propositional functions determine the same class. By analogy with Whitehead and Russell's definition of cardinal number, we would, except for the obvious circle, define the class determined by a propositional function ϕ as the class of propositional functions equivalent to ϕ. On the basis of the postulate which asserts the existence of abstractions, we could define the class determined by a propositional function ϕ as the abstraction from ϕ with respect to equivalence, and so avoid the necessity of another postulate for the introduction of classes.

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* G. Peano, Formulaire de Mathématiques, vol. 3 (1901), p. 70.

† See B. Russell, *The Principles of Mathematics*, 1903, p. 166, where a similar postulate is proposed.

These considerations would be changed, of course, if we accepted the postulate proposed in the introduction to the second edition of the *Principia Mathematica*, that functions of propositions are always truth-functions, and that a function can occur in a proposition only through its values, for this implies that equivalent propositional functions are identical, and hence that classes and propositional functions may be regarded as the same. But the authors themselves are doubtful about this proposal and their arguments in support of it are not fully convincing. In fact "*A* asserts *p*," which they take as a crucial instance, and which is certainly not a truth-function, seems to be a function of the proposition *p*. The statement that *A* asserts *p* is correctly analyzed not as meaning that *A* utters certain sounds (the analysis proposed by Whitehead and Russell) but that *A* utters sounds which have a certain content of meaning, and it is this content of meaning which constitutes the proposition *p*, in the usual sense of the word proposition. The proposal under discussion can mean only that the word proposition is to be used in some quite different sense, in which case the *Principia Mathematica* would fail to give any account of propositions in the usual sense of the word.

Continuing our consideration of the contents of volumes II and III, we find of especial interest the construction of the system of inductive cardinals (positive integers) on the basis of logical postulates only, because it is well known how the real number system, the space of euclidean or projective geometry, and other important mathematical systems, can be constructed on the basis of the system of positive integers, apart from difficulties which arise in connection with the theory of logical types.

The success of this construction of the system of inductive cardinals depends largely on the adoption of the following definition: Consider all those classes of cardinal numbers which contain 0 and which have the property that if they contain ν then they contain $\nu+1$; the common part of all these classes is the class of inductive cardinals. The numbers 0 and 1 and the operation of addition have, of course, been previously defined. Apart from difficulties connected with the theory of types (which theory we hope to see supplanted or greatly modified) this definition gives us the complete system of positive integers with all its usual properties, something certainly not present in the original assumptions (except in the sense that these assumptions enable us to construct the system of positive integers in the way just described).

This definition seems to be the only one that will lead to the desired result. It might suggest itself to consider the cardinal numbers of those classes which are similar to no proper part of themselves, but, without the aid of the multiplicative axiom or a weakened form of it, it is not known how to prove that this class of cardinals is not more extensive than that of the inductive cardinals defined as above.

The axiom of infinity, which it is found necessary to assume in proving important familiar properties of the finite cardinals, is of interest because, as one of the authors has pointed out elsewhere,* the existence of infinite classes can be proved if we disregard restrictions imposed by the theory of types.

Of the remaining subjects treated, that of real numbers and the treatment of Dedekindian and continuous series are of greatest interest.

It is well known how to construct the system of real numbers once the series of positive integers is given, but the usual method takes no account of difficulties raised by the theory of types. According to their original scheme, the authors proposed to overcome these difficulties by means of the axiom of reducibility, but in the introduction to the second edition they abandon this axiom in favor of the postulate mentioned above, that functions of propositions are always truth-functions, and that a function can occur in a proposition only through its values. As a result of this substitution the system of real numbers can no longer be adequately dealt with by any method known to the authors.

If for no other reason than that it leads to important results to which the other does not, the axiom of reducibility seems to be distinctly preferable to the postulate which Whitehead and Russell now propose as a substitute for it.

It is to be hoped, however, that the question will ultimately be settled by the complete abandonment of the theory of logical types or by an alteration in it more radical than any yet proposed. For the theory of types, although adequate to obtain the results we require, provided that the axiom of infinity and the axiom of reducibility are admitted, introduces many complications and creates some awkward situations, one of them the following, referred to at the beginning of volume II of the *Principia Mathematica*. Having proved the theorems that we require about functions of the first $n-1$ types, then in order to obtain the same theorems about functions of the nth type we must make a new assumption of all our postulates, applying them to functions of the nth type instead of functions of some lower type, and must then prove all our theorems anew. We "see," by symbolic analogy, that this can always be done. But the statement that this is so is impossible under the theory of types.

The well known contradictions, of which a list is given in the first volume of the *Principia Mathematica*, must, of course, be dealt with, and the theory of types at present affords the best known method of doing this. Our hope is that another method can be found which will entail fewer complications.

ALONZO CHURCH†

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* B. Russell, *The Principles of Mathematics*, 1903, p. 357.

† National Research Fellow in Mathematics.

SHORTER NOTICES

Lehrbuch der Funktionentheorie. By L. Bieberbach. Band II: *Moderne Funktionentneorie.* Leipzig and Berlin, Teubner, 1927. vii+366 pp.

This work is the second volume of the set with the above title. Volume I on the elements of the theory of functions of a complex variable was reviewed by the present writer in this Bulletin (vol. 28 (1922), p. 467). The present volume treats of the more recent developments of the subject. The work is confined almost entirely to the functions of a single complex variable, with particular attention to certain special aspects. Functions of several variables in the complex domain are not treated. The author's own pet problem of conformal mapping, however, does not receive an unduly large proportion of space, though the scope of the entire book is not as complete as Bieberbach's article in the Encyclopaedie der Mathematischen Wissenschaften.

While Bieberbach in the main uses function theoretic methods without leaning very heavily upon real analysis, nevertheless, the use of very recent developments in the theory of functions of a real variable, starting with integrals of Lebesgue, have when combined with the older classical methods, given such elegant theorems as that of Fatou.

The eight chapters carry the following headings and indicate the topics treated. I. Conformal mapping. II. The elliptic modular functions. III. Bounded functions. IV. Uniformization. V. Picard's theorem. VI. Entire functions. VII. Analytic continuation. VIII. The Reimann zeta function.

In addition to the researches of the author, recent results of the following writers are presented: Carathéodory, Carlson, Fabry, Fatou, Hadamard, Hardy, Hoheisel, Harnack, Jensen, Jentzsch, Julia, Landau, Nevanlinna, Phragmen, Ostrowski, Schottky, Szegö, Vitali, Wigert, Wiman, and others.

Julia in his Borel monograph of 1923 gives Porter joint credit with Vitali for the important theorem to the effect that to every set of analytic functions uniformly bounded in a regular domain, there belongs an infinite subsequence having an analytic limit function. This theorem is closely related to the work of Osgood, Stieltjes, Schottky, and Montel. Similarly, Porter seems to have been the first to have pointed out that for any series with its circle of convergence as a natural boundary there exist certain "sub-series" which may be continued outside of the original circle of convergence. These are called "over-convergents" and ascribed by Bieberbach to Jentzsch. Finally, Bieberbach and others have failed to recognize that Porter in the Annals of Mathematics of 1907 assigned to his polynomial convergents of a given series, all the properties encompassed in Montel's normal families of functions.

The reviewer desires to bestow the very highest praise on the style of the present text. The author has a facile pen and achieves a happy result. The work is indispensable for any one interested in present day developments in analytic functions.

H. J. ETTLINGER

Récréations Mathématiques et Problèmes des Temps Anciens et Modernes par W. Rouse Ball. Deuxième edition française traduite d'après la quatrième édition anglaise. Troisième partie avec additions de MM. Margossian, Reinhart, Fitz Patrick, et Aubry. Paris, Hermann, 1927. 2+363 pp. Paper, 21 fr.

The first edition of Ball's *Mathematical Recreations and Problems of Past and Present Times* appeared in 1892, the fourth edition in 1905, and the tenth edition in 1922 with the title *Mathematical Recreations and Essays.* Since the English publisher has nothing to hide, the latest edition always contained the dates of all previous editions. This is in marked contrast to the method of the French publisher who in this case as in the cases of other works recently reprinted seems to attempt to give the impression that the works were first published in 1927. As a matter of fact the three volumes of the second French edition of the *Recreations* were published in 1907, 1908, and 1909 respectively. The volume under review is a facsimile of the one issued in 1909 with the exception that on the title page and front cover "1909" has been changed to "1927," and on the last page, before the back cover, "Saint-Armand, Cher.—Imprimerie Bussière" has been changed to: "Reproduit par les procédés Dorel, 45 Rue de Tocqueville—Paris XVII^e." The defects of such a reproduction are noticeable. No further review of this work is called for. It may be remarked, however, that the three volumes of the original second French edition contained so much new material, they are well worth having in a library along with the English editions. The first French edition, in one volume, appeared in 1898 and was a translation of the third English edition (1896).

There was also an Italian translation by Gambioli, from the fourth English edition, which appeared in 1910.

R. C. ARCHIBALD

A Treatise on the Mathematical Theory of Elasticity. By A. E. H. Love. Fourth edition. Cambridge University Press, 1927. xviii+643 pp.

Love's *Elasticity* first appeared 35 years ago in two volumes. The second edition in 1906 was entirely rewritten and put into a single volume. This edition was reviewed in this Bulletin by F. R. Sharpe (vol. 16, p. 90). The third edition appeared in 1920, when, owing to the disturbance in the distribution of scientific literature during the war, a considerable amount of material which had been published was not available for incorporation in that edition. We now have the fourth edition, which like the third, is not really a major change from the second, but does bring the bibliography up to date and includes a considerable amount of new material and some revisions of the old. There is more about problems of the plate (thick and thin), a note on the deduction of the stress-strain relations from molecular theory (lattices), a revision and simplification of the theory of the equilibrium of a sphere (geophysics).

Those of us who are interested in mechanics have cause to be very thankful that Love's *Elasticity* and Lamb's *Hydromechanics* are kept in print and kept up to date. There is not the equivalent of either in any other language. We have also to be thankful that the authors have not yielded to the temptation to expand the volumes out of all proportion; it would have been so easy to have become encyclopedic, to have gone into the engineering applications, to have developed various parts of physics not really germane to the central thought, and even to have descended from the high path of the mathematical theory to the slough of multiplying hypotheses and inadequately justified approximations.

Some fifteen years ago I taught parts of Love's *Elasticity* to a group of graduate students at the Massachusetts Institute of Technology who had had courses on strength of materials, on structures, on applied mechanics, and on advanced calculus. The book was not easy to read with them although they had a good knowledge of the physical phenomena with which it deals; for highly trained mathematical students it might be easier in some respects, but I fear that with the lack of physical and practical insight often found in such students it would be for them not much but mathematics. One does not come instinctively by the feeling for the facts of elasticity as he does for those of particle or rigid mechanics. It would be a real service if the author could collaborate with some leading theoretical engineer to produce a text which should be thoroughly sound, and while serving for ordinary instruction, might lead naturally up to this great classic.

EDWIN B. WILSON

Quadratic and Linear Tables. Lt.-Col. Allan J. C. Cunningham. London, Francis Hodgson, 1927. xii+170 pp.

The first 55 pages of this volume is an extension to the limit 125,683 of the table of quadratic partitions issued in 1904 by the same author. The use of this table has been explained in the previous volume which has been reviewed in this Bulletin.

A small table follows giving the representation of certain numbers by the form x^2+462y^2, and furnishing a partial list of primes between 10,000,873 and 10,099,681. Since 462 is an "idoneal" number these primes are called by the misleading title "idoneal primes." Since we already have tables giving the complete list of primes in this range the value of this and the little table following it by T. B. Sprague is somewhat doubtful.

For the worker in the theory of numbers many of the remaining tables in the book will be welcomed. Thus we have a table giving the last four possible digits of square numbers which will help identify large squares: tables of solutions of quadratic congruences; quadratic residues and non-residues; tables which give the linear forms for a in the congruence $a^2-b^2 \equiv N \pmod{p}$ which are valuable in the application of Fermat's method of factorization by the difference of squares. The table for the congruence $a^2+b^2\equiv N \pmod{p}$ could easily have been derived from this one. These tables have been published recently also by M. Kraitchik. A further table giving solutions of the equation $ax-by=1$ occupies some 22 pages.

D. N. LEHMER

An Introduction to the Theory of Infinite Series. By T. J. I'A. Bromwich. Second edition, revised with the assistance of T. M. Macrobert. London, Macmillan and Company, 1926. xv+535 pp.

The first edition of this book, which appeared in 1908, has been of great use to mathematicians and students of mathematics having a direct or indirect interest in the field of infinite series. No more comprehensive account of the fundamental results in this field could be found elsewhere between two covers, and yet the work throughout was very readable and involved no undue demands in the way of prerequisite mathematical knowledge. During nearly twenty years it has remained a standard text and reference work in the subject, and the only work of its type in English.

In view of these characteristics of the first edition, it was rather to be expected that a second edition would in the main involve no very sweeping changes and no extensive rewriting. The one topic, namely summable divergent series, where the rapid development of the subject since the first printing would have necessitated considerable alteration of the text, has been omitted in the present edition, save for a brief account of its historical origin. The reviewer is inclined to regret this omission and to feel that the justification offered in the preface, lack of space, is not entirely adequate. No great addition to the number of pages in the book would have been required in order to bring up to date such topics in the field of summable series as were treated in the first edition. While these topics have a much smaller ratio to the total available literature than in 1908, yet they are fundamental and constitute a good introduction to the subject.

Among the new topics introduced in the present edition may be found discussions of quasi-uniform convergence, existence theorems regarding solutions of linear differential equations, the asymptotic development of Bessel's functions, trigonometric series including Gibbs' phenomenon, and an account of Napier's invention of logarithms. These additions are all well justified on the ground of their fundamental importance and add to the value of the book. There is every reason to believe that the second edition will prove as useful during the next two decades as the first edition has during the past two.

C. N. Moore

Einführung in die konforme Abbildung. By Ludwig Bieberbach. Second edition. Sammlung Göschen. Berlin, Walter de Gruyter, 1927. 130 pp.

In this little pocket volume, the author presents, in a thorough and painstaking manner, the fundamentals of the principal topics which arise in the theory of the conformal representation of one region of the complex plane on another through the use of analytic functions of a complex variable. The style is lucid and clear and the material well arranged. In so small a space, a surprisingly large amount of material has been presented. The pedagogical excellence of the book is particularly to be commended. It is an excellent text in its subject.

C. H. Sisam

History and Synopsis of the Theory of Summable Infinite Processes. By L. L. Smail. University of Oregon Publications, vol. 2, No. 8, Feb. 1925. 175 pp.

This monograph gives a detailed analysis of the literature on divergent series, from the beginnings of the modern theory, in 1880, to 1922. About two hundred fifty papers on divergent series were produced in this period. These are listed here, in chronological sequence, and the contents of each indicated. The abstracts are, in most cases, very explicit, and permit one to get at the substance of the papers with great rapidity. The monograph will undoubtedly be found an invaluable work of reference by many investigators.

J. F. RITT

Elementary Algebra. Part II. By F. Bowman. London, Longmans, Green and Co., 1927. viii+431 pp.

Except for the first four chapters, which treat the binomial, the exponential, and the logarithmic series, this book is designed to answer certain questions relating to the theory of equations which arose in Part I. The subjects developed are those usually treated of in elementary texts on the theory of equations, including the elements of determinants and linear equations. The author introduces the complex variable and proves the fundamental theorem of algebra. In this he does not confine himself to the bare necessities but includes considerable interesting material on algebraic functions and the geometry of the complex plane. The calculus is studiously avoided, the derivative being introduced under the name *gradient function.* There is a chapter on graphs, frequent graphical illustrations, numerous examples and many excellent sets of exercises.

L. T. MOORE

Harmonic Curves. By William F. Rigge, S.J. Published by the Creighton University, Omaha, Nebraska, 1926. 8vo. 213 pp.

While Father Rigge's volume does not perhaps call for special notice from the point of view of higher mathematics; it is particularly attractive to those who are interested in the construction of "conventional" designs. There are numerous diagrams which have been reproduced without change from those actually obtained from his machine, and these show remarkable accuracy in closing complicated symmetric figures. The machine referred to is one which has been gradually developed over an interval of nine years. It contains many components giving harmonic curves and these are greatly increased in number by interchangeable gears. Since it also contains "linkage" components it can draw many types of cycloids and cardioids as well as compound harmonic curves. The author in his preface says that if we consider periods only, it is capable of drawing 7,618,782,498 curves! An interesting additional feature is the production of stereoscopic figures. The elementary equations of the simpler curves are worked out in detail.

E. W. BROWN

NOTES

The October, 1927, number of the American Journal of Mathematics (volume 49, No. 4) contains the following papers: *New results in elimination*, by F. Morley and A. B. Coble; *Numerical functions of multipartite integers and compound partitions*, by E. T. Bell; *Lagrange resolvents in euclidean geometry*, by L. M. Blumenthal; *The ovals of the plane sextic curve*, by J. H. McDonald; *On isometric systems of curves and surfaces*, by C. E. Weatherburn; *A characteristic property of certain sets of trigonometric functions*, by M. H. Stone; *The normal probability function and general frequency functions*, by M. H. Stone; *On* (3, 3) *and higher point correspondences*, by T. R. Hollcroft; *Contributions to the theory of conjugates*, by E. P. Lane; *A certain general type of Neumann expansions and expansions in confluent hypergeometric functions*, by R. F. Graesser; *A modern presentation of Grassmann's tensor algebra*, by H. Barton. The frontispiece is a portrait of Oskar Bolza.

The September, 1927, number of the Annals of Mathematics (series 2, volume 28, No. 4) contains: *Generalized derivatives*, by A. J. Maria; *A Cremona group isomorphic with the group of the twenty-seven lines on a cubic surface*, by H. L. Black; *Transformations of the Kummer criteria in connection with Fermat's last theorem* (second paper), by H. S. Vandiver; *Theorems on deducibility* (second paper), by C. H. Langford; *On some new types of non-differentiable functions*, by A. N. Singh; *On the problem of coloring maps in four colors*, II, by C. N. Reynolds, Jr.; *Division algebras of order sixteen*, by R. Garver; *Note on a theorem concerning continuous curves*, by W. L. Ayres; *Development of functions in a system of approximately orthogonal functions*, by W. D. Cairns; *Concerning the summability of double series of a certain type*, by G. M. Merriman; *Double elliptic geometry in terms of point, order and congruence*, by D. A. Flanders; *The replacement theorem and related questions in the projective geometry of paths*, by T. Y. Thomas; *On types of knotted curves*, by J. W. Alexander and G. B. Briggs; *A problem in minima*, by D. Jackson; *On the developments of arbitrary functions in series of Hermite's and Laguerre's polynomials*, by J. V. Uspensky; *Parallelism and transversality in a subspace of a general* (*Finsler*) *space*, by J. H. Taylor; *On the form of differential equations of a system of paths*, by A. Church; *Differential invariants of relative quadratic differential forms*, by T. Y. Thomas and A. D. Michal.

The editors of the Journal de Mathématiques announce that the volumes for 1928 and 1929 will be dedicated to the eminent mathematicians Paul Appell and Émile Picard, as a tribute upon the occasion of their scientific jubilees. It is obvious that these volumes, to which many eminent mathematicians will contribute, will have unusual importance. The edition will be limited, and the volumes will not be reprinted.

Additional circulars of information and blanks for abstracts of papers for the Congress of Mathematicians to be held at Bologna next September, have been sent to all those whose names are on the list of prospective participants, as received at the office. Additional copies may be obtained upon request from the office of this Society, 501 West 116th St., New York City.

The British Association for the Advancement of Science will meet at Glasgow, September 5–12, 1928, under the presidency of Sir William Bragg. Professor A. W. Porter will be president of Section A (mathematical and physical sciences).

At the annual meeting of the London Mathematical Society, held November 10, 1927, the following officers were elected: Professor G. H. Hardy, president; Professor A. S. Eddington, Mr. R. H. Fowler, and Professor G. B. Jeffery, vice-presidents; Dr. A. E. Western, treasurer; Professor Harold Hilton, librarian; Professor G. N. Watson and Mr. F. P. White, secretaries.

A meeting commemorating the bicentenary of the death of Sir Isaac Newton was held at the American Museum of Natural History, New York City, November 25–26, 1927, under the auspices of the History of Science Society in collaboration with the American Astronomical Society, the American Mathematical Society, the American Physical Society, and the Mathematical Association of America. The following papers were read: *Introductory address on Newton's life and work*, by D. E. Smith; *Newton and optics*, by D. C. Miller; *Newton's philosophy of gravitation with special reference to modern relativity ideas*, by G. D. Birkhoff; *Newton's influence on the development of astrophysics*, by W. W. Campbell; *Newton and dynamics*, by M. I. Pupin; *Newton as an experimenter*, by P. R. Heyl; *Developments following from Newton's work*, by E. W. Brown; *Newton's twenty years delay in announcing the law of gravitation*, and *Newton's early study of the Apocalypse*, by Florian Cajori; *Newton's work in alchemy and chemistry*, by L. C. Newell; *Newton's place in the history of religious thought*, by G. S. Brett; *Newton in the mint*, by G. E. Roberts; *Newton's first disciple in the American colonies*, by F. E. Brasch.

The Paris Academy of Sciences announces the award of the following prizes for 1927: the Francœur prize to Georges Cerf, for his work in partial differential equations; the Montyon prize in mechanics to Dimitri Sensaud de Lavaud, for his work, based on theoretical considerations, on the prevention of oscillations of the front wheels of automobiles; the Poncelet prize to Henri Villat, for his work in the mechanics of fluids; the de Pontécoulant prize to Emile Paloque, for his work on the analytic theory of the motion of the Trojan planets; prizes in navigation to André Courtier, for his work in hydrography and in the prediction of tides, to Pierre Changeux, for his *Etude Général sur la Dynamique des Navires ou Appareils Utilisées en Navigation Maritime et Aérienne;* and to Edouard Davaux, for his *Cours d'Electrotechnique*; the Hébert prize to Pierre Sève, for his work on alternating currents; the Pourat prize to Antoine Magnan, for his works entitled *Les Charactéristiques des Oiseaux suivant le Mode de Vol, leur Application à la Construction des Avions*, and *Le Vol à Voile avec Contribution à* *l'Etude Expérimentale de la Physique et de la Mécanique des Fluides*; the Petit d'Ormoy prize in mathematics to Ernest Vessiot, for his work as a whole; the Saintour prize to Stanislas Zaremba, for his work in mathematical analysis; a prize from the Gegner foundation to Francisque Dumont, for his work in geometry; a prize from the Becquerel foundation to Louis de Broglie, for his work in wave mechanics, atomic structure, and radiation. The Bordin prize, for a memoir on the dynamics of non-holonomic systems, was not awarded, and the subject was withdrawn.

The First Congress of Polish Mathematicians was held at Lwow, September 8–12, 1927, under presidency of Professor Sierpinski. The total attendance was about 200 including many foreign mathematicians. Over 100 papers were read at the Congress.

The Nobel prize in physics for 1927 has been awarded jointly to Professor A. H. Compton, of the University of Chicago, and Professor C. T. R. Wilson, of Cambridge University.

The Hopkins prize of the Cambridge Philosophical Society for the period 1921–25 has been awarded to Dr. J. H. Jeans, for his work on the theory of gases, on radiation, and on the evolution of stellar systems.

The Bruce medal of the Astronomical Society of the Pacific has been awarded to Dr. W. S. Adams, of Mount Wilson Observatory.

The Franklin Institute has voted to award its Cresson gold medal for 1928 to Professor Vladimir Karapetoff, of Cornell University, "in consideration of the inventive ability, skill in design, and detailed theoretical knowledge of kinematics and electrical engineering displayed in the development of computing devices."

Dr. Irving Langmuir, of the General Electric Company, has been awarded the Perkin medal for 1928.

Cambridge University has conferred the honorary degree of doctor of science on Paul Painlevé, professor of analytical and celestial mechanics at the Sorbonne, now French minister of war.

Sir Ernest Rutherford has been elected president of the Royal Society of London. He has also been elected a foreign associate of the Paris Academy of Sciences.

Professor S. Banach has been appointed ordinary professor of mathematics at the University of Lwow.

Dr. Heinrich Behnke, of the University of Hamburg, has been appointed professor of mathematics at the University of Münster.

Professor Martin Brendel, of the University of Frankfurt a.M., has retired.

Dr. Werner Heisenberg, of the University of Copenhagen, has been appointed professor of theoretical physics at the University of Leipzig.

Dr. C. Kuratowski, of the University of Warsaw, has been appointed professor of mathematics at the Technical College of Lwow.

Dr. Alexander Ostrowski, of the University of Gottingen, has been appointed professor of mathematics at the University of Basel.

Professor Theodor Pöschl, of the German Technical School at Prague, has been appointed professor of mechanics at the Karlsruhe Technical School.

Professor G. Albanese, of the University of Catania, has been transferred to the professorship of descriptive geometry at the University of Palermo.

Professor E. Bompiani, of the University of Bologna, has been transferred to the professorship of descriptive geometry at the University of Rome.

Dr. L. Fantappiè, of the University of Rome, has been appointed associate professor of algebraic analysis at the University of Cagliari.

Miss P. Nalli, professor of the calculus at the University of Cagliari, has been transferred to a professorship of the calculus at the University of Catania.

Mrs. M. Piazzola Beloch and Dr. A. Tonolo have been appointed to associate professorships of analytic geometry and the calculus at the University of Ferrara.

Dr. N. Spampinato, of the University of Catania, has been promoted to an associate professorship of descriptive geometry.

Professor A. B. Coble, of the Johns Hopkins University, has accepted a professorship of mathematics at the University of Illinois, where he had been prior to the present academic year.

Professor B. F. Dostal, of the University of Michigan, has been appointed assistant professor of mathematics at the University of Florida.

Professor G. H. Hardy, of Oxford, will be in residence at Princeton during the first half of the academic year 1928–29, and will give a course entitled *Chapters in the theory of functions.* During the same period, Professor Oswald Veblen, of Princeton University, will be in residence at Oxford, and will give a course entitled *Tensors and other differential invariants.*

Dr. L. S. Hill has been appointed assistant professor of mathematics at Hunter College.

At the University of Illinois, Professors Arnold Emch and Olive C. Hazlett have been granted leave of absence for the coming academic year, for study in Europe.

Associate Professor Lao G. Simons has been promoted to be full professor and head of the department of mathematics at Hunter College.

Professor Mary E. Sinclair, of Oberlin College, has been granted leave of absence for the year 1927–28, and is in residence at the University of Miami, Coral Gables.

Dr. V. B. Teach has been appointed assistant professor of mathematics at the Armour Institute of Technology.

Dr. S. Timoshenko, of the research department of the Westinghouse Company, has been appointed professor of applied mathematics at the University of Michigan. He is succeeded at the Westinghouse Company by Dr. A. Nádai, of Göttingen.

Dr. D. V. Widder has been appointed associate professor of mathematics at Bryn Mawr College.

Professor E. I. Fredholm, of the University of Stockholm, known for his fundamental work in integral equations, died August 17, 1927, at the age of sixty-one.

Dr. Emil Müller, professor of descriptive geometry at the Vienna Technical School, died September 1, 1927.

Dr. R. A. Herman, lecturer in mathematics at Cambridge University, died November 29, 1927, at the age of sixty-six.

Mr. H. M. Taylor, F. R. S., fellow of Trinity College, Cambridge, is dead.

Professor F. J. Dick, of the Theosophical University, Point Loma, died May 25, 1927. Professor Dick was a member of the American Mathematical Society.

Dr. G. A. Osborne, professor emeritus of mathematics at the Massachusetts Institute of Technology, died November 20, 1927, at the age of eighty-eight. Professor Osborne had been a member of the American Mathematical Society since 1891.

Professor H. E. Russell, of the University of Denver, died May 31, 1927. Professor Russell was a member of the American Mathematical Society.

Mr. William Young, actuary of the New York Life Insurance Company, died October 23, 1927, at the age of fifty-two. Mr. Young was a member of the American Mathematical Society.

NEW PUBLICATIONS

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BERGFELD (E.). Die Axiome der euklidischen Geometrie psychologisch und erkenntnistheoretisch untersucht. München, C. H. Becksche Verlagsbuchhandlung, 1927.

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BÜRKLEN (O. T.). Mathematische Formelsammlung. Vollständig umgearbeitete Neuausgabe von F. Ringleb. (Sammlung Göschen.) Berlin, de Gruyter, 1927. 241 pp.

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CUNNINGHAM (A. J. C.). Quadratic and linear tables. London, Hodgson, 1927. 2+170 pp.

DEAUX (R.). Cours de géométrie analytique. 3 volumes. Mons, Delporte, 1927. 158+87+38 pp.

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ENRIQUES (F.). Questioni riguardanti le matematiche elementari raccolte e coordinate. Parte III: Numeri primi e analisi indeterminata. Massimi e minimi. Bologna, Zanichelli, 1927. 492 pp.

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FRASER (D. C.). Newton's interpolation formulas. London, Layton, 1927. 95 pp.

FRICKE (R.). See KLEIN (F.).

FUETER (R.). Vorlesungen über die singulären Moduln und die komplexe Multiplikation. 2ter Teil. Leipzig, Teubner, 1927. 6+360 pp.

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HONECKER (M.). Logik. Eine Systematik der logischen Probleme. Berlin, Dümmler, 1927. 194 pp.

KLEIN (F.) und FRICKE (R.). Vorlesungen uber die Theorie der automorphen Funktionen. 2te Auflage. 1ter und 2ter Band. Leipzig, Teubner, 1926. 16+634+14+668 pp.

LAMÉ (G.). Examen des différentes méthodes employées pour résoudre les problèmes de géométrie. (Réimpression.) Paris, Hermann, n.d. 124 pp.

LETTENMEYER (F.). Systeme linearer Differentialgleichungen unendlich höher Ordnung mit Polynomen beschränkten Grades als Koeffizienten. (Habilitationsschrift, Universität München.) München, Buchdruckerei F. Straub, 1927. 52 pp.

LINDOW (M.). Numerische Infinitesimalrechnung. Berlin, Dümmler, 1927. 8+176 pp.

MACLEOD (A. H. D.). Sur diverses questions se rapportant à l'étude du concept de réalité. Paris, Hermann, 1927. 11+239 pp.

MINKOWSKI (H.). Diophantische Approximationen. Eine Einfuhrung in die Zahlentheorie. 2te Auflage. Leipzig, Teubner, 1927. 8+236 pp.

MONTEL (P.). Leçons sur les familles normales de fonctions analytiques et leurs applications. Paris, Gauthier-Villars, 1927. 306 pp.

NIEWENGLOWSKI (B.). Questions d'arithmétique. Paris, Vuibert, 1927. 8+225 pp.

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RINGLEB (F.). See BÜRKLEN (O. T.).

RUNNING (T. R.). Graphical mathematics. New York, Wiley, 1927. 8+91 pp.

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SPEISER (A.). Die Theorie der Gruppen von endlicher Ordnung mit Anwendungen auf algebraische Zahlen und Gleichungen sowie auf die Kristallographie. 2te Auflage. (Die Grundlehren der Mathematischen Wissenschaften, Band V.) Berlin, Springer, 1927. 10+251 pp.

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VEBLEN (O.). Invariants of quadratic differential forms. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 24.) Cambridge, University Press, 1927. 8+102 pp.

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THE FEBRUARY MEETING IN NEW YORK

The two hundred fifty-ninth regular meeting of the American Mathematical Society was held at Columbia University, on Saturday, February 25, 1928, extending through the usual morning and afternoon sessions. Attendance included the following eighty-six members of the Society:

R. C. Archibald, R. G. Archibald, J. W. Arnold, Beetle, Benton, Bratton, Carrie, E. T. Carroll, Alonzo Church, Clark, Clutz, Cramlet, Dehn, Derby, Doermann, Dresden, Engstrom, Feld, Fiske, Fite, D. A. Flanders, Fort, Frink, Gale, Gehman, Gill, Glenn, Gronwall, C. C. Grove, L. S. Hill, Hille, Himwich, Hofmann, Hollcroft, Joffe, M. I. Johnson, R. A. Johnson, Kasner, Kenny, Kline, Knebelman, Koopman, J. P. Kormes, Mark Kormes, Langman, Lefschetz, Lehr, Littauer, L. T. Moore, T. W. Moore, Moyle, Mullins, F. H. Murray, C. A. Nelson, Ore, Pepper, Pfeiffer, Pixley, Post, Raudenbush, Rawles, Reddick, M. S. Rees, Robertson, Roos, Rutt, Schoonmaker, Seely, Serghiesco, Simons, Smail, Virgil Snyder, J. M. Thomas, T. Y. Thomas, Tracey, Veblen, Wedderburn, Weida, Weisner, H. S. White, Whittemore, W. M. Whyburn, Widder, W. A. Wilson, J. W. Young, Margaret M. Young.

There was no meeting of the Council or of the Trustees.

President Snyder presided at the morning session, relieved in the afternoon by Vice-President Young.

Titles and abstracts of the papers read at this meeting follow below. The papers of Gergen, Ghosh, Prasad, Trjitzinsky, G. T. Whyburn, and Zarycki were read by title. Mr. Ghosh was introduced by the Secretary, and Dr. Zarycki by Professor Dunham Jackson.

1. Dr. Louis Weisner: *Polynomials $f[\phi(x)]$ reducible in fields in which $f(x)$ is irreducible.*

Given a polynomial $f(x)$ with coefficients in R and irreducible in R, the author determines all polynomials $\phi(x)$ with coefficients in R such that $f[\phi(x)]$ is reducible in R.

2. Dr. R. G. Archibald: *The impossibility of a separation of types of linear odd divisors of binary quadratic forms.*

This paper develops theorems on the existence of solutions of quadratic congruences in two unknowns and utilizes these results to prove the impossibility of a separation of types of linear odd divisors of binary quadratic forms. Tables are given in Legendre's *Théorie des Nombres* of the quadratic divisors and of the linear odd divisors of a form t^2+au^2 for certain integral values of a. In many cases the forms of the linear odd divisors are paired off in sets with a single quadratic divisor. This paper is concerned with the remaining cases in Legendre's Tables IV–VII in which the forms of the linear odd divisors of t^2+au^2 are paired off in sets with more than one quadratic divisor. If the linear odd divisor $4ax+\alpha$ corresponds to two or more quadratic divisors of t^2+au^2, the question is raised whether it is possible to find a positive integer n such that all the divisors of t^2+au^2 contained in each of $4any+(j4a+\alpha), j=0, 1, \cdots, n-1$, are represented, respectively, by one and only one of the quadratic divisors of t^2+au^2. By the method here developed, it is found that such a separation is impossible.

3. Professor J. I. Tracey and Dr. L. T. Moore: *Covariant conditions for multiple roots of a binary form.*

This paper gives a method whereby a covariant may be formed which vanishes when an n-ic has a p-fold root, a p_1-fold root, a p_2-fold root, $\cdots$, a p_r-fold root simultaneously. It also enables one to determine the other systems of equality among the roots for which the same covariant will vanish.

4. Professor O. E. Glenn: *On the generalization of the galoisian complex domain.*

For every integral modulus n there exists a form $f(x)=x^2-x+r$, with r integral, having no integral roots modulo n. Assume an imaginary ι such that $f(\iota)\equiv 0 \pmod{n}$. Any polynomial in ι with integral coefficients is reducible by $\iota^2=\iota-r$ to $\alpha\iota+\beta$, where α, β are integral residues of n. These n^2 complex numbers form a domain Δ in which a congruence whose coefficients are residues of n, $F(x)=a_0x^m+a_1x^{m-1}+\cdots+a_m\equiv 0 \pmod{n}$, may have N roots and, if n is composite, $N<$, $=$, or $>m$. This paper, in its present form, is concerned with necessary and sufficient matrix conditions, analogous to the algebraic theory of Stuyvaert and Dines, in order that two polynomials F may have a common root in Δ.

5. Dr. Stefan Serghiesco: *On the number of roots of a system of n equations.*

In a previous paper presented to the Society, the author introduced an integral containing certain differential invariants formed from a given system of n equations, and showed that, in a special case, this integral gives the number of common roots of the system of equations. This present paper contains a further extension of this theory, and shows that by means of the same integral we may obtain a formula giving the general solution of the problem. The formula, which is a difference of two integrals, and the method are entirely different from those used in the Picard-Kronecker theory of the same problem.

6. Dr. W. M. Whyburn (National Research Fellow): *On self-adjoint, ordinary differential equations of the fourth order.*

The general, real, self-adjoint, ordinary differential equation of the fourth order and the second-order equation with coefficients that are complex functions of a real variable are studied through a consideration of the pair of real, second-order differential equations

$$\frac{d^2y}{dx^2}+p(x)y=q(x)z,\ \frac{d^2z}{dx^2}+p(x)z=r(x).y$$

Simple conditions are derived under which the integral curve in the yz-plane will be essentially spiriliform and will make an arbitrarily large number of circulations about the origin as x ranges over the entire real axis. Related integral equations are studied, and used to develop comparison theorems of the type obtained by Birkhoff (Annals of Mathematics, (2), vol. 12, pp. 103–127) for third-order systems. Several other properties of the system are exhibited through their relation to similar properties of the second-order equation $u''+p(x)u=0$. A dynamical interpretation is used for the equations and the hypotheses employed.

7. Professor Tomlinson Fort: *A series resembling factorial series.*

The author considers series of the type

$$\sum_{n=1}^{\infty}c_n\lambda_1\lambda_2\cdots\lambda_n/((z+\lambda_1)(z+\lambda_2)\cdots(z+\lambda_n)),$$

where $0\leqq\lambda_{n-1}\leqq\lambda_n\rightarrow\infty$. He proves a variety of theorems concerning convergence and summability. The series in question is seen to bear a relation to the factorial series,

$$\sum_{n=1}^{\infty}c_n n!/(z(z+1)\cdots(z+n)),$$

resembling that borne by the Dirichlet series, $\sum_{n=1}^{\infty}c_n e^{-\lambda_n z}$, to the special Dirichlet series $\sum_{n=1}^{\infty}c_n n^{-z}$.

8. Professor D. V. Widder: *The singularities of functions defined by the Stieltjes integral* $\int_0^\infty e^{-st}d\alpha(t)$.

Seeking to generalize power series and Dirichlet series by replacing the discrete set of numbers λ_n that appears in such series, $\sum_{n=1}^{\infty}a_n e^{-\lambda_n s}$, by a continuous set, the author considers a Stieltjes integral of the form $f(s)=\int_0^\infty e^{-st}d\alpha(t)$, where $\alpha(t)$ is a function of bounded variation in every finite interval, $0\leqq t\leqq R$. Functions of this form, in which $\alpha(t)$ has a continuous derivative, were first studied by Laplace, the function $\alpha'(t)$ having been designated as the determining function, $f(s)$ as the generating function. The present paper undertakes, for the first time, the study of the singularities of such functions, as determined by the generating function. The most important result states that if the function $f(s)$ has singularities at points a, and if the function $\phi(s)=\int_0^\infty e^{-st}d\beta(t)$ has singularities at points b, then $F(s)=\int_0^\infty e^{-st}\alpha(t)d\beta(t)$ has singularities at most at points $a+b$ and b, under certain conditions imposed on the rates of increase of $f(s)$ and $\phi(s)$

on the vertical lines of the complex plane. From further developments it is found that the well known theorems of Hadamard, Hurwitz, and Faber, three of the most fundamental theorems of analytic continuation, appear as special cases under the present theory.

9. Dr. T. H. Rawles: *Two classes of periodic orbits with repelling forces.*

The purpose of this paper is to obtain certain classes of periodic orbits of a system consisting of one body of very great mass which attracts two mutually repellent bodies of very small mass. It will be shown that if certain conditions of symmetry are imposed the orbits of the two small bodies must lie in two parallel planes or else be coplanar. Expressions for these two types will be obtained.

10. Professor W. A. Wilson: *Some properties of upper semi-continuous collections of continua.*

Let $M= \Sigma[X]$ denote the sum of a set of disjoint bounded continua in a plane, let t denote any point of a circumference C, and let there be a (1, 1) correspondence between the elements of M and the points of C such that $X=f(t)$ is upper semi-continuous in C. R. L. Moore has shown that if no element X separates the plane, then M divides the plane into two domains whose frontiers are parts of M. By imposing the condition that $X=f(t)$ is a *minimal* upper semi-continuous function, that is, that there is no function $Y=g(t)$ defined over C such that each Y is a sub-continuum of X and some $Y\neq X$, it is shown that M is the frontier of both its complementary domains. If, in addition, the condition that the elements X do not separate the plane is removed, M is still the common frontier of two complementary domains and the frontier of each other complementary domain is a part of some element. Finally, if $f(t)$ is a minimal upper semi-continuous function defined over a segment $a\leqq t\leqq b$, and no element separates the elements $X_a=f(a)$ and $X_b=f(b)$, M is a continuum irreducible between the sets X_a and Y_b.

11. Mr. J. J. Gergen: *On accessible points on the boundary of a three-dimensional region.*

The author is engaged in generalizing to three dimensions some of the results of W. F. Osgood and E. H. Taylor established in *Conformal transformations on the boundaries of their regions of definition* (Transactions of this Society, vol. 14 (1913), p. 277). The chief result of the present paper is that an accessible point on the boundary of a simply connected three-dimensional open region, on which the Green's function vanishes, corresponds at most to a point set of zero measure on the unit sphere, when the interior of the region is transformed into the interior of the unit sphere by means of the correspondence set up by the Green's functions and their orthogonal trajectories. Examples can be given to show that this is the most general result that can be obtained merely in these terms, without restricting the character of the boundary point beyond mere accessibility.

12. Professor Ganesh Prasad: *On the nature of θ in the mean-value theorem of the differential calculus.*

If $f(x)$ is a single-valued function which is finite and continuous in an interval (a, b) the ends being included, then the relation $f(x+h)-f(x) = hf'(x+\theta h)$, $0<\theta<1$, holds for every value of x and h for which the interval $(x, x+h)$ is in the interval (a, b), provided that *either* $f'(x)$ exists for every internal point of (a, b) *or* a certain less restrictive condition is satisfied. The object of the present paper is to consider θ as a function of h, x being taken a constant, say 0, and to show (1) that θ is not necessarily single-valued, (2) that, even if we fix upon a special group of values of θ, which may be called a single-valued function of h, such a function is not necessarily continuous, and (3) that, even if θ is continuous, it is not necessarily differentiable.

13. Mr. Suddhodan Ghosh: *On the solution of the equations of elastic equilibrium suitable for elliptic boundaries.*

A solution in elliptic coordinates of the two-dimensional problem of equilibrium of an elastic isotropic body in the absence of body forces is given in the present paper, and applied to two problems of plane strain. A single-valued solution is found, suitable for the case of an elliptic cylindrical cavity in an infinite solid, and also a many-valued solution applicable to the state of strain in an elliptic cylindrical shell which has suffered dislocation due to a triangular axial fissure. Love has given an outline of a method for solving problems in plane strain in elliptic coordinates, but his method is only applicable to cases where the surface displacements are given; it is shown that the method developed in this paper lends itself easily to the construction of the solution of the particular problem dealt with by Love.

14. Professor W. J. Trjitzinsky: *Functions with assigned initial values.*

The problem of constructing indefinitely differentiable functions of a real variable with assigned initial-values at a point has been undertaken by de la Vallée Poussin in connection with the study of quasi-analytic functions; he gives the representation in the form $\sum \alpha, \cos m,x$. In the present paper the problem is treated more completely; representations are given (1) in a series of power series, (2) a series of infinite integrals, and (3) with assigned initial values at a singular point, which is exhibited as a limiting point of a sequence of poles. An asymptotic representation is given in connection with the second method; also, applying a formula of Darboux, a more general asymptotic expansion is derived.

15. Professor G. T. Whyburn: *Concerning the cuttings of a continuous curve.*

Let M be any bounded plane continuous curve, and if R is an open subset of M, let $F(R)$ denote the boundary of R with respect to M. In this paper the following results are proved. (1) There exist in M, an irreducible cutting K of M, and a component H of K such that no point of H is accessible from any component of $M-K$. (2) If R is any connected open subset of M, then each component H of $F_m(R)$ is "accessible" from R, that is, if A is any point of R, there exists a continuum N containing both A and H, which is a subset of $R+H$. (3) If K is an irreducible cutting of M between two points A and B of M, and R_c is any component of $M-K$ which contains neither A nor B, then $F_m(R_c)$ contains points of not more than two components of K. (4) Suppose K is an irreducible cutting of M which itself has at least two components and is such that $M-K$ has at least three components. Then if any component of K is decomposable, K consists of exactly two points.

16. Dr. H. P. Robertson (National Research Fellow): *Invariants of contact transformations.*

A tensor calculus applicable to contact transformations is developed, and by means of it a theory of the invariants of such transformations constructed. The Hamiltonian theory of dynamics is identified with the invariant theory of a single scalar.

17. Professor Einar Hille: *Note on the behavior of certain power series on the circle of convergence with application to a problem of Carleman.*

The paper contains a study of the properties of convergence of power series of the form $\sum f(n) \exp\, [2i\, a(n)] z^n$, where $f(n)$ and $a(n)$ are subjected to certain restrictions. There exist power series of this type which are continuous in and on the unit circle and are such that $\sum [f(n)]^r$ converges for no $r<2$. This paper will appear in an early issue of the Proceedings of the National Academy.

18. Professor Solomon Lefschetz: *Duality relations in analysis situs.*

Given two complexes C_n and $C_\lambda \subset C_n$, where C_λ includes the boundary F of C_n, there may be defined certain simultaneous invariants of the pair which generalize the Betti numbers. These numbers are shown to satisfy duality relations that include directly or indirectly all those now known in analysis situs, notably the relations apparently wholly unrelated due to Poincaré and to Alexander. The same relations are extended to the case of any closed set $G \subset C_n$, with $F \subset G$.

19. Dr. Miron Zarycki: *General properties of Cantorian coherences.*

The author develops on a postulational basis that part of point set theory which concerns coherences and adherences. The postulates are those underlying the usual algebra of logic and three others which become familiar theorems of point set theory when the undefined terms are given a particular interpretation. These three postulates are shown to form an absolutely independent set.

20. Professor J. R. Kline: *Concerning the dimension of a continuous curve every subcontinuum of which is a continuous curve.*

The author proves that every continuous curve, which has the property that every one of its subcontinua is a continuous curve, must be one-dimensional in the sense of Menger and Urysohn.

21. Professor Edward Kasner: *General theory of element-point transformations.*

The author studies the general transformation T from a lineal element (x, y, y') of a first plane I to a point (X, Y) of a second plane II. To a point of I corresponds a curve of II. To a point of II corresponds a series of elements in I. By eliminating y' from $X=X(x, y, y')$, $Y=Y(x, y, y')$, we obtain a related contact transformation Γ. To a curve in I corresponds by T a curve in II, but to a curve in II corresponds a field of elements and therefore ∞^1 curves in I. Special types of transformation are studied, and in particular the type arising in the author's theory of polygenic functions.

22. Professor Edward Kasner: *Additive groups of transformations.*

The author considers sets of point transformations which form a group in a sense different from the usual sense. Instead of combining two transformations T_1, T_2 by the usual multiplication T_1T_2, he considers vectorial addition. If T_1 transforms the point P into P_1, and T_2 transforms P into P_2, then the sum of the vectors PP_1 and PP_2 gives PP_3 and the transformation from P to P_3 is defined as T_1+T_2. For example, affine transformations form a group in this sense, but projective transformations do not. Attention was called to additive groups incidentally in connection with Darboux transformations by the author in 1909 (see Transactions of this Society, vol. 15, p. 212, last foot note). Many new examples are now given. in particular the set of all harmonic transformations $X=\phi(x, y)$, $Y=\psi(x, y)$, where ϕ and ψ obey the Laplace equation. (See Proceedings of the National Academy, vol. 14 (1928), pp. 75–82, last paragraph.)

23. Dr. Lulu Hofmann and Professor Edward Kasner: *Homographic circles.*

This paper will appear in full in an early issue of this Bulletin.

24. Professor Edward Kasner: *The increment ratio of any two polygenic (or non-analytic) functions.*

In earlier papers (see Science, vol. 66 (1927), p. 581 and abstracts in this Bulletin), the author studied the derivative dw/dz, where $w=\phi(x, y) +i\,\psi(x, y)$, the components ϕ and ψ being arbitrary. In the present paper he studies dw_1/dw_2, where w_1 and w_2 are any such polygenic functions. To each point $z=x+iy$ corresponds a circle in the derivative plane, as in the simpler case, but the correspondence between the directions $m=dy/dx$ and the points of the circle is now a general homographic correspondence instead of being uniform (rate $-2:1$). The only cases in which dw_1/dw_2 is uniform are when w_2 is an analytic function of $x+iy$ or of $x-iy$, in the latter case the rate being $(2:1)$. If all the derivative circles reduce to points, w_1 must be an analytic function of w_2. To all the points of the z plane corresponds a congruence of homographic clocks. If a congruence is given arbitrarily, necessary and sufficient conditions are obtained for its being identifiable with a derivative congruence.

25. Professor T. R. Hollcroft: *The lines of an algebraic surface.*

In this paper, the number of invariants necessary and sufficient for a surface to contain a line or a given system of lines is found. From this is obtained the maximum number of independent lines or lines of a given system lying on an algebraic surface of given order. These limits hold when the surface is otherwise non-singular. When the surface has multiple points, lines, or curves, and these are accompanied by sets of lines on the surface, the lines of such sets do not account for additional invariants. The monoid is an example of this.

ARNOLD DRESDEN
Associate Secretary

A CORRECTION

On page 139 of the March-April issue (vol. 34), the last sentence in the abstract of paper No. 17 is incorrect and should have been omitted.

RESONANCE IN THE SOLAR SYSTEM*

BY E. W. BROWN

While the title of this address implies limited applications of resonance phenomena, it is necessary, in order to develop those applications, to consider the foundations on which the theory of resonance rests. The foundations are partly physical and partly mathematical since we have to consider not only the phenomena but the symbolic representation of them. At the very beginning of the discussion it is seen that questions concerning the degree of accuracy of physical measurements are involved in a fundamental manner. A wide range of topics has thus to be brought into the argument and some considerable time must be spent on quite elementary details if the ideas to be developed are to be made clear. The physicist will recognize applications in many directions, but I shall confine the latter mainly to those which arise in the motions of bodies within the solar system.

In order to make clear the questions with which we have to deal, it is necessary to understand what we mean by *periodicity*. Take the case of simple harmonic motion defined by

$$x = a \sin (bt - c).$$

In all physical problems, a, b, c are measured quantities deduced directly or indirectly from observation. If small errors occur in a, c only, the change in x is always small, however great t may be. But a small error in b induces finite changes in x when t is sufficiently great. As our power of measuring b is always limited, the time during which we can predict the value of x is also limited. Further, the conditions under which mechanical systems operate are such that *b* itself will not remain permanently the same, unless it be a universal constant of nature, and we know of no such constant as far as the motions of finite masses are concerned. The same result must follow for any periodic motion. The mathematical definition of periodicity is thus not sufficient. In the applications we must regard it as an average or temporary phenomenon. This point of view has to be emphasized when, in celestial mechanics, we use the phrase "mean angular velocity" which implicitly assumes that such a mean exists and is determinate. Where we are dealing with phenomena over a limited time interval, the matter is not usually so important, but when our interval becomes very great as in questions of cosmogony it must be taken into account.

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* The fifth Josiah Willard Gibbs Lecture, read at Nashville, December 28, 1927, before a joint session of the American Mathematical Society and the American Association for the Advancement of Science.

The word *resonance* is usually used to describe the phenomena which are observed when, in some mechanical system in motion, the periods of oscillation of two of its parts, not rigidly connected with one another, become the same. It would seem at first sight that the definition has no meaning, for the slightest change in our measures of the periods would abolish the resonance. This difficulty partly disappears when we learn that, in general, it is not necessary to have the periods exactly equal in order to produce the phenomena; it is only necessary that their difference shall be less than some finite amount. In the language of mathematics, the phenomena of resonance belong, not to a point, but to a finite range.

When we remember the fundamental importance of resonance in all physical observations—indeed our capacity to see and hear and therefore to observe depends on its existence—it is somewhat strange how little concerning the actual phenomena is to be found in the textbooks and treatises which deal with physical problems. Partly this is due to the fact that damping due to friction occurs in our mechanical systems and damping prevents the appearance of the essential characteristics of the motion. Partly also it is due to the general use of linear systems of differential equations which give results differing fundamentally from those of non-linear systems. In the resonance problems of celestial mechanics we have no damping, the equations are not linear in general even in a first approximation, and every "constant" present is a measured constant and subject to both error and change. Since a considerable degree of delicate adjustment is required to produce resonance, it is evident that these factors must be taken into consideration when discussing the mathematical representation of the phenomena.

Let us take the linear system

$$x + n^2 x = a \sin (pt + \alpha).$$

This represents a motion which is the sum of two harmonic oscillations of periods $2\pi/n$, $2\pi/p$, as long as $n \neq p$. Since the amplitude of the second oscillation is $a \div (n^2 - p^2)$, this amplitude increases indefinitely as $n-p$ approaches zero. When $n-p=0$ exactly, the solution changes its character and can be represented by two harmonic terms one of which has an amplitude of the form bt where b is finite. But from the physical point of view there is no discontinuity in passing from $n-p$ very small to $n-p=0$: the oscillations increase in amplitude as $n-p$ approaches zero.

For a linear system with several variables

$$\ddot{x}_i + \sum_j a_{ij} x_j = 0, \qquad (i.j = 1, 2, \cdots, n),$$

in which the constants a_{ij} are such that each coordinate can be represented as the sum of n harmonic terms with n different periods, it can be shown that a similar increase in the amplitudes occurs, when two of the periods become nearly equal. An exception to this result arises when we can, by a linear change of the variables, separate the system into two independant sets in one of which $i, j = 1, \cdots, p$ and in the other, $i, j = p+1, \cdots, n$.

These results, however, are of little value when the systems are not linear. Both the physical characteristics of the ensuing motions and their mathematical representation are changed. We cannot lay down any general type of equation, as in the linear case, and apply it to the particular cases which may be presented. The better plan therefore seems to be to take a special type and discuss it in some detail. But before doing so, I must try and give some indications of the difficulties which planetary and satellite motions in the solar system present, and of the methods which at present are in use to attack them.

The problem of the motion of three or more bodies under the Newtonian laws of motion and gravitation is so complicated that I cannot hope even to give an outline which shall be intelligible in the course of an hour's talk. And it would take me far longer to lead to the particular phenomena which form the subject matter of this lecture by any logical process even if the mere steps were simply outlined and explained. I shall therefore content myself with presenting the problems involved in the form of mathematical or physical analogies which will have the additional advantage of corresponding to simple physical mechanisms with which everyone is familiar.

I have two main ideas in view in discussing the question of resonance. One, the more immediate, is an attempt to deal with a set of observed phenomena which have not as yet received a fully adequate explanation, or perhaps, to be a little more specific, to account for the phenomena on the law of gravitation only without invoking special hypotheses as to the original formation of the solar system. This first question is the apparent avoidance of resonance in certain cases of planetary motions and the apparent permanence of resonance in others. The course of the discussion is seen to lead inevitably to the two parts of the second question—the configuration of the system in the distant past and distant future. Can we learn anything about the first part, usually termed the origin of the solar system, from its present configuration? And to what extent can we predict the future? Has the solar system the property which is usually but rather loosely, termed "stability"?

"Stability" in the usual sense is a mathematical rather than a physical property of an actual mechanical system. It is not difficult to deal with when the system is at rest, or when we confine its motion to certain very simple configurations. When we come, however, to deal with the problem of three or more bodies, it can have almost any meaning we like to attach to it. One common meaning in celestial mechanics is the limitation of one of the elements, say the velocity or mean distance, to certain limits. If we had any exact knowledge of what those limits were, there would be some satisfaction in using the term.

Let me give an illustration. One of the theorems often quoted is that which refers to the "stability" of the eccentricities. It is said that it can be shown that the eccentricities of the planets always lie between certain limits. What does this assume? The variables which define the motion are changed from the ordinary coordinates to new variables which are those defining the position, shape, and size of what is called the osculating ellipse. If there were only two bodies involved, this ellipse would have fixed position, shape, and size. When other bodies attract, these elements vary. Suppose we confine the variations to the first powers of the changes from constancy, as is done in the mathematical discussions on stability. Then the total change is the sum of all the individual changes. The theorem in question simply refers to the individual changes as calculated in this manner; it is the usual "mathematical stability." Each of them is periodic and therefore limited in magnitude. The proof, however, breaks down if we proceed to higher powers of the changes when there is approximate or exact resonance between two or more of the periods present. The limits between which the eccentricities can vary may become so wide as completely to alter the general configuration of the system; from the point of view of the physicist the motion is ultimately unstable.

In the ordinary discussions on stability we ask what is the result of slightly changing the position and velocity at any instant. In resonance problems we have to ask what happens when we introduce small periodic disturbing forces; at least it is in this way that the problems of celestial mechanics often present themselves. The physicist accustomed to the phenomenon in most of his work is able to say at once that the oscillations of the system increase in amplitude, and this indeed may be regarded as the first general effect of resonance. What happens later depends on the construction of the system. In terrestrial mechanisms we are accustomed to observe the early increase of amplitude and the apparent change of period which accompanies it. Frictional or damping forces are, however, always present, and since these affect resonance phenomena in a fundamental manner it is not easy to use familiar illustrations in order to visualize what will happen in the absence of friction. In atomic systems, provided the periodic disturbing force is not too small, the description usually is confined to a statement that the type of motion changes. Atomic systems under their modern descriptions have certain resemblances to the problems of celestial mechanics. But there is this difference. In the atomic problems we observe the aggregate or average effect of millions of revolutions of the electrons about the nucleus but have no method of observing the positions in the orbits at any one time. In the solar system, we can observe the latter, but so far have not been able to calculate the result of millions of revolutions.

Let us now proceed to the physical analogy which will serve to illustrate some of the phenomena of resonance in the solar system. It depends on the fact that the mathematical expressions are somewhat similar when reduced to their simplest forms. As with most analogies, however, the correspondence is not complete and must not be pushed too far. The analog to be used is that of the motion of the common pendulum. But the pendulum I have in mind differs from that attached to a clock in that it can make complete revolutions. The mechanism is perhaps better thought of as a bicycle wheel mounted on a stand with ball bearings so as to be as nearly frictionless as the imagination can make it; the actual description supposes that there is an entire absence of friction. To one point of the rim an iron weight is clamped. The wheel can be set in motion in various ways and can be acted on by different types of forces; some of them can be supposed to be impressed by a moving magnetic field which can act on the iron weight.

When no external force acts the motion of such a pendulum is well known. If it be given a small tap it oscillates in a period which depends on the amplitude; there is no such thing physically as an infinitesimal tap which gives a period independent of the amplitude. The greater the amplitude, the longer the period. The limit of the latter is infinite when the pendulum just reaches the highest point. The motion with a larger tap changes its character; the pendulum executes complete revolutions in either sense, with a variation of velocity between the highest and lowest points.

Mathematically, the motion is expressed by the equation

$$\ddot{x} + k^2 \sin x = 0,$$

where x is the angle between the position of the pendulum at any time and its equilibrium position at the lowest point. As long as the pendulum is oscillating, its motion is expressed by

$$x = \text{periodic function of } t, \text{ period } 2\pi/n.$$

When it is making complete revolutions, the motion is given by

$$x = \pm nt + A + \text{periodic function of } t, \text{ period } 2\pi/n.$$

In each case the period is an arbitrary constant of the solution depending on the initial impulse. It approximates to $2\pi/k$ only when the impulse is very small. Separating the two solutions is one in which x is a discontinuous (indeterminate) function of t at the highest point.

When the motion is near this critical case, a very small change in the initial conditions will not only completely alter the physical character of the motion but it will do so in a finite time. That is to say, the nearer the pendulum to the critical case, the shorter is the interval during which we can predict its position with a given degree of accuracy. The comparison here is with the non-critical case where the accuracy of prediction is more nearly proportional to the accuracy with which we can measure the constants.

Suppose that while the pendulum is executing small oscillations a periodic force with a period nearly equal to $2\pi/k$ acts. We know what happens. The oscillations begin to increase and, after the amplitude reaches a maximum, begin to decrease again, the process being repeated as long as we are able to observe it. A force of this type might be expressed by

$$\ddot{x} + k^2 \sin x = a \sin (kt + \alpha).$$

The phenomena of resonance in this case may be described by saying that the amplitude increases until the average period is sufficiently different from $2\pi/k$ as to change the relative phases by 180°, when the amplitude begins to decrease again owing to the external force opposing the motion.

But if we ask whether this process is repeated indefinitely, the answer is not so simple. We may regard the system as having two periods, $2\pi/k$ and another period which approximately measures the time between maximum amplitudes. The motion cannot be exactly recurrent unless these two periods are in the ratio of two whole numbers. When this happens the equation must have a periodic solution. This would evidently be a special case and we are desirous of finding out what happens under *any* initial conditions. A way out of the difficulty is to suppose that we can always find a periodic solution.

Suppose that the two frequencies mentioned above were k, k'. The last hypothesis demands that we shall always be able to find two integers, i, j, such that $k/k' = i/j$. Suppose that we have found such a case and that we either slightly alter the initial conditions or, what amounts to the same thing, our measures from observation of k or k'. The ratio k/k' would then be i'/j', where i', j' are two new integers.

Now while the difference $i/j-i'/j'$ may be very small, the theory of convergents in continued fractions teaches that $i-i'$ and $j-j'$ will be widely different, and, in general the smaller the difference between the two fractions, the greater will be the differences between their numerators and their denominators. Since the whole periods of the motions are $2\pi i/k$, $2\pi i'/k$ in the two cases, the period is discontinuous; there is a finite change in it corresponding to an infinitesimal change in the initial conditions.

It may still happen, however, that the amplitude of x always remains less than some given quantity. Suppose we attempt to solve the equation by means of infinite series of harmonic terms. The arguments will be of the form $(ik\pm jk')t+\text{const.}$, where i, j are integers. The process of integration gives coefficients of the form $a_{ij}\div(ik\pm jk')$, where a_{ij} diminishes to zero as i or j approach infinity. In general, the values of a_{ij} change only by an infinitesimal amount when k or k' similarly changes, so that it would seem always possible so to choose the measures of k, k' that the series is finite or that it shall be infinite. This means that the motion is either physically indeterminate within a finite time, or that our assumption of a solution oscillating between finite limits must be abandoned.

The problem is similar to that of the integral

$$\int\sum a_{ij}\sin(ikt\pm jk't+\alpha_{ij})dt, \qquad (i,j=1,2,\cdots,\infty),$$

where $\sum|a_{ij}|$ is convergent. The result is infinite or indeterminate as long as k, k' are measured quantities. The only possible conclusion appears to be that, except for special cases, at some time in its motion the pendulum will begin to make complete revolutions. Incidentally, it is interesting to note that the small amount of friction present in the motion of the most carefully constructed pendulum is necessary in order that it may furnish an accurate measure of time.

Let us next consider what happens when complete revolutions are under way. The difficulties caused by the assumption of a doubly infinite series still appear to be present, that is to say, we cannot assume that the motion is expressed by $x = k_1 t +$ an oscillating function of t. In general, however, the average speed of the complete revolutions will not continuously increase but will reach a maximum and then diminish again. When they have reached the point where oscillations can take place, there are two possibilities, namely, oscillations, or complete revolutions in the opposite sense. But since a limitation of the phase of the motion is necessary in order that oscillations may be set up, the reversal of the complete revolutions will be the usual phenomenon and only after many reversals will oscillation be again set up. These, after continuing for a time, will again be changed to complete revolutions, the maximum speed of which is apparently not limited, but will itself have oscillations the extent of which will be greater at certain intervals, the longer the motion continues.

The times occupied by these various phases of the motion will depend on the values of the constants present in the motion. We can get some guide to these from the general principle that the body will remain longest in that phase of motion which has the greatest degree of relative stability, that is, where small changes in the constants will produce the least effect in a given interval of time. These appear in general to be either the configurations where the oscillations are at their minimum amplitude or where the revolutions have their maxima of average speed, or at both. In certain cases of celestial mechanics in which the mechanism is too complicated for the use of the pendulum as an approximate analogue, small oscillations are impossible, and for these the phases corresponding to the maximum speeds of revolution will be the configuration in which most of the time will be spent.

The general description of the motion of a pendulum under a periodic force $a \sin (pt+\alpha)$ is not essentially different from that under a force

$$a \sin (x + pt + \alpha),$$

which resembles more closely some of the cases in celestial mechanics, especially when p/k is small. This latter is the case of a periodic force which has nearly the same period as the pendulum for all amplitudes except for those which approach complete revolutions. It can be shown that unless a/pk is small, the latter state is in general set up in an interval comparable with $2\pi/p$. A relatively stable configuration is not reached until the average angular velocity of revolution considerably exceeds p. When a/pk is small, the time required to produce complete revolutions may be very long. The distinction between the cases a/pk large and small appears to have an important bearing on the asteroid problem.

It has been stated that, in general, the motions which have the greatest relative stability are those in which the average speed of revolution considerably exceeds p. The ordinary processes of approximation show that in this motion, the periodic change in the average angular velocity as the pendulum goes round varies as k^2. As the speed diminishes, this amplitude tends to vary as the first power of k. In the applications to celestial mechanics, the unit of time is the period of revolution of the body,* and in all such cases k^2 is a quantity small compared with unity. Thus k^2 is small compared with k. Since resonance corresponds to the case where the pendulum is oscillating, it is convenient to define a "range" attached to the resonance and to measure it by k. Thus resonance phenomena take place within the range k; outside this range ordinary oscillations or perturbations, as they are usually called, occur.

I have dealt hitherto with the cases where the motion is analogous to a pendulum which can make very small oscillations when no disturbing force acts, and have mentioned the possibility of cases of motion in which this analogy breaks down to the extent that small oscillations are impossible, while oscillations of finite amplitude do appear to be possible. It is evident that in this event the change from oscillations to complete revolutions under disturbing forces of the form $a\sin(x+pt+\alpha)$ will take place more rapidly and that the time during which oscillations are under way will be relatively shorter. This result may be put in a form more useful for the applications by the statement that the time spent in oscillation compared with that spent in complete revolutions is very much shorter, and consequently over any long interval the chance of observing oscillations is smaller.

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* This must not be confused with the period of revolution of the pendulum.

In the whole of this discussion it is assumed that there is a complete absence of frictional forces. The moment we introduce them, however small in amount, the physical results are changed fundamentally. It would seem that the most familiar resonance effect—the identity of the periods of rotation about its axis and revolution around the earth of the moon—could not persist without large oscillations (librations) in the absence of small viscous forces in the moon's mass. The first effect of a non-uniform change in the period of revolution is to increase the amplitude of these free oscillations or librations. But the lunar bodily viscous forces tend to damp them down if the change is very slow. It follows as a corollary that once this resonance has been set up, it may continue indefinitely. Bodily viscosity in the moon's mass therefore changes a configuration which is unstable in the presence of small periodic disturbing forces into one in which the stability is permanent. Thus the moon will always turn the same face to the earth. A similar effect keeps the Eulerian nutations of the earth's axis small.*

My main object, however, is the consideration of resonances amongst planetary and satellite motions where we assume a complete absence of frictional forces, each body being treated as a particle with complete rigidity. I shall first consider an asteroid of very small mass moving in a planetary orbit round the sun, and acted on by planets of large masses, Jupiter and Saturn for example, the latter however, having masses small compared with that of the sun.

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* H. Jeffreys, *The Earth*, p. 248.

If the asteroid were not acted on by other planets it would describe an ellipse about the sun with constant period $2\pi/n$ and the angle described after any time would be given by

$$v = w + \text{periodic function of } w, \quad w = nt + \text{const.},$$

the distance being a periodic function of w having a finite mean value. The "elements" of the ellipse are usually taken to be w, the mean distance, the eccentricity, and three further elements which define the position of the ellipse, and the position of the asteroid in the ellipse at some given moment. When Jupiter acts on the asteroid, one useful method is to find how these elements vary. Suppose Jupiter is moving in a fixed ellipse, the mass of the asteroid being too small to affect the motion of Jupiter sensibly. Then the motion of Jupiter is expressible as a function of $w' = n't + \epsilon'$, the other elements being constant.

The only way we have to find the motion of the asteroid under these circumstances is to proceed by some method of continued approximation. When we do this, the first approximation to w is given by an equation of the form

$$\ddot{w} = mn^2 \sum i a_{ij} \sin(iw + jw' + \beta_{ij}),$$

where $i, j = 0, \pm 1, \pm 2, \cdots$, giving an infinite series. The magnitudes of the coefficients mainly depend on a parameter e which is of the order of the eccentricities and inclination. In general, a_{ij} is divisible by $e^{|i+j|}$. The factor m is the ratio of the mass of Jupiter to that of the sun and is about .001.

Having decided what accuracy we need, we cut off the series at some definite values of i, j. We then put $w = nt + \epsilon$,

$w'=n't+\epsilon'$ in the right hand member and integrate, obtaining

$$w = nt + \epsilon - mn^2 \sum \frac{ia_{ij}}{(in + jn')^2} \sin (iw + jw' + \beta_{ij}).$$

This method gives a good approximation so long as $in+jn'$ is not too small.

Suppose, however, that $in+jn'$ in one term is so small that the corresponding coefficient is very large and that even the oscillation of the mean angular velocity, w, is large. Then evidently the approximation is useless.

To avoid the difficulty put

$$iw + jw' + \beta_{ij} = x + \pi \text{ or } x$$

according as a_{ij} is positive or negative, and omit all the other terms as being small compared with this. The equation for x then takes the form

$$\dot{x} + k^2 \sin x = 0.$$

Since the integral of this equation is

$$\dot{x}^2 = b + 2k^2 \cos x,$$

the value of the angular velocity is limited whatever be the initial conditions. So far the presence of this resonance term produces no infinity. But the character of the motion is changed similarly to that of the pendulum when it changes from making oscillations to complete revolutions.

Suppose now that Saturn is present. It is large enough to affect the motion of Jupiter sensibly and introduces terms of the form $a \sin(x+pt+p')$ where $2\pi/p$ is a period which may be very long compared with the period of revolution of the asteroid. If this is simply an addition to w' its effect is not usually difficult to deal with for reasons into which I shall not enter. But there are certain terms which produce large periodic changes in the eccentricity of Jupiter—the so-called secular terms— and these terms appear to compel us to deal with equations of the form

$$\dot{x} + k^2 \sin x = a \sin (x + pt + \alpha).$$

As long as a/pk is small, we can continue our approximations, and it may be that the higher approximations will avoid the phenomena which have been explained earlier, although this is unlikely. In any case, however, the oscillation, if started with x small, may continue for long periods of time before x can reach the values $\pm\pi$. But if a/pk is large, the oscillations must soon turn into complete revolutions and continue until the asteroid is well outside the resonance range. Once there, as explained earlier, it will at least remain for a long time even if it can get back at all.

From the point of view taken here, the most important group of asteroids is that known by the name "Trojan," from the circumstance that all its six known members have received names famous in the Iliad of Homer. The mean period of each of these round the sun is at present the same as that of Jupiter, that is, the resonance is due to the equality of n, n', and consequently the resonance terms are those for which $i=-j$. The attendant circumstances, namely, that they are at the same mean distance from the sun and that they oscillate about the third vertices of the two equilateral triangles which have Jupiter and the sun as the other two vertices, are incidental to this discussion, except in so far as they render the discussion possible. Laplace, a century ago, showed that an ideal exact solution of the problem of three bodies was a configuration in which all of them remained at the corners of an equilateral triangle while describing ellipses of the same size and shape. It has further been shown that under infinitesimal displacements from this ideal configuration these deviations will remain infinitesimal, under conditions as to the magnitudes of the masses which are here satisfied.

Now the set of deviations in which we are interested corresponds to the case of the pendulum performing infinitesimal oscillations. But the observed deviations are far from being of this character; in the case of one asteroid they correspond to an amplitude of the order of 20° from the vertical position.

If we continue the approximations on the supposition that Jupiter moves in a fixed elliptic orbit, there is nothing to indicate, within the extent to which these approximations can be carried with a reasonable amount of labor, that the configuration is otherwise than stable for very long intervals. When, however, we introduce the action of Saturn, both directly on the asteroid and indirectly through its effect on Jupiter, we introduce large perturbations, chiefly because these two planets, and therefore Saturn and an asteroid, have their periods nearly in the ratio 2:5. It can be shown that the principal part of this action—that arising in the mean angles w, w'—is transferred directly to the asteroid. But there are other portions constituting slow changes in the eccentricity of Jupiter which introduce terms corresponding to the disturbing force $a \sin(x+pt+\alpha)$ in the motion of the pendulum. The number a/kp for these terms is small as long as the eccentricity of the asteroid is not too great. In the case of one asteroid, Achilles, I was able nevertheless to show* that a very slight change in the conditions would produce large oscillations (a secondary resonance) in its eccentricity—so large indeed that they might bring the asteroid to a close approach to Jupiter. Changes of this character are always furnished by the attractions of other planets, granted sufficient time, so that the present apparent stability of the orbit may be lost in the course of time.

The next great group is that of the 2:1 ratio. The first approximation for this group with Jupiter alone acting is such that small oscillations are impossible but oscillations with finite amplitude probably can exist. When we include the action of Saturn, the analogue of the case in which a/kp is large comes into play. We therefore expect that the time spent in oscillations is small compared with that spent in making complete revolutions. The "range" of the resonance is approximately $(0.0007e)^{1/2}$ where e is the average eccentricity.† If e be 0.1, this gives us a range of period for resonance conditions of the order of fifty days about the half period of Jupiter. Within this range very few asteroids should be found at any one time. There is another feature about this resonance point which can be deduced from the equation, namely, that oscillations can only be set up from complete revolutions in one sense; this of course statistically cuts down the probable number by one half.

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* Transactions, Yale Observatory, vol. 3, p. 114.

† Monthly Notices, Royal Astronomical Society, vol. 72, p. 619.

In all resonance cases except that of the 1:1 ratio the range of the eccentricities is large. In the case of the 2:1 group, in whatever manner the asteroid be started, the eccentricity at some time must be at least 0.2.* Near the edge of the range is a resonance point 5:1 with Saturn, and though the range of this new resonance point is much smaller, it will add to the improbability of finding an asteroid within the 2:1 range at any time. These results agree well with the statistical data. A considerable number of asteroids are known which can be well interpreted as having been nearly uniformly distributed near this resonance point, but there is a complete (with one or two doubtful exceptions) absence of asteroids within the range. Further, there appear to be more than the average number just outside the range.

The equations for all the resonance points $(1+i):i$ are quite similar to those of 2:1 with a slow decrease in range as i increases. There are, however, few asteroids near these ranges, so that statistical results are not properly applicable. But with higher values of i, two facts must be noted. In the first place, the resonance points become more crowded together as i increases and they approach the 1:1 resonance as a limit. In the second place, the mean distances approach that of Jupiter with increase of i and a circumstance of which little account has been yet taken, namely, a very close approach to Jupiter becomes more probable. Such a close approach fundamentally changes the orbit and more particularly the eccentricity, so that asteroids which have experienced this phenomenon must be regarded as belonging to a somewhat different statistical group. The above considerations, however, indicate that the chance of an asteroid remaining near these higher resonance points diminishes rapidly with the increase of i, or in other words, the action of Jupiter and Saturn will clear them away much earlier than those near the 2:1 ratio. Finally, while the resonance points $i{:}j$ for small values of i, j are those most likely to be effective in showing resonance phenomena, the values of n'/n just outside these resonance ranges (except the 1:1 case) are the most stable since, when the ranges are taken into account, these positions are the least crowded of all by resonance phenomena.

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* Loc. cit., Monthly Notices, p. 629.

Besides the asteroids, there is a family of comets which have also periods nearly half that of Jupiter. On account of their large eccentricities the methods used for the asteroids are not available. But it would seem that the same general results will hold. During the writing of this paper, the conclusions reached were illustrated in a publication by Professor A. O. Leuschner* on the Pons-Winnecke Comet. This comet in 1819 had a mean daily angular velocity round the sun of 632″ which increased to 639″ in 1858 and then diminished to 604″ in 1915. Between that date and 1927 it crossed the 2:1 resonance point of 599″ and is now 591″. This extraordinarily rapid change is partly "incidental" and due, as Leuschner points out, to the fact that the aphelion distance and phase are such that, during the last century, the comet has approached Jupiter very closely every other revolution. Theory suggests that these close approaches cannot persist through very many more revolutions. The point, however, namely that a large perturbation can carry a body rapidly through the resonance region, seems to be exemplified in this case.

Other resonance points within the group of asteroids are treated in the same general manner. In the case of the 3:1 group, small oscillations appear to be possible. The range of the resonance is smaller and the statistical material less plentiful with a smaller region of resonance; however, the disturbing effects have more power; in fact, in our pendulum analogy the terms of the right-hand member are of the same order of magnitude as those on the left, and, as stated above, the time which an asteroid can spend in this region is correspondingly less, giving a smaller chance of finding an asteroid there at any one time.

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* Publications of the Astronomical Society of the Pacific, October, 1927.

The rings of Saturn furnish valuable statistical material. Here the millions of "stones" or "rocks" which must constitute those rings revolve round Saturn and resonances are caused by the action of its larger satellites. There is little doubt that the well known divisions of the rings are caused by resonance action. But unless my main thesis is correct, namely that the time spent in oscillation within the resonance region is very short compared with the time spent outside, it is difficult to account for these divisions. Unfortunately we cannot at present observe the motion of individual members of the rings. We can, however, obtain a measure of the number at a given distance from Saturn by photometric observations and this number should be indicated also by the theory. The mathematical treatment, while very difficult and full of pitfalls, does not appear to be outside the range of present possibilities. It must, however, be carried out on practical lines, that is, with full consideration of the observational data.

I have up to now dealt with cases in which the integrals of the problem of three or more bodies are of little or no assistance on account of the small masses of the disturbed bodies in comparison with those of the bodies which produce the perturbations. Outside the integrals which define the motion of the center of mass of the system and which play no part in this discussion, there are only four known, or, if we confine the motion to one plane, only two, namely, the integrals of energy and angular momentum.

Since the portion contributed by a body to any one of these integrals has the mass of that body as a factor, it is at once evident that very small changes in the momentum or energy of a large body can change fundamentally those of the small bodies, as far as these integrals are concerned. From the point of view taken here, an exact integral constitutes a limitation on the range of variation. This may also be interpreted by the statement that the existence or near existence of an integral lengthens the time during which a given configuration can persist. When the masses present in the integral are of the same order of magnitude we can make use of it. Thus the mutual action of Jupiter and Saturn can be treated with the use of these integrals because the actions of the other planets in the solar system affect the integrals only to a comparatively small degree in a limited time. Similarly, in testing the mutual actions of the satellites of Jupiter and those of Saturn, we can also make use of these integrals. Temporarily stable resonance conditions are therefore more probable, relatively to the number of bodies under observation, than in the asteroid problem. And in fact, such a relation between the motions of three of the inner satellites of Jupiter is known to exist at the present time. And there is another, of a more complicated character, between two of the satellites of Saturn.

The case of Jupiter and Saturn is an interesting one in this connection, because the ratio of their periods is nearly 2:5. Another is that of Neptune and Uranus whose periods are nearly 2:1. While they are well outside the resonance "ranges," it is not possible to assert that they never have been or never will be within these ranges. Neither case permits small oscillations within the ranges. It is indeed entirely possible that they illustrate again the fact mentioned above, namely, that the positions which have the highest temporary stability are those just outside the largest resonance ranges. This fact has been pointed out as an observational result by previous workers. My argument here has been an attempt to show that these are probably not configurations of permanent stability, but merely those in which a much longer time will be spent than within the resonance regions.

A few words in reference to the application of the theory of resonance to the orbit of the moon may conclude this part of my subject. It has, owing to its proximity to the earth, received far more attention from the celestial mechanician than any other body, and its motion has been calculated to a high degree of accuracy. This calculated motion, involving at least ten arguments derived independently from observation, contains many hundreds of linear combinations of these arguments and thousands of other combinations have been examined. In no case has a combination giving a resonance with an observable coefficient been found. And yet, owing to tidal friction we know that its mean angular velocity is subject to a real secular change. On the law of chance alone we should not expect this result. If, however, the result obtained above is correct, namely, that a resonance condition with a large periodic perturbation moves the body away from the region rapidly, the chances of finding it there are much diminished. The changes in mean angular velocity due to tidal friction would therefore not be steady over periods measured by millions of revolutions but would contain intervals where small changes took place with comparative rapidity, the longer intervals of slow change being similar to that which characterizes the motion of the moon at the present time.

What general conclusions can we draw from these configurations? The first and the most obvious is that our knowledge of the past or future of the solar system must be a function of the interval within which it has been under observation, so far as the law of gravitation alone is concerned. The absence of a knowledge of any integrals of the problem of three or more bodies, beyond those known, is partly responsible for this result. If we try to represent the motion by oscillating functions we are met with the difficulty that these give an indeterminate result beyond a certain degree of accuracy. It is possibly true that there are no secular changes in the strict sense in any of the linear coordinates, but it seems equally true that, owing to resonance, we cannot lay down any limit to the amplitudes of these oscillations and thus limit the changes of the present configuration. There appears to be no regular increase in these amplitudes; it becomes rather a question of probability than of calculation as to whether any particular configuration will be fundamentally changed within a very long interval of time. It seems necessary that the smaller bodies of the solar system should have orbits which are under continuous development and this necessarily carries with it a continuous development of the whole system.

One way out of the difficulty which has been sometimes suggested is the adoption of a "quantum" frequency which would permit of complete periodicity. This frequency, however, would have to correspond to a period much longer than the probable past history of the solar system. To be logical it should extend to the whole material universe and should therefore be independent of the amount of matter present, that is, it should be a universal constant of nature—a property of space in the sense of Einstein. The assumption has philosophic interest only since it appears to have no practical bearing on the actual problem on account of the time factor involved and of other reactions of matter which are continually altering the conditions of motion.

This last remark is equally applicable to assumptions concerning the existence of periodic solutions corresponding to an arbitrary set of initial conditions; such solutions, if they exist, will have periods too long for us to be able to say that the non-gravitational forces have not had a sensible effect in the period. It does not seem possible to avoid the conclusion that small changes in the motion at any time will ultimately produce finite changes in the configuration, the amount of the change being a function of the interval, as well as of the other conditions. This amounts to the statement that the solar system, and indeed any gravitational system consisting of several bodies, is always in a state of development for gravitational reasons alone. The question is therefore shifted to that of the speed of this development, and I have attempted to show that in this speed the phenomena of resonance play a considerable part. Accumulated evidence goes to show that the history of the solar system from its formation into discrete bodies is much greater than 10^9 years. During such an interval it appears to be doubtful whether we can deduce much as to the initial configuration from the present configuration, except what the integrals of energy and angular momentum may furnish.

Certain of the present statistical distributions appear to be the results of chance in the same sense that the result of flipping a coin is said to be due to chance. In other words, a limited amount of information is to be obtained from considerations of probability. Thus the few known asteroids of the Trojan group are still there because the initial conditions of a large number presumed to have been there originally are such that the chance of still finding some there is not too small. We find few or none in the 2:1 and 3:1 ranges and perhaps in other resonance ranges because the chance of finding one under oscillation conditions is very small. The dark rings of Saturn are partly denuded of members of the system for the same reason. I am inclined to believe that the remarkable "families" of asteroid orbits deduced by K. Hirayama* will ultimately be found to be the results of perturbation effects due to resonance points with large ranges, rather than to some original configuration in which the bodies were all portions of some single body; in fact that "families" are due to the relatively stable regions just outside the resonance regions with large ranges. They are observable because the laws of chance indicate that at any one moment more asteroids will be found there than elsewhere.

It has already been pointed out that under resonance conditions, and especially those of the 2:1 ratio, the eccentricities will vary greatly. If a large number of bodies were started near such a region, the chance of one of them getting close to Jupiter on account of large eccentricity is greatly increased. Of these a certain proportion will have their eccentricities so greatly altered as to move afterwards in cometary orbits. The periods will, in general be much less changed. On this basis, the numerous short-period comets with periods nearly half that of Jupiter would be regarded as a succeeding stage of the evolution of asteroids rather than a previous stage as suggested by Leuschner in his paper on the Pons-Winnecke Comet already referred to. The close approach to Jupiter would turn loose surface material on the asteroid into satellite orbits about the asteroid and thus perhaps furnish the material for the "tail", further loosening of the material resulting from the temperature changes taking place at each near approach to the sun. Some evidence in favor of this hypothesis is furnished by the apparent dissolutions of two or more comets which have taken place within historic times. It is difficult to account for this last fact except on the hypothesis that the supply of comets is being renewed. The same mode of development should apply to the rings of Saturn. There, however, an eccentric "stone" is liable to collide with other members of the system before it gets very close to one of the satellites, and thus a certain average circularity would be preserved.

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*Japanese Journal of Astronomy, Vol. 1, No. 3.

The conclusion that the calculus of probabilities is more likely to lead to further information than the logical processes of exact analysis is perhaps a confession of defeat. For probability itself is here nothing but the chance of our being correct or incorrect in any statement, so long as we suppose that a determinate set of initial conditions will always produce a determinate result. When, however, we remember that our conditions are never exactly determinate in the mathematical sense on account of the inevitable errors of measurement, and that a very small change in the measurements will ultimately produce a finite change in the calculated configuration, this procedure is perhaps the only way in which we can deduce the needed information.

I am very fully aware of the unsatisfactory nature of an address supposed to be of a mathematical and therefore logical character which, at least partly, contains speculations as to mathematical or physical results which would seem to be capable of being logically deduced from the data. The reason for this procedure is the present condition of investigation in celestial mechanics. While a subject is rapidly advancing under the combined attacks of theory and experiment, speculations published without a sufficient background of observation are often harmful to progress because they diminish the interest of those who, equally capable of making the same speculations, take time to test them by experiment or calculation. But where a subject seems to have reached a condition where little progress is being made, and this I think is true as far as the gravitational development of the solar system is concerned, some speculation may be useful, if it indicates new avenues of approach or a more thorough exploration of old avenues. The speculations indulged in here have, however, a basis of physical and mathematical analogies and of experience gained from special examples. The processes followed have indicated how certain observed phenomena can be explained and have indicated also certain limits to the amount of information which can be obtained from the methods at present in use. The arguments lack precision; one question raised is whether it is possible to obtain precision. Our capacity to calculate and measure is limited, and it is this capacity which seems to be a part of the general question. In any case, it cannot be laid aside when we desire to obtain information as to the remote past or remote future of the solar system.

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SOME CANONICAL FORMS AND ASSOCIATED CANONICAL EXPANSIONS IN PROJECTIVE DIFFERENTIAL GEOMETRY*

BY E. B. STOUFFER

1. *Introduction.* A simplification of the methods of approach to any branch of mathematics is always very desirable. This is particularly true in a geometry, where extensive analytical machinery must be set up before geometric results can be obtained. This paper is a contribution to the simplification of Wilczynski's methods of attack upon plane and space curves in projective differential geometry.†

A canonical form for the fundamental differential equation associated with the curve is determined. It leads at once to a complete and independent system of invariants and covariants in their canonical form. The determination of the corresponding system in the general form involves only simple substitutions. An associated canonical expansion for the equation or equations of the curve is obtained by very direct methods and the geometrical significance of the corresponding triangle or tetrahedron of reference becomes easily evident. Because of the method of their derivation, the fundamental invariants and covariants obtained are those which have an immediately evident geometrical significance.

The methods here employed may be applied to surfaces, both curved and ruled in ordinary space and may also be extended to geometry in hyperspace. The resulting simplifications are in some cases quite remarkable. These results will be presented in later papers.

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* Part of a paper read upon invitation of the Program Committee at a meeting of the Southwestern Section of the Society, St. Louis, November 26, 1927.

† See Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, Chapters 2, 3 and 13.

In a series of papers* published in recent years, Sannia has made a study of plane and space curves by methods related somewhat to those used in this paper. He employs the absolute calculus but the results obtained below show that its use is unnecessary in order to gain real simplicity. Moreover, the methods of Sannia by no means extend themselves so naturally and so simply to surfaces in ordinary space and to hyperspace.

2. *Plane Curves*. It is well known that all the projective differential properties of plane curves may be studied by means of a single differential equation of the form

$$y''' + 3p_1y'' + 3p_2y' + p_3y = 0, \tag{A}$$

where differentiation is with respect to the independent variable x and where p_i are functions of x. If the homogeneous coordinates of a curve C are three functions $y_i(i=1, 2, 3)$ of x then the coefficients p_i may be so determined that y_i form a fundamental set of solutions of (A), provided merely that C is not a straight line. Moreover, any other set of fundamental solutions $z_i(i=1, 2, 3)$ of (A) may be expressed in the form

$$z_i = \sum_{j=1}^{3} c_{ij}y_j, \qquad (i = 1,2,3), \quad |c_{ij}| \neq 0.$$

Consequently all the curves determined by (A) in this way are projective transformations of C.

However, the parametric representation of C is not unique. A transformation of the independent variable of the form

$$\bar{x} = \phi(x), \tag{1}$$

and of the dependent variable of the form

$$y = \lambda\bar{y}, \tag{2}$$

where ϕ and λ are arbitrary functions of x, will not change the curve. Moreover, the form of (A) will not be changed by the above transformations but it will be converted into an equation with new coefficients. A function of the new coefficients and the new dependent variable and their derivatives which is equal, except for a factor, to the same function of the original coefficients and the original dependent variable and their derivatives is said to be a relative *covariant*. If the function does not contain the dependent variable or its derivatives, it is said to be an *invariant*.

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* Rendiconti dei Lincei, (5), vol. 31, 1° sem., pp. 450–454 and pp. 503–506; (5), vol. 31, 2° sem., pp. 17–19 and pp. 432–434; Annali di Matematica Pura ed Applicata, (4), vol. 1, pp. 1–18; (4), vol. 3, pp. 1–25.

The effect of the transformations is important. From (1) we have

$$(3)\quad \begin{cases} y' = \dfrac{dy}{d\bar{x}}\phi', \; y'' = \dfrac{dy^2}{d\bar{x}^2}(\phi')^2 + \dfrac{dy}{d\bar{x}}\phi'', \\[2ex] y''' = \dfrac{d^3y}{d\bar{x}^3}(\phi')^3 + 3\dfrac{d^2y}{d\bar{x}^2}\phi'\phi'' + \dfrac{dy}{d\bar{x}}\phi''', \end{cases}$$

whence, by substitution into (A), we obtain a new equation

$$(4)\quad \frac{d^3y}{d\bar{x}^3} + 3\bar{p}_1\frac{d^2y}{d\bar{x}^2} + 3\bar{p}_2\frac{dy}{d\bar{x}} + \bar{p}_3 y = 0,$$

where

$$(5)\quad \begin{cases} \bar{p}_1 = \dfrac{1}{\phi'}\left(p_1 + \dfrac{\phi''}{\phi'}\right), \\[2ex] \bar{p}_2 = \dfrac{1}{(\phi')^2}\left(p_2 + p_1\dfrac{\phi''}{\phi'} + \dfrac{\phi'''}{3\phi'}\right), \; \bar{p}_3 = \dfrac{1}{(\phi')^3}p_3. \end{cases}$$

Likewise the transformation (2) gives

$$(6)\quad \begin{cases} y' = \bar{\lambda}\bar{y}' + \bar{\lambda}'\bar{y}, \; y'' = \bar{\lambda}\bar{y}'' + 2\bar{\lambda}'\bar{y}' + \bar{\lambda}''\bar{y}, \\ y''' = \bar{\lambda}\bar{y}''' + 3\bar{\lambda}'\bar{y}'' + 3\bar{\lambda}''\bar{y}' + \bar{\lambda}'''\bar{y}, \end{cases}$$

whence by substitution into (4) we obtain a new equation in $\bar{y}$ whose coefficients $\overline{P}_i$ are expressed by the equations

$$(7)\quad \begin{cases} \bar{\lambda}\overline{P}_1 = \bar{\lambda}\bar{p}_1 + \bar{\lambda}', \; \bar{\lambda}\overline{P}_2 = \bar{\lambda}\bar{p}_2 + 2\bar{\lambda}'\bar{p}_1 + \bar{\lambda}'', \\ \bar{\lambda}\overline{P}_3 = \bar{\lambda}\bar{p}_3 + 3\bar{\lambda}'\bar{p}_2 + 3\bar{\lambda}''\bar{p}_1 + \bar{\lambda}''', \end{cases}$$

where differentiation is with respect to $\bar{x}$.

All of the above is well known theory.* However, for the purposes of the present paper it is important to notice that equations (3), (5), (6), (7) show the following significant fact. *If by means of* (1) *and* (2) *equation* (A) *is transformed into a canonical form which is preserved if and only if*

$$\phi'' = 0, \quad \frac{d\bar{\lambda}}{d\bar{x}} = 0, \tag{8}$$

then the new variable $\bar{y}$ *and the new coefficients* $\bar{P}_i$ *and their derivatives are determined except for multiplication by certain factors.*

The resulting variable $\bar{y}$ and its derivatives will each be the canonical form of a relative covariant and must each determine a unique point in the plane. Likewise, the resulting coefficients $\bar{P}_i$ and their derivatives must each be the canonical form of a relative invariant. We shall see later that the determination of the expressions for these invariants and covariants in terms of the coefficients and variables of (A) will involve only simple substitutions by means of equations (3), (5), (6), (7).

The determination of the transformations which give a canonical form to our equations may be made in many ways and many different canonical forms may be obtained. It is evident that each such canonical form must lead directly to an equation for the curve in the form of an expansion of one non-homogeneous coordinate in terms of the other. Since such an expansion determines the geometrical properties of the curve, we naturally desire that the canonical form of (A) be such as to simplify the process of obtaining these properties from the associated expansion.

We now assume that our fundamental differential equation has been transformed into such a canonical form with coefficients $\bar{P}_i$ and variables† $\bar{x}$ and $\bar{y}$ and proceed to calculate the corresponding canonical expansion. It is an important fact that we can proceed just as if the conditions to be imposed were actually known.

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* See Wilczynski, loc. cit., Chapter III.

† In the remainder of this paper a bar above a letter will indicate that it represents a function of $\bar{x}$ and that differentiation of the function is with respect to $\bar{x}$.

Let $\bar{y}_i (i=1, 2, 3)$ be the coordinates of a regular point* y on C. We may assume without loss of generality that this point y is given by the value $\bar{x}=0$ of the parameter. Then by the general theory of differential equations the coordinates $\bar{Y}_i$ of any point $\bar{Y}$ on the curve in the neighborhood of $\bar{y}$ may be expressed as a power series in $\bar{x}$ of the form

$$\bar{Y} = \bar{y}(0) + \bar{y}'(0)\bar{x} + \bar{y}''(0)\frac{\bar{x}^2}{2} + \bar{y}'''(0)\frac{\bar{x}^3}{3!} + \cdots.$$

When we start to substitute for $\bar{y}'''(0), \bar{y}^{\text{iv}}(0), \cdots$ by means of the assumed canonical form of our differential equation, we see at once that a great simplification will take place if $\bar{P}_1=0$. The first of equations (7) shows that we can make $\bar{P}_1=0$ by choosing $\bar{\lambda}$ to satisfy the equation $\bar{\lambda}\bar{p}_1+\bar{\lambda}'=0$. Moreover, equations (5) and (7) show that the condition $\bar{P}_1=0$ will be maintained if (8) is satisfied. We shall therefore assume that $\bar{P}_1=0$ is one of the conditions imposed in order to give us our canonical form. Thus the canonical form is of the nature

$$\bar{y}''' + 3\bar{P}_2\bar{y}' + \bar{P}_3\bar{y} = 0, \tag{9}$$

where the coefficients are determined by the substitution of $\bar{\lambda}' = -\bar{\lambda}\bar{p}_1$ into (7) and have the form

$$\left\{ \begin{aligned} \bar{P}_2 &= \bar{p}_2 - \bar{p}_1' - \bar{p}_1^2, \\ \bar{P}_3 &= \bar{p}_3 - 3\bar{p}_1\bar{p}_2 + 2\bar{p}_1^3 - \bar{p}_1''. \end{aligned} \right. \tag{10}$$

The substitution from (5) into (10) gives

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* Throughout this paper a point with coordinates $\alpha_i (i=1, 2, 3)$ in the plane or with coordinates $\alpha_i (i=1, 2, 3, 4)$ in space will be denoted simply by α when no confusion can arise.

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(11)\qquad \begin{cases} \overline{P}_2 = \dfrac{1}{(\phi')^2}\left(P_2 - \dfrac{2}{3}\psi' + \dfrac{1}{3}\psi^2\right), \\[2ex] \overline{P}_3 = \dfrac{1}{(\phi')^3}(P_3 - 3\psi P_2 + 3\psi\psi' - \psi^3 - \psi''), \end{cases}
$$

where $\psi = \phi''/\phi'$ and where P_2 and P_3 are the same functions of p_i as are $\overline{P}_2$ and $\overline{P}_3$ of $\bar{p}_i$. The expressions for the derivatives of $\overline{P}_2$ and $\overline{P}_3$ are obtained from (11) by simple differentiation. The particular value of ψ which gives the canonical form is as yet undetermined.

Substitution from (9) into the above expansion now gives*

$$
(12)\qquad \begin{cases} \overline{Y} = \bar{y} + \bar{y}'\bar{x} + \bar{y}''\dfrac{\bar{x}^2}{2} - (3\overline{P}_2\bar{y}' + \overline{P}_3\bar{y})\dfrac{\bar{x}^3}{3!} \\[1ex] \quad - [3\overline{P}_2\bar{y}'' + (3\overline{P}_2' + \overline{P}_3)\bar{y}' + \overline{P}_3'\bar{y}]\dfrac{\bar{x}^4}{4!} \\[1ex] \quad - [(6\overline{P}_2' + \overline{P}_3)\bar{y}'' + (2\overline{P}_3' + 3\overline{P}_2'' - 9\overline{P}_2^2)\bar{y}' \\[1ex] \quad + (\overline{P}_3'' - 3\overline{P}_2\overline{P}_3)\bar{y}]\dfrac{\bar{x}^5}{5!} \\[1ex] \quad - [(9\overline{P}_2'' + 3\overline{P}_3' - 9\overline{P}_2^2)\bar{y}'' \\[1ex] \quad + (3\overline{P}_3'' + 3\overline{P}_2'' - 36\overline{P}_2\overline{P}_2' - 6\overline{P}_2\overline{P}_3)\bar{y}' \\[1ex] \quad + (\overline{P}_3'' - 9\overline{P}_2'\overline{P}_3 - \overline{P}_3^2 - 3\overline{P}_2\overline{P}_3')\bar{y}]\dfrac{\bar{x}^6}{6!} + \cdots. \end{cases}
$$

Since we are assuming that we have a canonical form, $\bar{y}_i$, $\bar{y}_i'$, $\bar{y}_i''$ are the homogeneous coordinates of three fixed points not on a straight line. Consequently, the coordinates of any point in the plane may be expressed in the form $x_1\bar{y}_i + x_2\bar{y}_i' + x_3\bar{y}_i''$. It follows that the coordinates of the point may be taken to be (x_1, x_2, x_3). The coordinates x_i of $\overline{Y}$ in this new coordinate system may be read directly from equation (12) and have the form of expansions in powers of $\bar{x}$.

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* It is to be understood that in the coefficients of this expansion and those that follow $\bar{x} = 0$.

If we now pass to non-homogeneous coordinates we obtain the following results:

$$(13)\quad \begin{cases} \xi = \dfrac{x_2}{x_1} = \bar{x} - \dfrac{1}{2}\overline{P}_2\bar{x}^3 - \dfrac{1}{8}(\overline{P}_2' - \overline{P}_3)\bar{x}^4 \\ \qquad + (3\overline{P}_3' - 3\overline{P}_2'' + 9\overline{P}_2)\dfrac{\bar{x}^5}{5!} + \cdots, \\ \eta = \dfrac{x_3}{x_1} = \dfrac{\bar{x}^2}{2} - \dfrac{1}{8}\overline{P}_2\bar{x}^4 - (6\overline{P}_2' - 9\overline{P}_3)\dfrac{\bar{x}^5}{5!} \\ \qquad + (9\overline{P}_2^2 + 12\overline{P}_3' - 9\overline{P}_2'')\dfrac{\bar{x}^6}{6!} + \cdots. \end{cases}$$

The elimination of $\bar{x}$ from these equations gives us the desired expansion

$$(14)\quad \begin{cases} \eta = \dfrac{1}{2}\xi^2 + \dfrac{3}{8}\overline{P}_2\xi^4 + \dfrac{1}{40}(3\overline{P}_2' - 2\overline{P}_3)\xi^5 \\ \qquad + (9\overline{P}_2'' - 6\overline{P}_3' + 405\overline{P}_2^2)\dfrac{\xi^6}{6!} + \cdots. \end{cases}$$

We return now to the problem of the determination of our canonical form. It is easy to see from equations (11) that the term $3\overline{P}_2' - 2\overline{P}_3 = \bar{\theta}_3$ in the coefficient of ξ^5 in (14) is such that

$$\bar{\theta}_3 = \frac{1}{(\phi')^3}(3P_2' - 2P_3) = \frac{1}{(\phi')^3}\theta_3.$$

Consequently, assuming that θ_3 does not vanish identically,* we can make $\bar{\theta}_3 = 1$ by selecting $(\phi')^3 = \theta_3$. It is evident that the two conditions $\bar{\theta}_3 = 1$ and $\overline{P}_1 = 0$ are preserved only if $\phi'' = 0$, $\bar{\lambda}' = 0$. We then have for a canonical form of (A)

$$(15)\qquad \frac{d^3\bar{y}}{d\bar{x}^3} + 3\overline{P}_2\frac{d\bar{y}}{d\bar{x}} + \overline{P}_3\bar{y} = 0,$$

with $\bar{\theta}_3 = 1$, and for the associated canonical expansion

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* It is easily seen that the identical vanishing of θ_3 is the condition that C be a conic. The projective differential properties of conics are of no interest.

$$(16)\qquad \eta = \frac{1}{2}\,\xi^2 + \frac{3}{8}\,\overline{P}_2\xi^4 + \frac{1}{40}\,\xi^5 + \frac{9}{16}\,\overline{P}_2\,\xi^6 + \cdots .$$

The coefficients $\overline{P}_2$ and $\overline{P}_3$ and their derivatives are all canonical forms of relative invariants. In order to obtain their general form it is only necessary to substitute the value of ψ obtained from $(\phi')^3=\theta_3$ into equations (1) and their derivatives. That the expressions thus obtained are relative invariants becomes evident if we take any two equations of form (A) which are equivalent under (1) and (2) and reduce each of them to our canonical form. The coefficients thus obtained can differ only by a factor and consequently the expressions for these coefficients in terms of the original coefficients will have the same form and can differ only by a factor.

That the invariants thus obtained together with θ_3 form a complete system is evident. In fact, since $3\overline{P}_2' - 2\overline{P}_3 = 1$ we actually need only the invariants arising from $\overline{P}_2$ and its derivatives together with θ_3 in order to have a complete system. That all the invariants in the latter complete system are independent is also evident.

Exactly the same type of argument shows that $\bar{y}$, $\bar{y}'$, $\bar{y}''$ are the canonical forms of relative covariants which may be expressed in terms of the original coefficients and variables by means of (3), (5), (6), (7) with the now known values of ϕ'' and λ' substituted. That the covariants thus obtained are independent is obvious. That they form with the complete system of invariants a complete system of covariants follows from the fact that the higher derivatives of $\bar{y}$ than the second can be replaced by means of equation (15).

We have thus obtained a complete and independent system of invariants and covariants from our canonical form by a process which involves nothing more difficult than direct substitutions.

The three fundamental covariants $\bar{y}$, $\bar{y}'$, $\bar{y}''$ determine precisely the three vertices of the tetrahedron of reference. The geometric determination of these vertices is not difficult and gives results obtained by Wilczynski and Sannia. Equation (16) gives, by the method of undetermined coefficients, for the equation of the conic osculating C at $\bar{y}$

$$2x_3(x_1 - (3/2)\bar{P}_2x_3) - x_2^2 = 0, \tag{17}$$

and for the equation of the cubic osculating C at $\bar{y}$ and having a node at that point

$$10x_2x_3(x_1 - (3/2)\bar{P}_2x_3) - 5x_2^3 - 2x_3^3 = 0. \tag{18}$$

The tangent at (1, 0, 0), which is the point $\bar{y}$, to one branch of the cubic (18) is the line $x_2=0$ and the pole of this line with respect to the conic (17) is (0, 1, 0). Since (0, 1, 0) is the point $\bar{y}'$ and (0, 0, 1) is the point $\bar{y}''$, it is evident that the latter vertex is the point of intersection of the line $x_2=0$ with the tangent at $\bar{y}'$ to the curve generated by $\bar{y}'$.

One of the tangents from (0, 1, 0) to the conic (17) is $x_3=0$. The other tangent meets the cubic (18) in three distinct points, any one of which points may be used to determine the unit point of our system of coordinates.

Equation (15) shows at once the geometrical significance of the vanishing of the two fundamental invariants $\bar{P}_2$ and $\bar{P}_3$. If $\bar{P}_3=0$ the curve generated by the point $\bar{y}'$ is a plane curve. If $\bar{P}_2=0$ the tangent at $\bar{y}''$ to the curve generated by that point passes through $\bar{y}$, that is, the curve generated by $\bar{y}''$ is the envelope of the line $x_2=0$.

It is interesting to note at this point that essentially the canonical expansion used by Wilczynski is obtained if the vertex (0, 0, 1) is transformed to the second point of intersection of the line $x_2=0$ with the osculating conic.

All the machinery necessary for the study of the projective differential properties of plane curves has now been set up.

3. *Space Curves.* The projective differential properties of space curves can be studied by means of a differential equation of the fourth order of the form

$$y^{(iv)} + 4p_1y''' + 6p_2y'' + 4p_3y' + p_4y = 0. \tag{B}$$

The transformations of the variables which are permitted are exactly (1) and (2). In fact the methods employed and the equations derived in order to obtain the canonical expansion are obvious extensions of the methods and equations for plane curves. In place of equation (14) above we obtain in this case two equations

$$(19)\quad \begin{cases} \eta = \dfrac{1}{2}\xi^2 - \dfrac{1}{4}\overline{P}\xi^4 - 2(3\overline{P}_2' - 8\overline{P}_3)\dfrac{\xi^5}{5!} \\ \qquad - (6\overline{P}_2'' - 16\overline{P}_3' + 10\overline{P}_4 - 36\overline{P}_2{}^2)\dfrac{\xi^6}{6!} + \cdots, \\ \zeta = \dfrac{1}{6}\xi^3 - \dfrac{1}{20}\overline{P}_2\xi^5 - \dfrac{1}{180}(3\overline{P}_2' - 14\overline{P}_3)\xi^6 + \cdots. \end{cases}$$

In these equations $\overline{P}_2$, $\overline{P}_3$, $\overline{P}_4$ are the coefficients of the canonical form of equation (B) concerning which we know as yet only that $\overline{P}_1 = \bar{p}_2 - \bar{p}_1' - \bar{p}_1^2 = 0$. The condition which shall be imposed to produce $\bar{p}_i$ from p_i remains to be determined. In the above equations

$$\xi = \frac{x_2}{x_1}, \quad \eta = \frac{x_3}{x_1}, \quad \zeta = \frac{x_4}{x_1},$$

where the homogeneous coordinates (x_1, x_2, x_3, x_4) of a point in space follow from the coordinates of the point expressed in the form

$$x_1\bar{y}_i + x_2\bar{y}_i' + x_3\bar{y}_i'' + x_4\bar{y}_i'''.$$

Equations (19) do not indicate directly the conditions to be imposed in order to obtain a canonical form. In order to link our canonical form with the geometry we shall first calculate the osculating cubic, the simplest osculating curve other than the tangent line. Since a space cubic is determined by six points, the canonical expansion for the cubic osculating C at $\bar{y}$ must agree with (19) up to and including the fifth powers of ξ.

It is easily seen that the parametric representation of the osculating cubic can be put into the form

$$(20)\qquad \begin{cases} x_1 = 1 + a_1 t + b_1 t^2 + c_1 t^3, \\ x_2 = t + b_2 t^2 + c_2 t^3, \\ x_3 = b_3 t^2, \\ x_4 = c_4 t^3. \end{cases}$$

From these equations the expansions for $\eta = x_3/x_1$ and $\zeta = x_4/x_1$ in terms of $\xi = x_2/x_1$ are easily obtained. If the coefficients of the necessary powers of ξ are equated to the corresponding powers in (19), we obtain for the parametric equations of the osculating cubic

$$(21)\qquad \begin{cases} x_1 = 1 + \dfrac{9}{10}\overline{P}_2 t^2 - \dfrac{1}{30}(3\overline{P}_2' - 8\overline{P}_3)t^3, \\ x_2 = t + \dfrac{7}{10}\overline{P}_2 t^3, \qquad x_3 = \dfrac{1}{2}t^2, \qquad x_4 = \dfrac{1}{6}t^3, \end{cases}$$

and for the corresponding expansions for the cubic,

$$(22)\qquad \begin{aligned} \eta &= \frac{1}{2}\xi^2 - \frac{1}{4}\overline{P}_2\xi^4 - 2(3\overline{P}_2' - 8\overline{P}_3)\frac{\xi^5}{5!} + \frac{1}{200}\overline{P}_2{}^2\xi^6 + \cdots, \\ \zeta &= \frac{1}{6}\xi^3 - \frac{1}{20}\overline{P}_2\xi^5 - \frac{1}{90}(3\overline{P}_2' - 8\overline{P}_3)\xi^6 + \cdots . \end{aligned}$$

It now becomes evident that the projections of C and the osculating cubic from (0, 0, 0, 1), or $\bar{y}'''$, upon the osculating plane $x_4 = 0$ will have contact of the sixth order if and only if the coefficients of ξ^6 in the first equation of (19) and the first equation of (22) are equal, i.e., if and only if

$$\overline{M} = 15\overline{P}_2'' - 40\overline{P}_3' + 25\overline{P}_4 - 81\overline{P}_2{}^2 = 0.$$

But the equations for the space curve corresponding to (11) for the plane curve show that

$$\overline{M} = \frac{1}{(\phi')^4}(M + 15\psi\theta_3),$$

where*

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* The expressions for $\overline{P}_i$ in terms of $\bar{p}_i$ are obtained by the substitution of $\bar{\lambda}' = -\bar{\lambda}\bar{p}_1$ into the equations for the space curve corresponding to (7)

$$M = 15P_2'' - 40P_3' + 25P_4 - 81P_2^2, \quad \theta_3 = 3P_2' - 2P_3.$$

It follows at once that, if $\theta_3 \neq 0$, we can make $\overline{M}=0$. The conditions $\overline{M}=0$ and $\overline{P}_1=0$ determine our canonical form.

The coefficients $\overline{P}_2$, $\overline{P}_3$, $\overline{P}_4$ and their derivatives are the canonical form of relative invariants. There exist, of course, the relations obtained from $\overline{M}=0$ and from the differentiation of this equation. The expressions for the invariants in terms of the original coefficients again involve only simple substitutions, use being made of the now known values of ψ and $\overline{\lambda}'$. The questions of the completeness and of the independence of the system are answered precisely as for plane curves.

The expressions $\bar{y}$, $\bar{y}'$, $\bar{y}''$, $\bar{y}'''$ are canonical covariants which may be expressed in terms of the original coefficients and variables by direct substitutions. The covariants thus obtained, together with a complete system of invariants, give a complete system of covariants.

The vertices of the tetrahedron of reference are given by the four fundamental covariants $\bar{y}$, $\bar{y}'$, $\bar{y}''$, $\bar{y}'''$. If $\theta_3 \neq 0$, the plane $x_3=0$ is evidently the principal plane* of C and its osculating cubic. The osculating cubic meets $x_3=0$ in $(1, 0, 0, 0)$ and also in the point $(3\overline{P}_2' - 8\overline{P}_3, -21\overline{P}_2, 0, -5)$. The osculating plane to the cubic at this point intersects the tangent to C at $\bar{y}$ in the point $(0, 1, 0, 0)$. The totality of osculating planes to the cubic intersects $x_4=0$ in lines which envelope a conic, called the osculating conic, whose equation is

$$72\overline{P}_2 x_3^2 - 40x_1x_3 + 15x_2^2 = 0, \quad x_4 = 0.$$

The polar of $(0, 1, 0, 0)$ with respect to this conic is the line $x_2=0, x_4=0$. The tangent at $(0, 1, 0, 0)$ to the curve generated by $\bar{y}'$ intersects $x_2=0$, $x_4=0$ in $\bar{y}''$ which is $(0, 0, 1, 0)$, and

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for the plane curve. The equations can be found in Chapter 13 of Wilczynski's book. The expressions for P_i in terms of p_i are of the same form as those for $\overline{P}_i$ in terms of $\bar{p}_i$.

* Halphen, Journal de l'Ecole Polytechnique, vol. 28, p. 25.

the tangent at this point to the curve generated by $\bar{y}''$ intersects $x_3=0$ in $\bar{y}'''$ which is (0, 0, 0, 1). The vertices of the tetrahedron of reference are thus completely determined geometrically. These are essentially the results obtained by Wilczynski and Sannia for the location of the vertices.

The canonical form of (B),

$$\bar{y}^{(iv)} + 6\bar{P}_2\bar{y}'' + 4\bar{P}_3\bar{y}' + \bar{P}_4\bar{y} = 0$$

shows at once the geometrical significance of the vanishing of the fundamental invariants $\bar{P}_2$, $\bar{P}_3$, $\bar{P}_4$. In fact, the curve generated by $\bar{y}'$ is plane if and only if $\bar{P}_4=0$, and the tangent at $\bar{y}'''$ to the curve generated by this point intersects the line joining $\bar{y}$ and $\bar{y}''$ if and only if $\bar{P}_3=0$ and the line joining $\bar{y}$ and $\bar{y}'$ if and only if $\bar{P}_2=0$.

It is easy to make a transformation of coordinates which gives the canonical expansion used by Wilczynski. It is only necessary to change the vertex (0, 0, 1, 0) into the point where the line $x_2=0$, $x_4=0$ intersects the osculating conic and the vertex (0, 0, 0, 1) into the point where the osculating cubic intersects $x_3=0$.

If $\theta_3=0$, the principal plane of C and its osculating cubic is the osculating plane. This fact becomes evident if we observe that the coefficients of ξ^6 are equal in the second expansions of (19) and (22) if $\theta_3=0$. In this case it is easy to verify that the tangent plane at $\bar{y}$ to the quadric osculating C at $\bar{y}$ is the plane $x_3=0$ if

$$\bar{\theta}_4=\bar{P}_4-2\bar{P}_3'+(6/5)\bar{P}_2''-(81/25)\bar{P}_2^2=1.$$

Since $\bar{\theta}_4=\theta_4/(\phi')^4$, we can make $\bar{\theta}_4=1$ by choosing $(\phi')^4=\theta_4$, if $\theta_4\neq 0$. A canonical form for (A) is thus determined in this case and associated with it, just as above, we have two expansions, and a geometrically determined tetrahedron of eference.

If both θ_3 and θ_4 vanish, the curve C is a cubic and has no projective differential properties of interest.

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ON THE SEPARATION OF THE PLANE BY A CONTINUUM*

BY R. L. MOORE

In his paper, *On the separation of the plane by irreducible continua*,† W. A. Wilson obtains the following theorem. Let F be the union of two bounded continua H_1 and H_2 having these properties: H_1 and H_2 are irreducible about $A+B$; $H_1 \cdot H_2 = A+B$ where A and B are continua and $A \cdot B = 0$; H_1 and H_2 contain subcontinua C_1 and C_2 respectively such that $\alpha = C_1 \cdot C_2 \cdot A \neq 0$, $\beta = C_1 \cdot C_2 \cdot B \neq 0$, C_1 and C_2 are irreducible between α and β and $F = C_1 + C_2$. Then F cuts the plane and is the frontier of exactly two components of its complement.

In the present paper I will establish two theorems which together yield more information than Wilson's theorem.

THEOREM 1. *If A and B are two mutually exclusive continua and the bounded continua H_1 and H_2 are both irreducible about*‡ $A+B$, *and* $H_1 \cdot H_2 = A+B$, *then* $H_1 + H_2$ *is not the boundary of more than two domains.*

PROOF. Let F denote the point set $H_1 + H_2$. With the aid of the fact that $H_1 - (A+B)$ and $H_2 - (A+B)$ are§ connected, it may be easily seen that there exists an inversion of the plane about some point of the complement of F such that if, for each point set M, $\overline{M}$ denotes the image of M under this inversion, then no bounded complementary domain of $\overline{A}$ or $\overline{B}$ contains a point of $\overline{F}$. If, to each of the continua $\overline{A}$ and $\overline{B}$, all its bounded complementary domains are added and

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† This Bulletin, vol. 33 (1927), pp. 733–744.

‡ If A is a closed subset of a continuum C and no proper subcontinuum of C contains A then C is said to be irreducible about A. See W. A. Wilson, loc. cit., Definition I.

§ See W. A. Wilson, loc. cit., Lemma I.

the continuum so obtained is regarded as an element and each point that does not belong to A or to B is regarded as an element, the collection G of elements so obtained is an upper semi-continuous* collection. Furthermore, no element of the collection G separates the plane and together they fill up the plane. It follows that, if the elements of G are regarded as points, all theorems of the analysis situs of the plane hold true, in the space so obtained, in the sense described, in detail, in my paper *Concerning upper semi-continuous collections of continua.*† Hence, by a theorem used in a similar connection in the proof of Theorem 2, there exists a simple closed curve J of elements of G which separates $\overline{H}_1-(\overline{A}+\overline{B})$ from $\overline{H}_2-(\overline{A}+\overline{B})$. If, for each value of i $(i=1,2), t_i'$ denotes the segment of J whose extremities are the elements of G that contain $\overline{H}_1$ and $\overline{H}_2$, respectively, and D_i denotes the complementary domain of $\overline{F}$ that contains t_i, then, if $\overline{F}$ is the boundary of a domain, that domain must contain a point of t_1 or of t_2, and therefore must be identical with D_1 or with D_2.

THEOREM 2. *If α and β are two mutually exclusive closed point sets and each of the bounded continua C_1 and C_2 is irreducible from* ‡ *α to β, and $C_1 \cdot C_2=\alpha+\beta$, then C_1+C_2 is the boundary of at least two distinct domains.*

PROOF. Let F denote the point set C_1+C_2. With the help of the fact that $C_1-(\alpha+\beta)$ and $C_2-(\alpha+\beta)$ are§ connected

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* See my paper, *Concerning upper semi-continuous collections of continua which do not separate a given continuum,* Proceedings of the National Academy, vol. 10 (1924), pp. 356–360.

† Transactions of this Society, vol. 27 (1925), pp. 416–428. In the first line of the statement of Theorem 24 on page 427 of this paper, the word "bounded" should be inserted between the words "closed" and "point".

‡ If K and L are two closed and mutually exclusive point sets, the continuum H is said to be irreducible from K to L if it contains at least one point of K and at least one point of L but contains no proper subcontinuum that does so. See Anna M. Mullikin, *Certain theorems relating to plane connected sets,* Transactions of this Society, vol. 24 (1922), pp. 144–162. Wilson uses the term *irreducible between* instead of *irreducible from.*

§ See Anna M. Mullikin, loc. cit.

it may be shown that there exists an inversion of the plane, about some point of the complement of F, such that, using the notation indicated in the above proof of Theorem 1, no bounded complementary domain of any maximal connected subset of $\bar{\alpha}+\bar{\beta}$ contains any point of $\bar{F}$. To each maximal connected subset X of $\bar{\alpha}$ add all the bounded complementary domains of X and call each point set so obtained an element. Let G_α denote the set of all such elements. Let G_β denote a set of elements determined in the same way from the maximal connected subsets of $\bar{\beta}$. Let G denote the upper semi-continuous collection whose elements are the elements of G_α, the elements of G_β, and the points that belong to no element of G_α or G_β. Let H_1 denote the set of elements $\overline{C}_1-(\bar{\alpha}+\bar{\beta})+G_\alpha+G_\beta$ and let H_2 denote the set $\overline{C}_2-(\bar{\alpha}+\bar{\beta})+G_\alpha+G_\beta$. The set of elements $G_\alpha+G_\beta$ is a totally disconnected set of elements and it consists of all elements common to the two closed, connected and bounded sets H_1 and H_2. Let Q denote a simple closed curve enclosing H_1+H_2. There exists an arc AB such that (1) A belongs to Q, (2) B belongs to $\overline{C}_1-(\bar{\alpha}+\bar{\beta})$ or to $\overline{C}_2-(\bar{\alpha}+\bar{\beta})$ and (3) B is the only element of AB that belongs to H_1+H_2. Suppose B belongs to $\overline{C}_1-(\bar{\alpha}+\bar{\beta})$. Then $\overline{C}_2-(\bar{\alpha}+\bar{\beta})$ lies in a bounded complementary domain of $Q+AB+H_1$. Call this domain D. There exists* a simple closed curve J of elements of G such that (1) J contains at least one element of $G_\alpha+G_\beta$ and encloses at least one element of $\overline{C}_2-(\bar{\alpha}+\bar{\beta})$, (2) $(J+G_\alpha+G_\beta)-(G_\alpha+G_\beta)$ is a subset of D and it contains no element of $\overline{C}_2-(\bar{\alpha}+\bar{\beta})$. With the help of the fact that $\overline{C}_2-(\bar{\alpha}+\bar{\beta})$ is connected and that every element of $G_\alpha+G_\beta$ is a limit element of $\overline{C}_2-(\bar{\alpha}+\bar{\beta})$ it easily follows that J contains $G_\alpha+G_\beta$ and encloses the whole of $\overline{C}_2-(\bar{\alpha}+\bar{\beta})$. Furthermore, $\overline{C}_1-(\bar{\alpha}+\bar{\beta})$ is wholly without J. It is easy to see that the curve J contains two segments t_1 and t_2 such that (1) for each i $(i=1, 2)$ t_i contains no element of G_α or of G_β but the end elements of t_i belong to G_α and to G_β respectively and

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* See this Bulletin, vol. 33 (1927), p. 521, Abstract 38.

(2) t_1 and t_2 belong to different complementary domains of H_1+H_2. The segments t_1 and t_2 are point sets in the ordinary sense* and they are connected subsets of the complement (in the ordinary sense) of $\overline{F}$ and each of the sets $\bar{\alpha}$ and $\bar{\beta}$ contains at least one limit point of t_1 and at least one limit point of t_2. In the space of the ordinary points of the plane, for each i let D_i denote the complementary domain of $\overline{F}$ that contains t_i. The continuum $\overline{F}$ is the boundary both of D_1 and of D_2. For let N denote the boundary of D_1. If N does not contain $\overline{C}_2$ then $N\cdot\overline{C}_2$ contains no connected subset containing both a point of some element of G_1 and a point of some element of G_2. Hence there exists an arc A_1A_2 from a point A_1 of t_1 to a point A_2 of t_2 and lying, except for these points, wholly in the interior of J and having no point in common with $N\cdot\overline{C}_2$. Since $\overline{C}_1-(\bar{\alpha}+\bar{\beta})$ lies wholly without J it has no point in common with A_1A_2. Thus A_1A_2 has no point in common with N. But A_1 and A_2 belong to different complementary domains of N. Thus the supposition that N does not contain $\overline{C}_2$ has led to a contradiction. In a similar way it may be shown that N contains $\overline{C}_1$. Therefore $\overline{F}$ is the boundary of D_1. By a similar argument it may be shown to be the boundary of D_2. It follows that F is the boundary of at least two domains.

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* This does not imply that they are necessarily subsets of *arcs* whose elements are all points in the ordinary sense. It is possible, for example, that t_1 may be the set of all points of the graph of $y=\sin(1/x)$ that lie between the lines $x=0$ and $x=1$. This point set is not a subset of any arc whose elements are all points but it is a segment of an arc of which the only element which is not a point is the straight line interval from $(0,1)$ to $(0,-1)$.

A CHARACTERISTIC PROPERTY OF A CO-SET

BY G. A. MILLER

In view of the fundamental importance of co-sets in the theory of groups it may be of interest to note here the following characteristic property of a co-set. *A necessary and sufficient condition that a set of group operators* $s_1, s_2, \cdots, s_l$, *which is not itself a group be a right co-set is that it involve at least one operator such that if the inverse of this operator is multiplied on the right into every operator of the set the resulting products constitute a group.* In this theorem the word right may be replaced by the word left. A proof of this theorem results directly from the fact that if the two sets

$$s_1 s_\alpha^{-1},\ s_2 s_\alpha^{-1},\ \cdots,\ s_l s_\alpha^{-1}, \quad \text{and} \quad t_1,\ t_2,\ \cdots,\ t_l,$$

where α is one of the numbers $1, 2, \cdots, l$, are identical and constitute a group, then the latter set multiplied on the right by s_α is a right co-set. It may be noted that if the former of these sets is a group for one value of α it must be the same group for every one of these values.

It should also be noted that if the operators $s_1, s_2, \cdots, s_l$ satisfy the condition that when every one of them is multiplied on the right by the inverse of one of them, the resulting operators constitute a group, then they must also constitute a group when every one of them is multiplied on the left by the inverse of one of them, and vice versa. A necessary and sufficient condition that these two groups be the same is that at least one of the operators $s_1, s_2, \cdots, s_l$ transform the group $t_1, t_2, \cdots, t_l$ into itself, and if this condition is satisfied then this group is transformed into itself by every one of the operators $s_1, s_2, \cdots, s_l$. It must therefore be an invariant subgroup of the group generated by these operators.

If the set of operators $s_1, s_2, \cdots, s_l$ has the property that $s_1 s_\alpha, s_2 s_\alpha, \cdots, s_l s_\alpha$ is a group for a certain value of α then

s_α^{-1} must be found in this set. Hence it results from what was said above that the set is a right co-set which involves the inverse of one of its operators. It does not follow, however, that $s_1s_\alpha, s_2s_\alpha, \cdots, s_ls_\alpha$ is a group for every value of α from 1 to l if it is a group for some one of these values. If it is a group for every one of these values then this group must be a subgroup of index 2 under the group generated by $s_1, s_2, \cdots, s_l$. Moreover, this set must include the inverse of every one of its operators. It is obvious that similar remarks apply to the set if $s_\alpha s_1, s_\alpha s_2, \cdots, s_\alpha s_l$ constitute a group.

If a set of group operators $s_1, s_2, \cdots, s_l$ has the property that it as a whole is unchanged when it is multiplied on the right by an operator t which is not necessarily contained in the set, then it is composed either of co-sets or of augmented co-sets with respect to the group generated by t. If it is also unchanged when it is multiplied on the right by an operator t' which is not generated by t, then it must be composed either of co-sets or of augmented co-sets with respect to the group generated by t and t'. In general, if the set is unchanged as a whole when it is multiplied on the same side successively by various operators then it must be composed either of co-sets or of augmented co-sets with respect to the group generated by these multipliers. In particular, l mus tbe divisible by the order of the group generated by these multipliers. It should be noted that this is an extension of Lagrange's fundamental theorem, since in the case when $s_1, s_2, \cdots, s_l$ is a group it is obvious that it as a whole is unchanged when it is multiplied by the generators of an arbitrary subgroup, and hence the order of a group must be divisible by the order of each of its subgroups according to the more general theorem just noted.

When the multiplying operators are contained in the set $s_1, s_2, \cdots, s_l$ it results from the preceding paragraph that the set includes the group generated by the multipliers. If it is not a group itself it must therefore be composed of augmented co-sets with respect to the group generated by these multipliers. It may be noted that these considerations are frequently useful in determining whether a given set of operators constitutes a group when we can readily find whether the set involves the group generated by certain of its operators. The fact that a set of operators which involves the square of every one of them and also the product of every pair of them does not necessarily constitute a group results directly from operators involved in the symmetric group of order 6. It is necessary to add that it involves these products when the factors are taken in both of the possible orders.

From what precedes it results that *if a set of operators has the property that it as a whole remains unchanged when all the operators of this set are multiplied on the same side by various operators, then it either contains all the operators of the group generated by these multipliers or it contains no operator of this group.* In the former case it is either this group or it consists of augmented co-sets with respect to it, while in the latter case it must be composed of one or more co-sets with respect to the same group. Hence, a characteristic property of co-sets and of augmented co-sets is that the totality of the operators involved therein remains unchanged when all of these operators are multiplied on the same side by a given operator. If such a totality is composed of right co-sets or augmented right co-sets it is not necessarily also composed of left co-sets or of augmented left co-sets.

THE UNIVERSITY OF ILLINOIS

A NEW NORMAL FORM FOR QUARTIC EQUATIONS*

BY RAYMOND GARVER

1. *Introduction.* There are a number of well known normal forms to which the general quartic equation may be reduced by means of a Tschirnhaus transformation, without requiring the solution of any equation of higher than the third degree. The reduction to $y^4+c_3y+c_4=0$ or to $y^4+c_2y^2+c_4=0$ requires no comment. Bring first showed how to obtain the form $y^4+c_1y^3+c_4=0$, by applying a simple reciprocal transformation (which is reducible to a Tschirnhaus transformation) to the first form above.† The reduction to the binomial form $y^4+c_4=0$ by an ordinary third degree transformation leads to a sixth degree equation, but Lagrange was able to show (by an a priori proof) that this sextic will factor into three quadratics whose coefficients are themselves roots of cubic equations.‡ The required factorization would, however, be difficult actually to obtain; the necessary transformation may be obtained more conveniently in another way.§

I wish in this paper to discuss a new normal form, $y^4+c_2y^2=0$. I first give an a priori proof, based on that of Lagrange for the transformation to binomial form, that the reduction does not involve the solution of any equation of higher than the third degree. Then, using a theorem of Hermite and certain results of Cayley, I consider the setting up of the necessary transformation.

2. *The a Priori Proof.* If we apply the transformation

$$(1) \qquad y = x^3 + k_2x^2 + k_3x + k_4$$

to the general quartic equation

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* Presented to the Society, April 7, 1928.

† Bring's work was published in 1786. It is reproduced in Grunert's Archiv, vol. 41 (1864), pp. 105–112.

‡ See his *Oeuvres*, vol. III, Paris, 1869, pp. 284–295. His result was originally published in 1770.

§ See this Bulletin, vol. 33 (1927), No. 6.

$$x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0, \tag{2}$$

the conditions that the transformed equation in y lack its second, fourth and fifth terms are, as can easily be verified by making use of Newton's relations,

$$\sum y = 0,\ \sum y^3 = 0,\ \left(\sum y^2\right)^2 - 2\sum y^4 = 0. \tag{3}$$

But the first of these equations is linear in k_2, k_3, k_4, the second is of the third degree, while the third is of the fourth degree. Hence if we eliminate two of the k_i, say k_3 and k_4, between these equations, we shall expect to be led to a final equation of the twelfth degree in k_2.

The problem is to show that k_2 can be found without requiring the solution of any equation of higher than the third degree. To do this, it is necessary to express k_2 as a function of the roots x_1, x_2, x_3, x_4 of the given quartic, on the assumption that the transformed equation is $y^4+c_2y^2=0$. The roots of this latter equation are 0, 0, and, let us say, r and $-r$. We must then have, from (1),

$$\left\{\begin{aligned} 0 &= x_1^3 + k_2x_1^2 + k_3x_1 + k_4, \\ 0 &= x_2^3 + k_2x_2^2 + k_3x_2 + k_4, \\ r &= x_3^3 + k_2x_3^2 + k_3x_3 + k_4, \\ -r &= x_4^3 + k_2x_4^2 + k_3x_4 + k_4. \end{aligned}\right. \tag{4}$$

It is now a matter of simple algebra to solve these equations for k_2; the result may be written in the form

$$k_2 = -\frac{(x_1^3+x_2^3-x_3^3-x_4^3)(x_1-x_2)-(x_1^3-x_2^3)(x_1+x_2-x_3-x_4)}{(x_1^2+x_2^2-x_3^2-x_4^2)(x_1-x_2)-(x_1^2-x_2^2)(x_1+x_2-x_3-x_4)}. \tag{5}$$

We may now find the number of different values which k_2 can assume by making the 24 possible permutations on the roots x_1, x_2, x_3, x_4. If the reader goes through this work, he will find that k_2 can take on only six different values, $-M/P$, $-M'/P'$, $-M''/P''$, $-N/Q$, $-N'/Q'$, $-N''/Q''$, where M, N, P, Q have the values

$$
(6)\quad \begin{cases}
M = (x_1^3 + x_2^3 - x_3^3 - x_4^3)(x_1 - x_2) \\
\qquad\qquad - (x_1^3 - x_2^3)(x_1 + x_2 - x_3 - x_4), \\
N = (x_1^3 + x_2^3 - x_3^3 - x_4^3)(x_3 - x_4) \\
\qquad\qquad - (x_3^3 - x_4^3)(x_1 + x_2 - x_3 - x_4), \\
P = (x_1^2 + x_2^2 - x_3^2 - x_4^2)(x_1 - x_2) \\
\qquad\qquad - (x_1^2 - x_2^2)(x_1 + x_2 - x_3 - x_4), \\
Q = (x_1^2 + x_2^2 - x_3^2 - x_4^2)(x_3 - x_4) \\
\qquad\qquad - (x_3^2 - x_4^2)(x_1 + x_2 - x_3 - x_4),
\end{cases}
$$

where M', N', P', Q' are obtained from M, N, P, Q by interchange of the subscripts 2 and 3, and where M'', N'', P'', Q'' are obtained from M, N, P, Q by replacing the subscripts 2, 3, 4 by 4, 2, 3, respectively. We thus see that k_2 can depend only on an equation of the sixth degree; that is, the twelfth degree equation mentioned above must, in this case, reduce to one of the sixth degree.

We now wish to go further, and show that this sixth degree equation can be broken up into quadratic factors, whose coefficients are themselves roots of equations of degree not higher than the third. Consider the quadratic equation $k_2^2+tk_2+u=0$, whose roots are $-M/P$ and $-N/Q$, and whose left-hand member is therefore a priori a factor of the left-hand member of the sextic. We have

$$
(7)\qquad t = \frac{M}{P} + \frac{N}{Q} = \frac{MQ + NP}{PQ}, \qquad u = \frac{MN}{PQ}.
$$

It can now be verified that t and u each take on only three different values under the 24 permutations on the roots x_1, x_2, x_3, x_4. These values are

$$
(8)\quad \begin{cases}
t_1 = \dfrac{MQ + NP}{PQ}, & u_1 = \dfrac{MN}{PQ}, \\[2ex]
t_2 = \dfrac{M'Q' + N'P'}{P'Q'}, & u_2 = \dfrac{M'N'}{P'Q'}, \\[2ex]
t_3 = \dfrac{M''Q'' + N''P''}{P''Q''}, & u_3 = \dfrac{M''N''}{P''Q''}.
\end{cases}
$$

The equation in t having the roots t_1, t_2, t_3 has, moreover, coefficients which are rational functions of the coefficients of the given equation; this follows because the elementary symmetric functions of the t's are symmetric functions of the x's, as can again be verified easily. A similar statement can be made concerning the cubic equation in u having the roots u_1, u_2, u_3.

The sextic in k_2 thus has a quadratic factor whose coefficients are roots of cubic equations; in fact it clearly has three such factors and may be written

$$(9) \qquad (k_2^2 + t_1k_2 + u_1)(k_2^2 + t_2k_2 + u_2)(k_2^2 + t_3\ {}_2 + u_3) = 0.$$

This completes, except for certain details which can be filled in easily enough, the a priori demonstration that the general quartic equation can be reduced to the form $y^4+c_2y^2=0$ without solving any equation of higher degree than the third. This demonstration does not, however, lead to a convenient determination of the necessary transformation.

3. *The Transformation.* The use of a special form of the Tschirnhaus transformation introduced by Hermite makes the computation of the transformed equation in y relatively simple.[*] This depends on certain invariantive properties which need not be stated explicitly; I shall merely say that his transformation for the quartic is of the form

$$(10) \quad y = (ax + b)B + (ax^2 + 4bx + 3c)C + (ax^3 + 4bx^2 + 6cx + 3d)D,$$

where B, C and D are the parameters, and where the original quartic is

$$(11) \qquad ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0.$$

Cayley[†] has computed the transformed quartic, as well as its invariants of the second and third degree. I make use of these last two expressions. If we use S and T to represent the quadratic invariant $ae-4bd+3c^2$ and the cubic invariant $ace+2bcd-ad^2-eb^2-c^3$ of (11), and if we put

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[*] Comptes Rendus, vol. 46 (1858), p. 961; *Oeuvres*, vol. 2, Paris, 1908, pp. 30–37.

[†] *Collected Papers*, vol. 4, Cambridge, 1891, pp. 375–394; vol. 5, 1892, 449–453.

$$(12)\quad \begin{cases} \phi = aB^2 + 4bBC + c(2BD + 4C^2) + 4dCD + eD^2, \\ \Delta = \dfrac{2}{3}(BD - C^2), \end{cases}$$

then the corresponding invariants of the transformed equation, which we may denote by S_y and T_y, are

$$(13)\quad \begin{cases} S_y = S\phi^2 + 18T\phi\Delta + 3S^2\Delta^2, \\ T_y = T\phi^3 + S^2\phi^2\Delta + 9ST\phi\Delta^2 + (54T^2 - S^3)\Delta^3. \end{cases}$$

I first show that the parameters B, C and D can be chosen so that the transformed equation will have $S_y^3-27T_y^2$ equal to zero. This expression is clearly of the sixth degree in ϕ and Δ, but it reduces to (S^3-27T^2) multiplied by the square of $(\phi^3-9S\phi\Delta^2-54T\Delta^3)$. Hence the condition $S_y^3-27T_y^2=0$ is a cubic in ϕ/Δ with known coefficients. If a root of this cubic is r, we have then to consider the equation $\phi=r\Delta$, which is a homogeneous, quadratic equation in B, C and D. One of these can be taken equal to zero, and the equation solved for the ratio of the other two.

Now it is well known that any quartic for which S^3-27T^2 is equal to zero can be reduced by a linear fractional transformation (which is reducible to a Tschirnhaus transformation) to the normal form $y^4+c_2y^2=0$. In fact it is not difficult to show that this reduction requires only the solution of quadratic equations, but I shall not reproduce the details of this calculation. Finally, since the product of two Tschirnhaus transformations is itself a Tschirnhaus transformation, we arrive at the desired result.

THE UNIVERSITY OF ROCHESTER

EXISTENCE THEOREMS FOR IMPLICIT FUNCTIONS OF REAL VARIABLES*

BY H. J. ETTLINGER

The classical theorems on implicit functions make use of the continuity of the given functions and their partial derivatives when all the variables are considered as independent.† The existence theorems established herein bring the implicit function theorems into line with the most recent developments of real variable theory.‡ Two of my students, W. M. Whyburn and J. H. Sturdivant, have made use of Theorem I in connection with studies of the properties of solutions of ordinary linear differential equations with summable coefficients.

In Theorem I sufficient conditions are given to ensure a single-valued continuous solution $y=y(x)$ of the relation $F(x, y)=0$. These conditions reduce the classical conditions considerably.

By introducing symmetry in x and y save in the final two hypotheses of Theorem II, sufficient conditions are given to ensure a single-valued absolutely continuous solution (with a summable derivative almost everywhere).

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* Presented to the Society, September 8, 1926.

† See Goursat-Hedrick, *Mathematical Analysis*, vol. 1, 1904, Chapter II p. 35ff. For a summary of the results and references to original sources, see Bliss, Princeton Colloquium Lectures, *Fundamental Existence Theorems*, delivered in 1909, published by the Society in 1913, New York.

A distinct lightening of the classical conditions for the existence of implicit functions is to be found in a paper by Hedrick and Westfall, Bulletin de la Société de France, vol. 44 (1916), pp. 1–14.

‡ See Carathéodory, *Vorlesungen über reelle Funktionen*, Leipzig, Teubner, 1918. Very recently (August, 1927) a second edition of this treatise has appeared with only a few additional references to more recent literature. The page references given in later footnotes to the present work apply equally well to the second (1927) edition.

THEOREM I. HYPOTHESIS. 1. *D_h is a neighborhood of a point (X, Y) in the (x, y) plane, $|x-X|\leqq h$, $|y-Y|\leqq h$, $h>0$. 2. $F(x, y)$ is defined in D_h and vanishes at (X, Y). 3. (a) $F(x, y)$ is absolutely continuous in y on $|x-X|\leqq h$, for every fixed x on $|x-X|\leqq h$; (b) $|F_y'(x, y)|\leqq M(y)$ for all values of x on $|x-X|\leqq h$, where $M(y)$ is summable in y on $|y-Y|\leqq h$; (c) $F_y'(x, y)$ is continuous in x for every fixed y almost everywhere on $|y-Y|\leqq h$. 4. $F(x, Y)$ is continuous in x on $|x-X|\leqq h$. 5. $F_y'(x, y)>0$ (<0)* for each fixed x on $|x-X|\leqq h$, almost everywhere on $|y-Y|\leqq h$.*

CONCLUSION. *There exists in D_k, $0<k\leqq h$, a unique single-valued continuous function $y=y(x)$, such that we have 1. $F(x, y(x))\equiv 0$ in D_k, 2. $Y=y(X)$, 3. $y(x)$ is continuous on $|x-X|\leqq k$.*

PROOF. By a theorem due to Carathéodory† it follows that $F(x, y)$ is continuous in (x, y) at (x, Y) for $|x-X|\leqq h$ and

$$F(x, y) = F(x, Y) + \int_Y^y F_t'(x, t)dt.$$

From hypothesis 5, it follows that for a fixed x, $F(x, y)$ is a monotonic increasing (decreasing) function of y on $|y-Y|\leqq h$, and since $F(X, Y)=0$,

$$F(X, y) = \int_Y^y F_t'(X, t)dt.$$

Hence $F(X, y)<0$ for $y<Y$, and $F(X, y)>0$ for $y>Y$, or $F(X, Y-h)<0$ and $F(X, Y+h)>0$.

Since $F(x, y)$ is continuous in x, it follows that there exists a neighborhood of (X, Y) contained in D_h such that $F(x, Y-k)<0$ and $F(x, Y+k)>0$ for every x on $|x-X|\leqq k$.

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* For the case (<0) all the subsequent inequality signs are reversed.

† Loc. cit., p. 678, Satz 5. The theorem made use of here is not explicitly stated by Carathéodory, but is implicit in the existence theorem for differential equations cited above. See my note, *On continuity in several variables*, this Bulletin, vol. 33 (1927), p. 37. Hypotheses 3 (*a*) and 3 (*b*) of Theorem I above should replace hypotheses (2) and (3) of the theorem of my note.

But for any fixed x on $|x-X|\leqq k$, $F(x, y)$ is monotonic increasing in y. Hence for each x on $|x-X|\leqq k$, there is one and only one value of y, $y=y(x)$, on $|y-Y|\leqq k$, such that $F(x), y(x)\equiv 0$. Now for $x=X$, we see that the corresponding value of y is Y. Finally by the very method* of obtaining $y(x)$, we see that $y(x)$ is continuous in x on $|x-X|\leqq k$.

THEOREM II. HYPOTHESIS. 1, 2, 3, 5 *remain as in Theorem* I. 4.(*a*) $F(x, y)$ *is absolutely continuous*† *in* x *on* $|x-X|\leqq h$, *for every fixed* y *on* $|y-Y|\leqq h$; (*b*) $|F_x'(x, y)|\leqq N(x)$ *for all values of* y *on* $|y-Y|\leqq h$, *where* $N(x)$ *is summable in* x *on* $|x-X|\leqq h$; (*c*) $F_x'(x, y)$ *is continuous in* y *for every fixed* x *almost everywhere on* $|x-X|\leqq h$. 6. $|F_x'(x, y)/F_y'(x, y)|\leqq K(x)$ *for every* y *on* $|y-Y|\leqq h$, *where* $K(x)$ *is summable in* x *on* $|x-X|\leqq h$.

CONCLUSION. 1. *There is one and only one solution of Theorem* I, $y=y(x)$, *which is absolutely continuous in* x *on* $|x-X|\leqq k$, *and* 2. $y_x'=-F_x'(x, y)/F_y'(x, y)$ *almost everywhere on* $|x-X|\leqq k$, *where* y_x' *is summable on* $|x-X|\leqq k$.

PROOF. Condition 4 of Theorem II carries with it condition 4 of Theorem I. From Theorem I we have a solution $y=y(x)$, such that $F(x, y(x))\equiv 0$. Let (x, y) and $(x+\Delta x, y+\Delta y)$ be any two points on $F(x, y(x))\equiv 0$. We may write the identity

$$\frac{\Delta y}{\Delta x}=-\frac{\dfrac{\int_x^{x+\Delta x}F_t'(t, y+\Delta y)dt}{\Delta x}}{\dfrac{\int_y^{y+\Delta y}F_t'(x, t)dt}{\Delta y}} \tag{1}$$

---

* This follows exactly as in the classical theorem, see Goursat-Hedrick, loc. cit., p. 37.

† Hypothesis 4 is exactly symmetric to hypothesis 3 with respect to x and y.

The right side of (1) has the limit*

$$-\frac{F'_x(x, y(x))}{F'_y(x, y(x))}, \tag{2}$$

except for a null set of x values on $|x-X| \leqq k$. Hence the left side of (1) approaches a limit almost everywhere on $|x-X| \leqq k$, or

$$y'_x = -\frac{F'_x(x, y(x))}{F'_y(x, y(x))}. \tag{3}$$

By a theorem due to Carathéodory† the numerator and denominator of (2) are measurable functions of x on $|x-X| \leqq k$, and hence summable by hypothesis 6. Hence by (3) $y(x)$ is absolutely continuous in x on $|x-X| \leqq k$.

The above theorems may be extended by the usual method of induction to a system of n functions in n dependent variables and m independent variables.‡

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* This follows from a generalization of Theorem III of my review, *Schlesinger on Lebesgue integrals*, this Bulletin, vol. 33 (1927) p. 111. A detailed proof is given by W. M. Whyburn in his dissertation as yet unpublished (offered to the Transactions of this Society).

† Loc. cit., p. 665, Satz 1.

‡ See Goursat-Hedrick, loc. cit., p. 45, etc.

THE CONVERGENCE OF DOUBLE FOURIER SERIES OF A CERTAIN TYPE*

BY G. M. MERRIMAN†

Occasion has arisen recently, in connection with double Fourier series whose coefficients satisfy certain conditions, for the development of a necessary condition for the convergence of such series. The present paper obtains such a condition, following a generalization to two variables of a theorem of Fatou.‡

Let $f(\alpha, \beta)$ be a periodic summable function whose double Fourier series, supposed for convenience to contain no terms independent of α or β, is given by

$$f(\alpha,\beta) \sim \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \begin{array}{c} a_{m,n} \cos m\alpha \cos n\beta + b_{m,n} \cos m\alpha \sin n\beta \\ + c_{m,n} \sin m\alpha \cos n\beta + d_{m,n} \sin m\alpha \sin n\beta \end{array} \right\},$$

which will be abbreviated as

$$(1) \qquad f(\alpha,\beta) \sim \sum_{1}^{\infty} \sum_{1}^{\infty} (a,b,c,d,\alpha,\beta)_{m,n} = \sum_{1}^{\infty} \sum_{1}^{\infty} A_{m,n}.$$

The integral of $f(\alpha, \beta)$, obtained by integrating (1) once with respect to each variable, is§

$$(2) \qquad g(\alpha,\beta) = \sum_{1}^{\infty} \sum_{1}^{\infty} \frac{1}{mn} (d, -c, -b, a, \alpha, \beta)_{m,n}.$$

We wish to prove the following theorem.

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* Presented to the Society, April 3, 1926.

† National Research Fellow.

‡ P. Fatou, *Séries trigonométriques et séries de Taylor*, Acta Mathematica, vol. 30 (1906), p. 385.

§ For the equality sign, see W. H. Young, *Multiple Fourier series*, Proceedings of the London Mathematical Society, (2), vol. 11, §§10 and 13.

THEOREM. *If $a_{m,n}$, $b_{m,n}$, $c_{m,n}$, and $d_{m,n}$ are $o(1/mn)$, and if $A^{-1,0}_{m,n}$ and $A^{0,-1}_{m,n}$ are $o(1/m)$ and $o(1/n)$ respectively,* then a necessary condition that* (1) *should be convergent is the existence of*

$$(3)\quad \lim_{x,y\to 0}\left\{\frac{1}{2xy}\left[g(\alpha+x,\beta+y)+g(\alpha-x,\beta-y)-g(\alpha+x,\beta)\right.\right.$$
$$\left.\left.-g(\alpha-x,\beta)-g(\alpha,\beta+y)-g(\alpha,\beta-y)+2g(\alpha,\beta)\right]\right\}=f(\alpha,\beta).$$

We first prove the following lemma.

LEMMA. *If* (1) *is convergent and its coefficients satisfy the conditions of the theorem, then there exists*

$$(4)\quad \lim_{x,y\to 0}\left\{\frac{1}{4xy}\left[g(\alpha+x,\beta+y)-g(\alpha+x,\beta-y)-g(\alpha-x,\beta+y)\right.\right.$$
$$\left.\left.+\,g(\alpha-x,\beta-y)\right]\right\}=f(\alpha,\beta).$$

We replace the fractional expression in (4) by

$$(5)\qquad \sum_1^\infty\sum_1^\infty A_{m,n}\,\frac{\sin mx}{mx}\,\frac{\sin ny}{ny},$$

and we desire to show that this expression converges to $f(\alpha,\beta)$ as $x, y\to 0$. Our procedure is to divide the sum in (5) into four sums, each one to be considered separately:

$$(6)\quad \sum_1^\infty\sum_1^\infty=\sum_{r+1}^\infty\sum_{s+1}^\infty+\sum_{r+1}^\infty\sum_1^s+\sum_1^r\sum_{s+1}^\infty+\sum_1^r\sum_1^s,$$

where r and s are the largest integers less than π/x and π/y respectively.

(i) Consider the first sum in (6). By hypothesis we have that $A_{m,n}$ is $o[1/(mn)]$, that is, for $m>M, n>N, mn\,|A_{m,n}|<\epsilon$, ϵ having been previously assigned. Hence, for $r>M$, $s>N$,

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* For convenience of notation, we put $A^{-1,-1}_{m,n}=A_{m,n}$, $A^{-1,0}_{m,n}=\sum_{n=1}^{n}A^{-1,-1}_{m,n}$, $A^{0,-1}_{m,n}=\sum_{m=1}^{m}A^{-1,-1}_{m,n}$, $A^{0,0}_{m,n}=\sum_1^m\sum_1^n A^{-1,-1}_{m,n}$.

$$\left|\sum_{r+1}^{\infty}\sum_{s+1}^{\infty}A_{m,n}\frac{\sin mx}{mx}\,\frac{\sin ny}{ny}\right|<\frac{\epsilon}{xy}\sum_{r+1}^{\infty}\sum_{s+1}^{\infty}\frac{1}{m^2n^2}$$

$$<\frac{\epsilon}{rsxy}<\frac{\epsilon}{(\pi-x)(\pi-y)}<\frac{\epsilon}{9}$$

if x, $y<.1$.

(ii) For the consideration of the second sum of (6) (and that of the third is wholly analogous to the analysis to be given here), we note that by the hypothesis $A_{m,n}^{-1,0}$ is $o(1/m)$, that is, $m\left|\sum_{n=1}^{n}A_{m,n}\right|<\epsilon$, for $m>M$. Again, $(\sin ny)/(ny)$ is a monotonically decreasing function of increasing n, and is always positive and less than unity since ny is between 0 and π. Therefore, by Abel's lemma,

$$\left|\sum_{r+1}^{\infty}\sum_{1}^{s}A_{m,n}\frac{\sin mx}{mx}\,\frac{\sin ny}{ny}\right|<\sum_{r+1}^{\infty}\frac{\epsilon}{m}\,\frac{|\sin mx|}{mx}$$

$$<\frac{\epsilon}{x}\sum_{r+1}^{\infty}\frac{1}{m^2}<\frac{\epsilon}{\pi-x}<\frac{\epsilon}{3}$$

if $x<.1$. Similarly,

$$\left|\sum_{1}^{r}\sum_{s+1}^{\infty}A_{m,n}\frac{\sin mx}{mx}\,\frac{\sin ny}{ny}\right|<\frac{\epsilon}{3}$$

if $y<.1$.

(iii) We divide the fourth sum of (6) into two parts: $\sum_1^M\sum_1^N$ and

$$\left\{\sum_{1}^{M}\sum_{N+1}^{s}+\sum_{M+1}^{r}\sum_{1}^{N}+\sum_{M+1}^{r}\sum_{N+1}^{s}\right\}.$$

In the second sum, we use the fact that $(\sin mx)/(mx)$ and $(\sin ny)/(ny)$ are both monotonically decreasing and are less in absolute value than unity, and the fact that the convergence of $\sum\sum A_{m,n}$ implies that, for M and N large enough,

$$\left|\left\{\sum_{1}^{M}\sum_{N+1}^{s}+\sum_{M+1}^{r}\sum_{1}^{N}+\sum_{M+1}^{r}\sum_{N+1}^{s}\right\}A_{m,n}\right|$$

can be made less than $2\epsilon/9$, whatever the values of r and s.*

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* See T. J. I'A. Bromwich, *Infinite Series*, §29 and §37, the latter containing the generalization of Abel's lemma which we have used.

In the first sum we note that as x, $y\rightarrow 0$, the sum approaches $\sum_1^M\sum_1^N A_{m,n}$.

Collecting the results of (i), (ii), and (iii), we find that as x, $y\rightarrow 0$, (5) converges to $f(\alpha, \beta)$, completing the lemma.

To prove the theorem we have only to combine (4) with

$$(7)\quad \lim_{x,y\rightarrow 0}\left\{\frac{1}{xy}\left[g(\alpha+x,\beta+y)+g(\alpha+x,\beta-y)+g(\alpha-x,\beta+y)\right.\right.$$
$$+g(\alpha-x,\beta-y)+4g(\alpha,\beta)-2g(\alpha+x,\beta)-2g(\alpha-x,\beta)$$
$$\left.\left.-2g(\alpha,\beta+y)-2g(\alpha,\beta-y)\right]\right\}=0,$$

which is a generalization to two variables of a result due to Riemann*.

Instead of requiring the existence of the limit (3), we can replace that condition by the requirement of the existence of (4), as is shown by our lemma. But the requirement of the existence of (4) immediately translates into that of the existence of

$$\frac{1}{4xy}\int_{-x}^{x}\int_{-y}^{y}f(\alpha+u,\ \beta+v)dudv\rightarrow f(\alpha,\ \beta)$$

as x, $y\rightarrow 0$, which in turn is equivalent to requiring the existence of

$$(8)\quad \frac{1}{4xy}\int_0^x\int_0^y[f(\alpha+u,\ \beta+v)+f(\alpha-u,\ \beta+v)$$
$$+f(\alpha+u,\ \beta-v)+f(\alpha-u,\ \beta-v)]dudv\rightarrow f(\alpha,\ \beta)$$

as x, $y\rightarrow 0$. Hence, we have the following corollary.

COROLLARY. *If the conditions of the theorem are satisfied, then a necessary condition for the convergence of* (1) *is the existence of* (8).

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* See Hilda Geiringer, *Trigonometrische Doppelreihen*, Monatshefte für Mathematik und Physik, 1918, p. 73.

TERNARY CHARACTERISTICS OF PRIMES*

BY E. T. BELL

1. *Introduction.* As will be seen from the proofs, the curious properties of primes given by the theorems of §2 are less abstruse than they seem at first sight. That they are obvious when once the clue to their derivation is given does not, however, detract from their interest.

Let μ be either of 1, 2, and λ a definite one of 0, 1. We shall denote the equation

$$x^2 - a[\lambda + (1-\lambda)p^\alpha]y^2 - bq^\beta z^2 = \mu^2 \tag{1}$$

by its *characteristic* $[a, b, \lambda]$. Thus $[a, b, 0]$ is

$$x^2 - ap^\alpha y^2 - bq^\beta z^2 = \mu^2, \tag{2}$$

and $[a, b, 1]$ is

$$x^2 - ay^2 - bq^\beta z^2 = \mu^2. \tag{3}$$

To avoid separate statements, we do not distinguish the cases $[a, b, \lambda]$, $\lambda = 0, 1$, until necessary. In (1), $p, q, \alpha, \beta, x, y, z$ are variable integers subject only to the following restrictions: p, q are primes>2; $\alpha>0$, $\beta>0$ are $\equiv 1 \pmod 4$; $x>0, y>0, z>0$; a, b are constant integers >0. If, subject to all these restrictions,

$$p = p', \quad q = q', \quad \alpha = \alpha', \quad \beta = \beta', \quad x = x', \quad y = y', \quad z = z'$$

is a set of values of $p, q, \alpha, \beta, x, y, z$ for which (1) is true, we shall call the matrix $(p', q', \alpha', \beta', x', y', z')$ a *solution* of (1). Note in particular that the definitions imply that p', q' in each solution are primes>2. The equation $[a, b, \lambda]$ is transcendental; α, β are variables. When $\lambda = 1$ we may omit p, α, p', α' from the definitions, since their retention is then trivial; a solution of $[a, b, 1]$ may be written $(*, q', *, \beta',$

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x', y', z'). Solutions being matrices, their equality is defined. Unequal solutions will be called *distinct*.

If p', q' in a given solution are such that, for m constant and >2,

$$p' \equiv p_0', \quad q' \equiv q_0', \quad \bmod m, \quad p_0' > 0, \quad q_0' > 0,$$

we shall say that the solution has the *residue* $(p_0', q_0')_m$ *modulo m.*

A set consisting of an infinity of distinct solutions is said to be *infinite.* Infinite sets are called *distinct* when each of the sets contains an infinity of solutions not in any of the others.

An integer $D > 0$ which determines the constants a, b, and λ, so that (1) has an infinite set of solutions will be called a *determinant* of $[a, b, \lambda]$. The precise way in which D determines $[a, b, \lambda]$ so as to have the stated property is immaterial for the moment.

Let c, n, r_j, s be constant integers, $c>0$, $n>2$, $r_j>0$ $(j=1,\cdots, s)$. Let the δ_j $(j=1,\cdots, s)$ be distinct primes >2 such that δ_j is prime to $c (j=1,\cdots, s)$ and

$$\delta_j \equiv r_j, \bmod n, \qquad (j = 1, \cdots, s);$$

and let the c_j be integers >0. Then, if every integer D of the form $c\delta_1^{c_1} \delta_2^{c_2}\cdots \delta_s^{c_s}$ is a determinant of $[a, b, \lambda]$, and if further the infinite sets of solutions appertaining to each pair of unequal determinants of this form are distinct, we shall call the set of all integers of the prescribed form a *discriminant* of $[a, b, \lambda]$, and write this discriminant $\{c, r_1^{c_1}, r_2^{c_2},\cdots, r_s^{c_s}\}_n$.

Our object is to find discriminants, to exhibit a characteristic $[a, b, \lambda]$ for a given discriminant, to assign the residues of the solutions in each case, and to classify the solutions.

The solutions in certain infinite sets will be classified by separation in finite odd numbers of distinct infinite sets, called *periods*, such that all the solutions in a given period are derivable from a fundamental one, called the *source* of the period. These periods depend upon the following sequences of integers; in the periods the integers in the sequences are all >0.

Let g, h denote constant integers both different from zero. Then (g, h) is called the *parameter* of the infinity of sequences $w_n (n=0, 1, \cdots)$ of integers defined by

$$w_{n+2} = gw_{n+1} - hw_n$$

and a pair (w_0, w_1) of initial integers. The particular pair of these sequences determined by the pairs of initial values

$$(w_0, w_1) = (0, 1), \quad (w_0, w_1) = (2, g)$$

are the *Lucas sequences* u_n, $v_n (n=0, 1, \cdots)$ respectively, for the parameter (g, h). All the periods in a given infinite set pertain to the same parameter (g, h); different determinants in a given discriminant give periods pertaining to different parameters. As the periods depend upon u_n, v_n, the numerous known properties of Lucas sequences can be applied to read off properties of solutions of $[a, b, \lambda]$. For convenience we add references concerning u_n, v_n.*

2. *Existence Theorems.* Among many others we can state the following three general theorems. These give considerable information regarding the situation described in §1.

There exist discriminants Δ such that, if D is a determinant in Δ, the following statements hold.

THEOREM I. *In addition to determining the characteristic* $[a, b, \lambda]$, *each D determines a unique matrix (T, U) of integers $T>0$, $U>0$, and an odd number ω of distinct matrices*

$$(p_j, q_j, \alpha_j, \beta_j, V_j, W_j), \quad (j = 1, \cdots, \omega),$$

in which V_j, W_j are integers >0, such that

$$\sigma_{jn} \equiv (p_j, q_j, \alpha_j, \beta_j, Uu_n, \mu V_j v_n/2, \mu W_j v_n/2),$$

$$(j = 1, \cdots, \omega; \quad n = 1, 2, \cdots),$$

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* Lucas, American Journal of Mathematics, vol. 1, pp. 184, 289; *Théorie des Nombres*, Chapter 18; Bachmann, *Niedere Zahlentheorie*, Kap. 2; Dickson, *History of the Theory of Numbers*, vol. 1, Chapter 17.

are ω *distinct infinite sets of solutions of* $[a, b, \lambda]$; *the* u_n, $v_n(n=0, 1, \cdots)$ *are the Lucas sequences for the parameter* $(2T/\mu, 1)$. *The infinite set* σ_{jn} $(n=1, 2, \cdots)$ *is a period* (*as described in* §1) *with the source*

$$\sigma_{j1} = (p_j, q_j, \alpha_j, \beta_j, U, TV_j, TW_j),$$

the solution σ_{jn} *being obtained from* σ_{j1} *by multiplying* U, TV_j, TW_j *by* u_n, v_n/v_1, v_n/v_1 *respectively. If* $\lambda=1$, *there is one and only one* (q_j, β_j) *for each* (V_j, W_j).

THEOREM II. *The* D *as in Theorem* I *also determines a modulus* m *such that all the solutions* $\sigma_{jn}(j=1, \cdots, \omega;\ n=1, 2, \cdots)$ *have the same residue modulo* m, *and this residue remains constant as* D *ranges over all determinants in* Δ.

THEOREM III. *The matrices* (T, U), *and hence also the periods described in Theorem* I, *are distinct for unequal determinants* D *in* Δ.

3. *Specific Theorems for* $\lambda=s=c_1=1$. The relevant parameters to be assigned are c, r, n, which determine Δ, since $s=1$; a, b, which fix $[a, b, \lambda]$ when $\lambda=1$, and m, q_0', which

n, r, c	a, b	m, q_0'	n, r, c	a, b	m, q_0'
(1) 4, 1, 2	1, 1	8, 1	(19) 24, 11, 1	1, 1	12, 7
(2) 8, 1, 1	4, 1	8, 5	(20) 24, 11, 1	1, 2	12, 5
(3) 8, 3, 2	1, 1	8, 5	(21) 24, 13, 1	6, 1	24, 7
(4) 8, 3, 3	4, 1	8, 5	(22) 24, 17, 5	2, 1	24, 11
(5) 8, 3, 4	1, 1	8, 3	(23) 24, 17, 5	18, 1	24, 19
(6) 8, 5, 1	2, 1	8, 3	(24) 24, 19, 1	9, 2	12, 5
(7) 8, 5, 4	1, 1	8, 3	(25) 40, 7, 5	1, 2	20, 13
(8) 12, 5, 4	1, 1	24, 19	(26) 40, 7, 5	1, 2	20, 17
(9) 12, 5, 4	9, 1	24, 11	(27) 40, 11, 1	5, 2	20, 3
(10) 16, 3, 1	8, 1	8, 5	(28) 40, 11, 1	5, 2	20, 7
(11) 16, 7, 1	1, 2	8, 3	(29) 40, 19, 1	5, 2	20, 3
(12) 16, 7, 1	2, 1	8, 5	(30) 40, 19, 1	5, 2	20, 7
(13) 16, 7, 1	4, 1	8, 3	(31) 40, 23, 1	1, 2	20, 3
(14) 16, 7, 2	1, 1	8, 5	(32) 40, 23, 1	1, 2	20, 7
(15) 24, 7, 1	2, 1	24, 5	(33) 40, 23, 5	1, 2	20, 13
(16) 24, 7, 2	1, 1	24, 13	(34) 40, 23, 5	1, 2	20, 17
(17) 24, 7, 2	3, 1	24, 11	(35) 48, 31, 1	18, 1	24, 13
(18) 24, 7, 2	9, 1	24, 5			

give the residue $(*, q_0')_m$, since here p_0' does not occur. We are concerned therefore in this case with §1(3) for assigned values of a, b and primes $q \equiv q_0'$ mod m; the corresponding values of n, r, c are useful only later in proving that the thirty-five shown in the table give instances of the theorems in §2 applied to §1(3).

From these we see, for example, from the second and third columns for (24), that

$$x^2 - 9y^2 - 2q^\beta z^2 = \mu^2,$$

in which the prime $q \equiv 5$ mod 12 has infinities of solutions in the sense of §1, with the properties stated in §2. By the first column the discriminant in this case is the class of all positive primes $\equiv 19$ mod 24. The list can be continued indefinitely, and likewise for the next two.

4. *Specific Theorem for* $\lambda = 0$, $s = 1$. As the case $\lambda = 0$ is not fundamentally different from $\lambda = 1$, we give but one example. The discriminant in the following is the class of all positive primes $\equiv 1$ mod 8; the primes p, q are both $\equiv 3$ mod 8:

$$x^2 - 2p^\alpha y^2 - q^\beta z^2 = \mu^2.$$

5. *Specific Theorems for* $\lambda = 1$, $s = 2$, $c_1 = c_2 = 1$. In each of the following the constant c is unity, so we shall not tabulate it. In the third column the significance of the parenthesis referring to values of q_0' is that one of the two values in the parenthesis is permissible for the accompanying modulus m. Thus we have the entry 40, (11, 19) in (4); hence q in

n, r_1, r_2	a, b	m, q_0'
(1) 8, 1, 3	1, 2	8, 5
(2) 24, 5, 5	12, 1	24, 13
(3) 24, 7, 7	12, 1	24, 13
(4) 40, 3, 7	10, 1	40, (11, 19)
(5) 40, 3, 23	10, 1	40, (11, 19)
(6) 40, 7, 27	10, 1	40, (11, 19)
(7) 40, 23, 27	10, 1	40, (11, 19)
(8) 120, 31, 31	60, 1	120, (61, 109)
(9) 120, 31, 79	60, 1	120, (61, 109)
(10) 120, 79, 79	60, 1	120, (61, 109)

§1(3) is a prime of one of the forms 11 or 19 mod 40 for this example. Since $c_1 = c_2 = 1$ it is necesary in the first column, referring to the discriminant, to give only n, r_1, r_2($c=1$, as stated).

These illustrate the fact that a given $[a, b, \lambda]$ may have, several discriminants. There are numerous examples in which $s > 2$, but we shall omit these and pass to the proofs.

6. *Proofs.* All of the preceding results become obvious on combining two simple remarks. The notation is as in §§1, 2. Take for (T, U) the fundamental solution of the Pellian equation $t^2 - Du^2 = \mu^2$. Resolve D into the form $ap^\alpha \xi^2 + bq^\beta \eta^2$, where ξ, η are integers > 0. The rest follows at once. The resolution of D, combined with the successive solutions of the Pellian equation, obtained in the usual manner from (T, U), furnish the periods in §2. There are in each instance ω resolutions of D of the prescribed kind, and hence at least one. That such resolutions exist is known from the ingenious method of Bouniakowsky, or independently from the general arithmetical formulas obtained by paraphrasing identities between elliptic and theta functions. The latter method provides an inexhaustible source of these results and of others of a similar nature relating to forms in more than two variables. Bouniakowsky's method was exploited by Liouville; the specific theorems in §§3–4 can be verified by comparing with Liouville's resolutions of the corresponding D in volumes 3–5 of the second series of his Journal. I believe that proofs for these have not been published; on another occasion I will supply the details.

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ON THE EXISTENCE OF LINEAR ALGEBRAS IN BOOLEAN ALGEBRAS*

BY ORRIN FRINK, JR.†

According to the definition to be found in Dickson's *Algebras and their Arithmetics*, pages 9–11, the following properties are characteristic of a linear algebra.

(1) *The elements of the algebra form an abelian group with respect to addition.*

(2) *Multiplication is distributive with respect to addition.*

(3) *The algebra has a finite basis; that is, a finite number of elements can be found such that every element of the algebra can be expressed as a linear combination of these basal units, with coefficients taken from the field over which the algebra is defined.*

With this definition in mind we wish to determine for what pairs of Boolean operations considered as the addition and multiplication operations the elements of a Boolean algebra will constitute a linear algebra. We are limited in our choice of an addition operation by the first property above to operations of the form $axy+a'xy'+a'x'y+ax'y'$, which Bernstein‡ has shown to be the only Boolean abelian group operations. To find suitable multiplication operations $pxy+qxy'+rx'y+sx'y'$, we seek those which are distributive with respect to the above. The condition that the first distributive law hold is found in Schröder's *Algebra der Logik* (vol. 2, p. 503) to be

$$a'(pq+rs)+(ad+a'd'+bc+b'c')(p'q+r's)+d(p'q'+r's')=0,$$

where a, b, c, d, and p, q, r, s are the discriminants of the addition and multiplication operations respectively. Here $a=d$, and $b=c=a'$; hence we get $p'q=r's=a'q=a's=ap'$

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† National Research Fellow.

‡ Transactions of this Society, vol. 26, p. 174.

$=ar'=0$, which can be written $p>a>q$, $r>a>s$. Similarly from the second distributive law we get $p>a>r$, and $q>a>s$. It follows that $q=r=a$, whence it is seen that multiplication must be commutative. Our multiplication operations then must be of the form $pxy+axy'+ax'y+sx'y'$, where $p>a>s$. Another way of writing this which involves only two parameters is $(a+b)xy+axy'+ax'y+ab'x'y'$, where $b=ps'$. It is to be noted that these operations are all associative, the condition being $p's+(p'+s)(qr'+q'r)=0$.

Before trying to satisfy the condition about the finite basis, we will look for *idempotent* elements, that is, elements e not equal to a such that $e^2=e$. Setting $y=x$ in the multiplication operation gives us $(a+b)x+ab'x'$. Equating this to x, we conclude that x is idempotent when and only when it is of the form $(a+b)u+ab'u'$, where u is arbitrary. On the other hand, an element n not equal to a such that $n^2=a$ is *nilpotent*. Equating the above expression for x^2 to a we see that all elements of the form $(a+b')u+abu'$ are nilpotent. If we substitute this expression for a nilpotent element in the multiplication operation we also see that the product of one of these nilpotent elements by any element of the algebra is a. Of course it should be remembered that a corresponds to zero in our algebra.

An important fact is that any element of the algebra can be expressed as the "sum" of an idempotent element and a nilpotent element. For we have $x=(ab'+xb)\oplus(ab+xb')$, where the sign $\oplus$ represents our addition operation as distinguished from logical addition. This is an illustration of the principal theorem on algebras (Dickson, loc. cit., p. 125), which states that any associative linear algebra is the sum of its maximum invariant nilpotent sub-algebra and a semi-simple algebra. The sum is here a direct sum, the cross products being zero.

In our case the field over which the algebra is supposed to be defined is not given to us in advance, as contemplated by the definition of linear algebras. However, an algebra which contains an idempotent element must contain a subalgebra which is isomorphic with the field. It can be seen that a and any idempotent e form a field of 2 elements. Bernstein* has shown that the only possible fields in Boolean algebras are of this kind, and in fact if a field contained besides a and e another element f, then since f^2 is idempotent it would have to equal e, since a field cannot contain two idempotents. In that case, however, $e \oplus f$ would be nilpotent, which is likewise impossible in a field. Since we are limited to finite fields. we see that if our algebra is to have a finite basis it must itself be finite. However, any Boolean algebra contains finite Boolean sub-algebras, and hence may contain linear algebras.

We have seen that our entire algebra is the direct sum of a zero algebra and an idempotent algebra. If either b or b' contains only a finite number of elements, then one of these two sub-algebras is finite and may have a finite basis. Suppose the idempotent algebra is finite and contains exactly 2^n elements of the form $ab'+kb$. The product of two such elements $ab'+xb$ and $ab'+yb$ is seen to be $ab'+(xy+axy'+ax'y)b$. We see from this that we may set up a correspondence which is preserved under multiplication, between the elements $ab'+kb$ of our idempotent algebra and the elements kb of a Boolean algebra† whose multiplication operation is $xy+axy'+ax'y$. This Boolean algebra can be represented by the subsets of a set of n elements $(e_1, e_2, \cdots, e_n)$, which combine under logical addition and multiplication. We now pick out the elements kb corresponding to the unit sets $(e_1), (e_2), \cdots, (e_n)$ and call them k_1b, $k_2b, \cdots, k_nb$. We then choose as basal units for the idempotent algebra the elements $ab'+k_ib$ $(i=1, 2, \cdots, n)$. The product of any two of these is a if they are distinct. Our field consists of the two elements a and $ab'+a'b$, the latter being the modulus of the algebra. It is seen that these elements correspond to the null set and the universal set in the above representation by means of subsets. To see how an element x of the linear algebra is represented as a linear combination of the basal units, we merely find to what subset of $(e_1, e_2, \cdots, e_n)$ x corresponds. The subscripts of the k's which appear in the linear combination are the same as those of the e's which appear in the subset.

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* Transactions of this Society, vol. 28 (1926), p. 654.

† N. Wiener, Transactions of this Society, vol. 18 (1917), p. 65.

No particular difficulty presents itself in picking a set of basal units for the zero algebra if it is finite. The elements form an abelian group with respect to addition, of order say 2^m, and we need merely select a set of m generators of the group for our basal units. The field, of order 2, must be taken outside the algebra, since the latter does not have a modulus. If the entire algebra is finite, in choosing a set of basal units for it we select the units for the sub-algebras separately, and again we must consider the field to be outside the algebra, since it has no modulus if the nilpotent part exists.

We may sum up our results thus far as follows.

The elements of a Boolean algebra form for the operations $axy+a'xy'+a'x'y+ax'y'$ *and* $(a+b)xy+ax'y+axy'+ab'x'y'$ *considered as the addition and multiplication operations respectively a commutative associative algebra which is the direct sum of a zero algebra and an algebra all of whose elements except the zero element are idempotent. If either of these sub-algebras is finite, it may be represented as a linear algebra over* $GF(2)$, *as may the entire algebra if it is finite.*

These results I consider important for Boolean algebra and the algebra of classes. Let $a=0$ and $b=1$, which gets rid of the zero algebra. Then our addition and multiplication operations become $xy'+x'y$ and xy respectively. In other words we may substitute the operation $xy'+x'y$ for logical addition in Boolean algebra. This operation, which I will for the time being represent with Peano by $x \circ y$, has been treated by many authors, but to Veblen* belongs the credit for introducing it as the addition operation to replace logical addition. In the analysis situs applications, where it is called "addition modulo 2," it plays a more important role than logical addition, and its use results in formulas which resemble more those of ordinary algebra.

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* Cambridge Colloquium Lectures, p. 9.

Two important properties of this operation are $a \oplus a = 0$, and $a' \circ a = 1$ for all values of a. From the first it follows that we may transpose terms from one side of an equation to the other. In virtue of the second property we can avoid entirely the use of the Boolean negation x'. Another important fact is that logical addition and addition modulo 2 are interchangeable when there is no overlapping between terms, as for example in the case of a Boolean expression in the completely expanded form. Thus consider a Boolean function of two variables $f(x, y) = axy + bxy' + cx'y + dx'y'$. Here we could substitute $\circ$ for $+$. If we wish to avoid the use of the negation sign we replace x' by $x \circ 1$ and y' by $y \circ 1$, obtaining $f(x, y) = pxy \circ qx \circ ry \circ d$, where $p = a + b + c + d$, $q = b + d$, and $r = c + d$. This may be considered the normal form for a function of two variables written in terms of addition modulo 2. To pass from an expression of the latter type to one involving logical addition, we may use the well known rule for obtaining the discriminants of a Boolean expression by giving the variables the values 0 and 1. As a final remark it is interesting to note that although we may speak of the logical sum of an infinite number of elements, the sum modulo 2 of an infinite number of elements is in general meaningless.

PRINCETON UNIVERSITY

THE DERIVATION OF ALGEBRAIC INVARIANTS BY TENSOR ALGEBRA*

BY C. M. CRAMLET†

1. *Review of Elements of Tensor Algebra.* The few simple laws of tensor algebra offer a basis for a very natural approach to the theory of algebraic invariants. In the study of algebraic invariants a ground form is chosen as, for example, the bilinear n-ary form $a_{\rho\sigma}u^\rho v^\sigma$‡ and the quantities u^r and v^s are transformed linearly by cogredient transformations such as

$$\bar{u}^r = u^\rho q_\rho^r. \tag{1}$$

If the equivalent transformed form be now written $\bar{a}_{\rho\sigma}\bar{u}^\rho\bar{v}^\sigma$, the transformation equations for the a's are

$$\bar{a}_{rs} = a_{\rho\sigma}p_r^\rho p_s^\sigma, \tag{2}$$

where the p's satisfy the equations

$$p_s^r q_\rho^r = p_\rho^r q_s^\rho = \delta_s^r, \qquad (= 0, r \neq s\,; = 1, r = s), \tag{3}$$

or what is equivalent, q_s^r is equal to the cofactor of p_s^r in the determinant $|p_s^\tau|$, divided by this determinant. In tensor algebra the two equations of transformation (1) and (2) illustrate two types of tensors, that is, sets of ordered functions which are transformed in this linear manner. The set of quantities u^ρ is called a *contravariant tensor of rank* 1 since the transformed quantities are each expressed linearly in the q's. The set of quantities a_{rs} with two lower indices illustrate a *covariant tensor of rank* 2. These terms *covariant* and *contravariant* are used differently in the classical theory of invariants. Here the terminology of tensor algebra will be used.

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‡ Repeated Greek letters are summed from 1 to n. Such a form is ordinarily written $a_{\rho\sigma}x^\rho y^\sigma$. In the theory of surfaces, with the constant values at a point, of a given tensor such as a_{rs} may be associated arbitrary vectors, in this case two, u^r and v^s. Thus when a general transformation of coordinates is made the vectors are transformed linearly, so the theory of algebraic invariants is applicable at the point.

The three laws of tensor algebra which follow directly from the defining transformation equations are as follows.

(a) *The sum of two tensors of the same kind is a tensor of that kind.*

(b) *The (outer) product of two tensors is a tensor whose rank is the sum of the ranks of these tensors.* For example, $a_{rs}v^t$ is a mixed tensor covariant of rank 2 and contravariant of rank 1. In the notation $T_{rs}^{\;t}$, the transformation law is

$$\bar{T}_{rs}^{\;t} = T_{\rho\sigma}^{\;\tau} q_\tau^t p_r^\rho p_s^\sigma.$$

(c) *If a covariant and contravariant index of a mixed tensor such as $T_{rs}^{\;t}$ are set equal and summed, as the notation $T_{r\alpha}^{\;\alpha}$ indicates, a tensor is obtained which has its contravariant and covariant rank each decreased by one.* The proof consists in setting $t=s=\alpha$ in the equation of (b) above, and making use of (3), from which we find

$$\bar{T}_{r\alpha}^{\;\alpha} = T_{\rho\sigma}^{\;\tau} q_\tau^\alpha p_r^\rho p_\alpha^\sigma = T_{\rho\sigma}^{\;\tau} p_r^\rho \delta_\tau^\sigma = T_{\rho\alpha}^{\;\alpha} p_r^\rho.$$

This is the transformation equation defining a covariant vector. The expression $a_{\rho\sigma}u^\rho v^\sigma$ having no unsummed indices is called a scalar. The process is called contraction or inner multiplication and the contracted or summed indices are said to cancel.

The symbol δ_s^r which occurs in (3) may be made a tensor by assigning the law of transformation to it that its indices suggest. Then $\bar{\delta}_s^r = \delta_\sigma^\rho p_s^\sigma q_\rho^r = p_s^\rho q_\rho^r = \delta_s^r$, by (3) and the new components are exactly equal to the old components. In general therefore the tensor defined by the equation

$$(4)\qquad \gamma^{r_1\cdots r_k}_{s_1\cdots s_k} = B^1\delta^{r_1}_{s_1}\delta^{r_2}_{s_2}\cdots\delta^{r_k}_{s_k} + B^2\delta^{r_2}_{s_1}\delta^{r_1}_{s_2}\cdots\delta^{r_k}_{s_k} + \cdots,$$

in which every permutation of the set r appears and the B's are arbitrary scalars, possesses this property. It has been proved* moreover that there are no other invariant tensors. The tensor $\delta^{r_1\cdots r_k}_{s_1\cdots s_k}$, obtained from (4) by assigning B the values ±1 according as the permutation on r of the δ's of which B is the coefficient is even or odd, since it transforms by invariance will (as does $\gamma^{r_1\cdots r_k}_{s_1\cdots s_k}$) satisfy the equation

$$\delta^{r_1\cdots r_k}_{s_1\cdots s_k} = \delta^{\rho_1\cdots\rho_k}_{\sigma_1\cdots\sigma_k} q^{\sigma_1}_{s_1} \cdots q^{\sigma_k}_{s_k} p^{r_1}_{\rho_1} \cdots p^{r_k}_{\rho_k}. \tag{5}$$

Setting $k=n$ and $r_1 \cdots r_n = 1 \cdots n$, this becomes

$$\delta^{1\cdots n}_{s_1\cdots s_n} = \delta^{1\cdots n}_{\sigma_1\cdots\sigma_k} p^{\sigma_1}_{s_1} \cdots p^{\sigma_n}_{s_n} \cdot \left| q_s^{\,r} \right|.$$

This is usually written as

$$\epsilon_{s_1\cdots s_n} = \epsilon_{\sigma_1\cdots\sigma_n} p^{\sigma_1}_{s_1} \cdots p^{\sigma_n}_{s_n} \cdot \left| q_s^{\,r} \right|, \tag{6}$$

and is called a covariant tensor† density of weight −1. This equation represents its transformation equation and shows these important facts: (1) it is invariant under a transformation of coordinates; (2) the indices are of tensor character, that is, indicate a certain law of transformation; (3) a factor Δ^{-1} must be introduced after a transformation of coordinates. Similarly from (5), setting the indices s equal to $1 \cdots n$, we have

$$\epsilon^{r_1\cdots r_n} = \epsilon^{\rho_1\cdots\rho_n} q^{r_1}_{\rho_1} \cdots q^{r_n}_{\rho_n} \cdot \Delta, \tag{7}$$

and $\epsilon^{r_1\cdots r_n}$ is called a contravariant tensor density of weight 1.

2. *Proof of the Fundamental Theorems.*‡ It will now be

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* P. Franklin, Philosophical Magazine, vol. 45, p. 998; C. M. Cramlet, this Bulletin, vol. 32 (1926), p. 212; T. Y. Thomas, Annals of Mathematics, vol. 27, No. 4 (1926), p. 548; C. M. Cramlet, Tôhoku Mathematical Journal, vol. 28, Nos. 3, 4 (1927), p. 242.

† See Veblen, *Invariants of Quadratic Differential Forms*, Cambridge Tract No. 24, 1927, Chapter I and Chapter II, §14.

‡ The problem of §2 was discussed in a series of conferences with Professor O. Veblen and Mr. W. Flexner. Thanks are due to Professor Veblen for constructive criticism which was helpful for a clear formulation of the problem. Mr. Flexner has worked out a proof of this problem by the use of the Cayley operator. The novel feature of his proof is the fact that the symbolic method is not used.

shown that the determination of all algebraic invariants reduces to an exercise of the three laws of tensor algebra.

Let us suppose that we have a system of forms determined by the coefficients (or covariant tensors) $a_{r_1 \cdots r_h}$, $b_{r_{h+1} \cdots r_s}$, $\cdots$, $c_{\cdots r_k}$. Assuming the theorems, which depend upon elementary considerations on the arbitrariness of these forms, that only functions homogeneous in each of the tensors separately need be considered, we know that a typical term of an algebraic invariant is

$$A_{r_1 \cdots r_k} \equiv a_{r_1 \cdots r_h} b_{\cdots r_s} \cdots c_{\cdots r_k},$$

and the invariant is representable as

$$I = K^{\rho_1 \cdots \rho_k} A_{\rho_1 \cdots \rho_k} \tag{8}$$

the K's being undetermined constants. An algebraic invariant must by definition be transformed according to the equation $\bar{I} = I \cdot F$, where $\bar{I}$ is exactly the same function of the coefficients of the transformed forms as is I of the original forms. Assuming further the well known theorem, which is a consequence of the assumption that I is rational, that F is an integral power of the modulus of the transformation, say Δ^w where $\Delta = |p_j^i|$, we may write this equation in the form

$$K^{\rho_1 \cdots \rho_k} \bar{A}_{\rho_1 \cdots \rho_k} = K^{\rho_1 \cdots \rho_k} A_{\rho_1 \cdots \rho_k} \Delta^w. \tag{9}$$

On the other hand, since A is a tensor, we have

$$K^{\rho_1 \cdots \rho_k} \bar{A}_{\rho_1 \cdots \rho_k} = K^{\rho_1 \cdots \rho_k} A_{\sigma_1 \cdots \sigma_k} p_{\rho_1}^{\sigma_1} \cdots p_{\rho_k}^{\sigma_k}. \tag{10}$$

The right members of these equations are identically equal for arbitrary p's; hence $k = wn$. Multiplying (6) by Δ and (9) by w ϵ's, we find, by direct substitution of this new (6) into this new (9),

$$\begin{aligned} \epsilon_{r_1 \cdots r_n} \cdots \epsilon_{\cdots r_k} K^{\rho_1 \cdots \rho_k} \bar{A}_{\rho_1 \cdots \rho_k} \\ = K^{\sigma_1 \cdots \sigma_k} A_{\sigma_1 \cdots \sigma_k} \epsilon_{\rho_1 \cdots \rho_n} \cdots \epsilon_{\cdots \rho_k} p_{r_1}^{\rho_1} \cdots p_{r_k}^{\rho_k}. \end{aligned} \tag{11}$$

Substituting in this

$$A_{\sigma_1\cdots\sigma_k} = \overline{A}_{\tau_1\cdots\tau_k} q^{\tau_1}_{\sigma_1} \cdots q^{\tau_k}_{\sigma_k},$$

we have

$$(12)\qquad {}_{\tau_1}\overline{A}_{\cdots\tau_k}\big[\epsilon_{r_1\cdots r_n} \cdots \epsilon_{\cdots r_k} K^{\tau_1\cdots\tau_k} - \epsilon_{\rho_1\cdots\rho_k} \cdots \epsilon_{\cdots\rho_k} K^{\sigma_1\cdots\sigma_k} q^{\tau_1}_{\sigma_1} \cdots q^{\tau_k}_{\sigma_k} p^{\rho_1}_{r_1} \cdots p^{\rho_k}_{r_k}\big] = 0.$$

If $\overline{A}$ is an arbitrary tensor its coefficient must vanish. But A may be a product of tensors, in which case the terms of the product are interchangeable, or, one of the tensors may determine a p-ic in which case its indices are interchangeable. There is no loss of generality in the invariant I defined by equation (8) if the K's are assumed to have the same symmetry as the A's. This amounts to replacing the constants associated with equal terms by their average. We may now set coefficients of $\bar{A}$ equal to zero as in the case where $\bar{A}$ is wholly arbitrary. The vanishing of these coefficients expresses the conditions that $\epsilon_{r_1\cdots r_n} \cdots \epsilon_{\cdots r_k} K^{r_1\cdots r_k}$ be an invariant tensor of the rank indicated by the indices, that is a tensor having identical components in all systems of coordinates. Using the theorem of §1 embodied in (4), we find

$$\epsilon_{r_1\cdots r_n} \cdots \epsilon_{\cdots r_k} K^{\rho_1\cdots\rho_k} = B^1 \delta^{\rho_1}_{r_1} \delta^{\rho_2}_{r_1} \cdots \delta^{\rho_k}_{r_k} + B^2 \delta^{\rho_2}_{r_1} \delta^{\rho_1}_{r_2} \cdots \delta^{\rho_k}_{r_k} + \cdots.$$

Multiplying through by $\epsilon^{r_1\cdots r_n} \cdots \epsilon^{\cdots r_k}$, r summed, we have

$$(13)\quad (n!)^w K^{\rho_1\cdots\rho_k} = B^1 \epsilon^{\rho_1\rho_2\cdots\rho_n} \cdots \epsilon^{\cdots\rho_k} + B^2 \epsilon^{\rho_1\rho_2\cdots\rho_n} \cdots \epsilon^{\cdots\rho_k} + \cdots.$$

Since $k = wn$, each term contains w ϵ's and the ρ's appear in all possible permutations. We have thus found that for I of (8) to be an invariant it is necessary that $K^{\rho_1\cdots\rho_k}$ be of the form (13). It is obviously sufficient, for when each term of (13) with the B's arbitrary is multiplied by inner multiplication with $A_{\rho_1\cdots\rho_k}$ the tensor indices cancel by inner multiplication according to tensor law (c), §1, and, since $\epsilon^{\rho_1\cdots\rho_n}$ is a tensor density of weight one, each term gives rise to a scalar density of weight w. Those solutions that do not satisfy the symmetry conditions imposed on K, that is, those which have two antisymmetric indices on an ϵ summed with two symmetric indices on an A, will vanish identically. We thus have the following theorem.

THEOREM 1. *An algebraic invariant of weight w is formed from a given system of covariant tensors by forming by outer multiplication a tensor A of rank wn and multiplying this by inner multiplication by w ϵ's, associating the indices of the ϵ's in any manner whatever with the indices of A. Moreover these constitute all algebraic invariants that can be formed from the given covariant tensors.*

The extension to the following general theorem is immediate.

THEOREM 2. *From a given system of covariant and contravariant tensors, a tensor $A^{s_1\cdots s_l}_{r_1\cdots r_k}$ may be formed by outer and inner multiplication. In this tensor, k and l are multiples of n, say, $k=w_1n$, $l=w_2n$ and $w=(k-l)/n$, all letters representing integers. All algebraic invariants may be formed by multiplying by inner multiplication w_1 of the contravariant ϵ's, $\epsilon^{r_1\cdots r_n}$ and w_2 covariant ϵ's, $\epsilon_{r_1\cdots r_n}$.* A few illustrations of the methods here described will be given.

3. *Algebraic Invariants of a System of Linear Forms.* It will require n tensors such as a_r to build up an nth rank tensor by outer multiplication. These n tensors can be represented by $a_{r|s}$. The bar is used to indicate that the preceding r does not designate a tensor character. The only possible algebraic invariant is $\epsilon^{\rho_1\cdots\rho_n}a_{r_1|\rho_1}\cdots a_{r_n|\rho_n}$.* This is merely the determinant of the system of forms. For a set of vectors $u_{r|}{}^s$ the algebraic invariant of weight -1 is

$$\epsilon_{\sigma_1\cdots\sigma_n}u_{r_1|}{}^{\sigma_1}\cdots u_{r_n|}{}^{\sigma_n}.$$

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* The non-tensor range is alternating here. The number of alternating ranges associated with n elements is always even. See my paper, *Applications of the determinant and permanent tensors to determinants of general class and allied tensor functions*, American Journal, vol. 49, No. 1 (January, 1927), p. 87.

4. *Symmetry Tensors.* When two or more indices of an element are summed with indices of the same δ or ϵ there results an algebraic invariant whose vanishing expresses a symmetry condition on the ground form. For example, if the vectors in the bilinear form $a_{\rho\sigma}u^\rho v^\sigma$ have but two components ($n=2$, the binary bilinear form), we may form the algebraic invariant $\delta^{\rho\sigma}_{rs}a_{\rho\sigma}$. The vanishing of this indicates that $a_{rs}=a_{sr}$ in all coordinate systems. The property remains true for any n but then we have a tensor that is not an algebraic invariant. The property of being skew symmetric is a property that carries over to any new form yet the condition is not expressible as an algebraic invariant but only as a tensor. This condition is expressed by the tensor equation

$$\pi^{\rho\sigma}_{rs}\, a_{\rho\sigma}=0,\ (\pi^{\rho\sigma}_{rs} = \delta^\rho_r\delta^\sigma_s + \delta^\sigma_r\delta^\rho_s),\quad \text{or}\quad a_{rs} = -\, a_{sr}.$$

In a new coordinate system this becomes $\pi^{\rho\sigma}_{rs}\bar{a}_{\rho\sigma}=0$, from which we find $\bar{a}_{rs}=-\bar{a}_{sr}$. The method of obtaining other types of symmetry conditions for tensors with the use of the δ or γ tensors is apparent.*

5. *Bilinear Forms.* With the two-index tensor a_{rs} the following invariant of weight one may be formed if n is even (equal to $2i$, say):

$$\epsilon^{\rho_1\sigma_1\cdots\rho_i\sigma_i}a_{\rho_1\sigma_1}\cdots a_{\rho_i\sigma_i}. \tag{14}$$

If we seek the invariants of weight two, the degree must equal n, for example

$$\epsilon^{\rho_1\cdots\rho_n}\epsilon^{\sigma_1\cdots\sigma_n}a_{\rho_1\sigma_1}\cdots a_{\rho_n\sigma_n}. \tag{15}$$

The set of indices either ρ or σ on the elements is called a range. If the tensor is not symmetric the indices of an ϵ need not be summed with a range. There will then be many invariants like (15). If, however, they are summed with a range, the second ϵ merely multiplies the invariant by $n!$.

6. *Systems of Bilinear Forms.* Explicit expressions for a

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* Weyl, *Space, Time, Matter*, 4th edition, pp. 57, 58.

system of bilinear forms are readily written. Represent the system by

$$\phi_{q|} = a_{q|\rho\sigma} u^{\rho} v^{\sigma}, \qquad (q = 1, 2, \cdots).$$

The following is a set of invariants of the forms:

$$\epsilon^{\rho_1 \cdots \rho_n} \epsilon^{\sigma_1 \cdots \sigma_n} a_{q_1|\rho_1\sigma_1} \cdots a_{q_n|\rho_n\sigma_n}. \tag{16}$$

The q's may be distinct or repeated (see reference in §3) and any of the elements may be outer products of elements of linear forms.

7. *Trilinear Forms.* With a given tensor a_{rst} it is only necessary to construct inner tensor products with the tensors $\epsilon^{\rho_1 \cdots \rho_n}$ so that all indices on the elements are summed. All such are algebraic invariants and all algebraic invariants are thus acquired. To illustrate in detail a simple case, choose $n=2$ and require umbral letters on an ϵ to be associated with indices of a range. Such invariants will also be invariants of the cubic, for which a_{rst} is symmetrical. Thus consider the invariant of degree 4 and weight 6 of the binary cubic

$$\begin{aligned} I &= \epsilon^{\alpha_1\alpha_3}\epsilon^{\beta_1\beta_3}\epsilon^{\alpha_2\alpha_4}\epsilon^{\beta_2\beta_4}\epsilon^{\gamma_1\gamma_4}\epsilon^{\gamma_2\gamma_3} a_{\alpha_1\beta_1\gamma_1} a_{\alpha_2\beta_2\gamma_2} a_{\alpha_3\beta_3\gamma_3} a_{\alpha_4\beta_4\gamma_4} \\ &= \epsilon^{\alpha_1\alpha_3}\epsilon^{\beta_1\beta_3} a_{\alpha_1\beta_1\gamma_1} a_{\alpha_3\beta_3\gamma_3} \cdot \epsilon^{\alpha_2\alpha_4}\epsilon^{\beta_2\beta_4} a_{\alpha_2\beta_2\gamma_2} a_{\alpha_4\beta_4\gamma_4} \cdot \epsilon^{\gamma_1\gamma_4}\epsilon^{\gamma_2\gamma_3}. \end{aligned} \tag{17}$$

The product of the first four quantities in this equation is a cubic determinant so the indices γ_1 and γ_3 are a permanent range. We may represent this product by $A_{\gamma_1\gamma_3}$. The whole theory of algebraic invariants might be translated into theorems regarding inner products of higher determinants. Further details of evaluating this invariant will be carried out:

$$I = \epsilon^{\gamma_1\gamma_4} A_{\gamma_1\gamma_3} A_{\gamma_4\gamma_2} \epsilon^{\gamma_2\gamma_3}.$$

The first three terms of this are again a determinant, hence the range $\gamma_2\gamma_3$ is alternating and yields when multiplied innerly with $\epsilon^{\gamma_2\gamma_3}$, $I = 2!\epsilon^{\gamma_1\gamma_4} A_{\gamma_1\gamma_3} A_{\gamma_4\gamma_2}$. This invariant is determined by the particular manner in which the indices of an ϵ are associated with the indices of the elements; thus schematically we may write 17 as

$$I = (13)(13)(24)(24)(14)(23) = (13)^2(24)^2(14)(23).$$

This expression is 2! times the invariant represented in the same manner by the symbolic method.

8. *Conclusion.* We have here laid down a basis for a development of the theory of algebraic invariants which is applicable to tensors of any type. Transition can be made to the symbolic method which now becomes established without the use of ideal vectors. The computation seems particularly more simple than for the symbolic method.

The ϵ symbols have been used throughout and the number of indices on an ϵ was always the same as the number of components of a vector in n-space or as the number of variables used to represent a point in $(n-1)$-space in homogeneous coordinates. If in any of the formulas we replace an $\epsilon^{r_1\cdots r_n}$ or $\epsilon_{s_1\cdots s_n}$ by a $\delta^{r_1\cdots r_k}_{s_1\cdots s_k}$ and allow k to be less than n, each algebraic invariant becomes an alternating tensor which has 1 distinct component for $k=n$ and more components for $k<n$. This observation shows rather clearly the position algebraic invariants occupy in tensor algebra and again among alternating tensors. The reason that algebraic invariants have been of greater interest lies in the fact that the vanishing of an algebraic invariant gives but one condition while the vanishing of any other tensor is a much stronger condition since it gives rise to as many conditions as the tensor has distinct components.*

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*F. L. Hitchcock has written a number of papers bearing upon the subject of this paper, in particular see *A new method in the theory of quantics*, Journal of Mathematics and Physics, vol. 4 (1925), p. 238. If his view point is adopted in this paper, the results remain true if the indices associated with each ϵ and with the indices of A that are multiplied innerly with this ϵ are cogredient, that is, we know that all invariants are obtained by these methods. Further we know that all such upon specialization by making all indices cogredient become algebraic invariants. Then, however, the question would still remain open as to whether these included all algebraic invariants.

TRANSVERSE SEISMIC WAVES ON THE SURFACE OF A SEMI-INFINITE SOLID COMPOSED OF HETEROGENEOUS MATERIAL*

BY H. BATEMAN

In his Adams' Prize Essay of 1910, A. E. H. Love† discussed the propagation of transverse seismic waves in a homogeneous superficial layer covering a homogeneous solid differing from the surface layer in rigidity and density. The analysis has been extended by K. Aichi‡ and E. Meissner§ to the case of a heterogeneous material stratified in horizontal layers. Aichi adopts an exponential law for the variation of secondary wave velocity|| with depth and obtains a solution involving Bessel functions. Meissner considers various laws of variation of density and rigidity, and in one case he also obtains a solution involving Bessel functions. In this paper, a solution is given for a heterogeneous material of such a kind that the functions occurring are all of an elementary nature.

Using the same notations as Aichi, but taking the positive direction of the axis of z upwards, the axis of x being in the direction of propagation, we assume that the transverse displacement η is given by the equation

$$\eta = Y(z) \cos (pt + fx), \tag{1}$$

and that the density ρ and rigidity μ of the material depend only on the coordinate z.

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* Presented to the Society, San Francisco Section, October 29, 1927.

† *Some Problems of Geodynamics*, Cambridge University Press, 1911, p. 160.

‡ Proceedings of the Physico-Mathematical Society of Japan, vol. 4 (1922), pp. 137–142.

§ Proceedings of the Second International Congress for Applied Mathematics, Zürich, 1926.

|| This is the quantity c defined by equation (5).

The components of stress across an area perpendicular to the axis of y are $\mu\partial\eta/\partial x$, 0, $\mu\partial\eta/\partial z$, respectively, and so the equation of motion

$$\rho\frac{\partial^2\eta}{\partial t^2} = \frac{\partial}{\partial x}\left(\mu\frac{\partial\eta}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial\eta}{\partial z}\right)$$

gives Meissner's equation

$$\frac{d}{dz}\left(\mu\frac{dY}{dz}\right) + (\rho p^2 - \mu f^2)Y = 0. \tag{2}$$

When the rigidity μ is constant, this equation reduces to the equation

$$\frac{d^2Y}{dz^2} + \left(\frac{\rho p^2}{\mu} - f^2\right)Y = 0, \tag{3}$$

used by Aichi.

In addition to the differential equation, Y must satisfy the boundary conditions $Y=0$ when $z=-\infty$, $dY/dz=0$ when $z=a$. The first of these conditions expresses simply that there is no deep penetration of the waves while the second condition expresses that there is no tangential stress at the free surface $z=a$.

Taking first the case of a constant value of μ, we write

$$\frac{\rho p^2}{\mu} = 2n^2 \operatorname{sech}^2 nz + k^2, \qquad f^2 = n^2 \operatorname{ctnh}^2 \omega + k^2, \tag{4}$$

where k^2, ω, and n are positive constants when p is given. Our law of variation of ρ thus apparently depends on the frequency of the waves, and is definite only when this frequency is assigned. We shall therefore regard this particular p as a constant, and we shall use a different symbol P when the frequency is varied.

It is easily seen that the differential equation and the first boundary condition are satisfied by the equation

$$Y = e^{nz \operatorname{ctnh} \omega} \operatorname{sech} nz \cosh(nz - \omega).$$

The second boundary condition is satisfied if

$$\sinh^2\omega \tanh^2 na - \sinh\omega\cosh\omega\tanh na + 1 = 0.$$

Regarding this as an equation for tanh na when ω is given, we find that if cosh $\omega > 2$ there are two real values of tanh na, both of which are positive. This means that at first ρ increases with the depth. Now c, the velocity of propagation of "S" waves, is given by the formula

$$c^2 = \frac{\mu}{\rho}. \tag{5}$$

Hence at first c decreases downwards reaching a minimum value at the plane $z=0$. Below this plane c increases with the depth and gradually approaches a limiting value which is greater than the value at the surface. The existence of a solution in the present case is particularly interesting because in Love's case a solution exists only when c is greater for the substratum than for the upper layer. Let b denote the value of c at the surface and v the velocity of the transverse waves represented by (1); then we have

$$v = \frac{p}{f}, \quad \frac{p^2}{b^2} = 2n^2 \operatorname{sech}^2 na + k^2, \quad f^2 = n^2 \operatorname{ctnh}^2 \omega + k^2\,;$$

but

$$2 \operatorname{sech}^2 na = \operatorname{ctnh}^2 \omega \mp \operatorname{ctnh} \omega (\operatorname{ctnh}^2 \omega - 4 \operatorname{csch}^2 \omega)^{1/2}\,;$$

therefore

$$\frac{p^2}{b^2} - f^2 = \mp\, n^2 \operatorname{ctnh} \omega (\operatorname{ctnh}^2 \omega - 4 \operatorname{csch}^2 \omega)^{1/2}.$$

This equation tells us that v is greater or less than b according as the lower or upper sign is given to the square root. When the upper sign is taken, the amplitude Y begins to increase downwards until it attains a maximum value at a level z given by the equation

$$2 \sinh \omega \tanh nz = \cosh \omega - (\cosh^2 \omega - 4)^{1/2}\,;$$

beyond this level the amplitude decreases steadily to zero. When the lower sign is taken the amplitude decreases steadily to zero from the surface downwards. To discuss the effect of a change of frequency, we consider the propagation of the disturbance

$$\eta = Y(z)\cos(Pt + Fx),$$

the law of variation of density with depth being the same as before. Writing

$$2P^2 = m(m+1)p^2, \quad 2F^2 = m(m+1)k^2 + 2\sigma^2 n^2,$$

where m and σ depend on P but p is constant as before, we find that Y must now satisfy the differential equation

$$\frac{d^2Y}{dz^2} + n^2 Y\left[m(m+1)\operatorname{sech}^2 nz - \sigma^2\right] = 0. \tag{6}$$

A particular solution is given by the hypergeometric function

$$Y = e^{n\sigma z} F(-m,\ m+1\ ;\ 1-\sigma\ ;\ \xi), \tag{7}$$

where

$$2\xi = 1 - \tanh nz.$$

When m is an integer, this function is a polynomial in ξ except for the exponential factor. If σ is positive, Y will satisfy the boundary condition at $z = -\infty$. The boundary condition at $z = a$ gives a relation between σ and the quantity $\tanh na$, which will be denoted by the symbol τ. In the cases $m = 1, 2, 3$, the relations are

$$\sigma^2 - \sigma\tau + \tau^2 = 1,$$

$$\sigma^3 - 3\sigma^2\tau + 6\sigma\tau^2 - 6\tau^3 = 4\sigma - 6\tau,$$

$$\sigma^4 - 6\sigma^3\tau + 21\sigma^2\tau^2 - 45\sigma\tau^3 + 45\tau^4 = 10\sigma^2 - 39\sigma\tau + 54\tau^2 - 9,$$

respectively. Portions of the curves represented by these equations are drawn roughly in Fig. 1, σ and τ being rectangular coordinates.

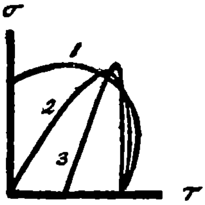

FIG. 1

For large values of m there are several possible values of σ corresponding to a given value of τ; thus when $m=3$ and $\tau=1$, we can have $\sigma=0$, 1, 2, or 3.

For all integral values of m, unity is a possible value of σ corresponding to $\tau=1$, and when τ is very nearly equal to one, there is a value of σ which is very nearly equal to one.

An approximate expression for this value of σ may be derived from equation (7) by differentiating with respect to z and retaining only first powers of $1-\tau$ after z has been put equal to a. The resulting equation,

$$0 = \sigma\left[1 - \frac{m(m+1)}{2(1-\sigma)}(1-\tau)\right] + (1-\tau^2)\left[\frac{m(m+1)}{2(1-\sigma)}\right],$$

gives approximately

$$\sigma = 1 + \frac{m(m+1)}{2}(1-\tau).$$

Using this value of σ, we have approximately

$$\begin{aligned} 2F^2 &= 2n^2 + m(m+1)\left[k^2 + n^2(1-\tau)\right] \\ &= 2n^2 + 2\frac{P^2}{p^2}\left[k^2 + n^2(1-\tau)\right]. \end{aligned}$$

Now the wave-velocity v and the group-velocity u are given by the formulas,*

$$\frac{1}{v} = \frac{F}{P} = (pP)^{-1}\left[n^2p^2 + P^2k^2 + P^2n^2(1-\tau)\right]^{1/2},$$

$$\frac{1}{u} = \frac{dF}{dP} = Pp^{-1}\left[k^2 + n^2(1-\tau)\right] \cdot \left[n^2p^2 + P^2k^2 + P^2n^2(1-\tau)\right]^{-1/2},$$

$$\frac{u}{v} = 1 + \frac{n^2p^2}{P^2k^2 + P^2n^2(1-\tau)}.$$

Since $1-\tau$ is positive the group-velocity is greater than the

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[*] H. A. Lorentz, *Problems of Modern Physics*, p. 17.

wave-velocity. This result is fundamentally different* from that obtained in Love's case of two adjacent homogeneous layers, and also from that obtained by Meissner for the case of a heterogeneous medium in which the density and rigidity vary continuously and monotonically. Some doubt may be felt with regard to the validity of our approximation and the differentiation with respect to P and it is hoped that the result will be checked by some independent method.

Turning now to the case in which the rigidity varies with the depth, we put

$$Z = \mu^{1/2} Y \ ;$$

the differential equation satisfied by Z is then

$$\frac{d^2 Z}{dz^2} + Z\left[\frac{oP^2}{\mu} - F^2 + \frac{1}{2\mu}\frac{d^2\mu}{dz^2} - \frac{1}{4\mu^2}\left(\frac{d\mu}{dz}\right)^2\right].$$

This differential equation may be reduced to the type already considered in the following cases:

(i) $\mu = \nu e^{-2hz}$, $$\frac{\rho p^2}{\mu} = 2\omega^2 \operatorname{sech}^2 nz + k^2,$$

$$F^2 = \frac{P^2 k^2}{p^2} + \sigma^2 n^2 + h^2,$$

(ii) $\mu = \nu \cosh nz$, $$\frac{\rho p^2}{\mu} = 2\omega^2 \operatorname{sech}^2 nz + k^2,$$

$$F^2 = \frac{P^2 k^2}{p^2} + \frac{n^2}{4} + \sigma^2 n^2,$$

where ω is an arbitrary constant.

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* The group velocity is generally less than the wave-velocity. This is shown very clearly in Meissner's diagrams but Meissner's definition of the group-velocity is different from ours.

ON CONTINUOUS CURVES IN n DIMENSIONS*

BY G. T. WHYBURN AND W. L. AYRES†

If M_1 and M_2 are subsets of a connected point set M, the subset K of M is said to *separate* M_1 and M_2 in M if $M-K$ is the sum of two mutually separated sets containing M_1 and M_2 respectively. R. L. Moore‡ has shown that in order that a plane continuum M be a continuous curve§ it is necessary and sufficient that for every two distinct points A and B of M there should exist a subset of M which consists of a finite number of continua and which separates A and B in M. Consider the following example: Let S_i $(i=1, 2)$ be the set of all points (x, y, z) in three dimensions such that $x=(-1)^i$, $-1\leqq y\leqq 1$, $0\leqq z\leqq 1$. Let R_0 be the set of all points (x, y, z) such that $-1\leqq x\leqq 1$, $-1\leqq y\leqq 1$, $z=0$. For each integer $n>0$, let R_n be the set of all points (x, y, z) such that $-1\leqq x\leqq 1$, $-1\leqq y\leqq 1$, $z=1/n$. Let

$$M = S_1 + S_2 + \sum_{n=0}^{\infty} R_n.$$

It is easy to see that every two points of M may be separated by a single subcontinuum of M and yet M is not a continuous curve. Hence the condition given by Moore is not sufficient in order that a continuum in n dimensions $(n>2)$ be a continuous curve. In this paper we give two modifications (Theorems 2 and 4) of Moore's theorem which hold in n dimensions.

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† National Research Fellow in Mathematics.

‡ *A characterization of a continuous curve*, Fundamenta Mathematicae, vol. 7 (1925), pp. 302–307.

§ We shall use the term *continuous curve* in the sense of a point set which is closed, connected and connected im kleinen. See R. L. Moore, *Concerning simple continuous curves*, Transactions of this Society, vol. 21 (1920), p. 347.

THEOREM 1.* *If M is a continuous curve in euclidean space of n dimensions, K is a bounded subcontinuum of M and ϵ is any positive number, then there exists a set L such that* (1) *$K+L$ is a continuous curve which is a subset of M,* (2) *every point of L is within a distance ϵ of some point of K,* (3) *L consists of a countable set of arcs of M, not more than a finite number of which are of diameter greater than any given positive number,* (4) *$L+K$ is non-dense at every point except those points at which K fails to be non-dense.*

PROOF. The set M is uniformly connected im kleinen over the set K.† Let δ_1, δ_2, δ_3, $\cdots$ be a sequence of positive numbers such that every two points of K whose distance from one another is less than δ_m can be joined by an arc of M whose diameter is less than $\epsilon/2m$. For each point p of K and each positive integer n, let C_{np} and C'_{np} be hyperspheres with center p and radii ϵ/n and $\epsilon/(2n)$ respectively.‡ By the Borel theorem, for each value of n there is a finite subset of the set $[C'_{np}]$,

$$C'_{np_{n1}},\ C'_{np_{n2}},\ C'_{np_{n3}},\ \cdots,\ C'_{np_{nn'}},$$

such that every point of K is in the interior of one of the sets $C'_{np_{ni}}$ for $1 \leqq i \leqq n'$. Since M is a continuous curve there are but a finite number,

$$M_{ni1},\ M_{ni2},\ M_{ni3},\ \cdots,\ M_{nim_n},$$

of the components§ of $M \cdot I(C_{np_{ni}})$ that contain points in the interior of $C'_{np_{ni}}$. For each n, i and j, let $[K_{nij}]$ be the set of components of $K\cdot M_{nij}\cdot I(C'_{np_{ni}})$. By the Zermelo postulate[*], there exists a set of points $[P_{nij}]$ such that each set K_{nij} contains just one point P_{nij} and each point P_{nij} belongs to just one component K_{nij}. In the set $[P_{nij}]$ there is a finite subset,

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* This theorem contains as a special case a theorem due to H. M. Gehman, *Concerning the subsets of a plane continuous curve*, Annals of Mathematics, vol. 27 (1925), pp. 29–46, Theorem 3.

† S. Mazurkiewicz, *Sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 1 (1920), p. 173.

‡ If p is a point and r a positive number, the *hypersphere* with center p and radius r is the set of all points of the space whose distance from the point p is r. If S is a hypersphere, $I(S)$ denotes the interior of S.

§ A connected subset of a point set H which is not a *proper* subset of any connected subset of H is called a *component* of H.

$$\overset{1}{P}_{nij},\ \overset{2}{P}_{nij},\ \overset{3}{P}_{nij},\ \cdots,\ \overset{k_1}{P}_{nij},$$[†]

such that every point of $[P_{nij}]$ is within a distance δ_1 of some point of this finite set. There exists an arc α^r_{nij} $(1\leqq r\leqq k_1-1)$ with end points P^r_{nij} and P^{r+1}_{nij} and lying wholly in M_{nij}. There exists a finite subset,

$$\overset{k_1+1}{P}_{nij},\ \overset{k_1+2}{P}_{nij},\ \cdots,\ \overset{k_2}{P}_{nij},$$

of the set $[P_{nij}]$ such that every point of $[P_{nij}]$ is within a distance δ_2 of some point of P^1_{nij}, P^2_{nij}, $\cdots$, $P^{k_1}_{nij}$. Let α^r_{nij} $(k_1\leqq r\leqq k_2-1)$ be an arc of M_{nij} with end points P^{r+1}_{nij} and some point of $\sum_{s=1}^{k_1} P^s_{nij}$. Continue this process indefinitely except that for $t>n$ we place the additional condition on $\alpha^r_{nij}(k_t\leqq r\leqq k_{t+1}-1)$ that it be of diameter less than $\epsilon/(2t)$. This can be done since any two points of K whose distance from one another is less than δ_t can be joined by an arc of M whose diameter is less than $\epsilon/(2t)$.

For each n, i and j, there is a countable set of arcs of M, $\overset{1}{\alpha}_{nij}$, $\overset{2}{\alpha}_{nij}$, $\overset{3}{\alpha}_{nij}$, $\cdots$, such that (a) each lies interior to a hypersphere of radius ϵ/n with a point of K as center, (b) only a finite number are of diameter greater than a given positive number, and (c) each has its end points on K. For each value of n the numbers i and j range over finite sets of values; hence the set of all arcs $[\alpha^r_{nij}]$ for a fixed value of n satisfy conditions (a), (b), and (c) above. And since all arcs $[\alpha^r_{nij}]$ for a fixed value of n are of diameter less than $2\epsilon/n$, the set of all arcs $[\alpha^r_{nij}]$ for all values of n satisfies the condition that only a finite number are of diameter greater than a given positive number. Let

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[*] E. Zermelo, *Untersuchung über die Grundlagen der Mengenlehre,* Mathematische Annalen, vol. 65 (1908), pp. 261–281.

[†] The symbol k_1 denotes a positive integer whose value depends on n, i and j.

$$L = \sum \alpha^{r}_{nij}, \quad \begin{array}{ll} 1 \leqq i \leqq n', & 1 \leqq j \leqq m_n, \\ 1 \leqq r < \infty, & 1 \leqq n < \infty. \end{array}$$

We have shown that L satisfies conditions (2) and (3) of our theorem. It remains to prove that (1) and (4) are satisfied. Since only a finite number of the arcs of L are of diameter greater than a given positive number and each has a point on the closed set K, every limit point of L which does not belong to L belongs to K. Thus $K+L$ is closed. Let P be any point of $K+L$. If P does not belong to K it is easy to see that $K+L$ is connected im kleinen at P, for the interiors of hyperspheres of sufficiently small radii and center P contain no point of K and points of only a finite number of arcs of L. If P is a point of K and η is any positive number, there is a hypersphere $C_{np_{ni}}$ which lies entirely in the interior of the hypersphere with radius $\eta/4$ and center P and such that $I(C'_{np_{ni}})$ contains P. Let M_{nij} be the component of $M \cdot I(C_{np_{ni}})$ containing P. There exists a positive number γ such that every point of K whose distance from P is less than γ lies in M_{nij}. There exists a number $\rho > 0$ such that every point p' of L whose distance from P is less than ρ lies on an arc $\alpha_{p'}$ of L, one of whose points e belongs to $K \cdot M_{nij}$ and such that the subarc $p'e$ of $\alpha_{p'}$ is of diameter less than $\eta/2$.* Let σ be the smaller of γ and ρ. Now let Q be any point of $K+L$ whose distance from P is less than σ. If Q belongs to K it belongs to M_{nij}. By the method of construction of L, there is a subset L' of L such that M_{ni_j} contains L' and $L'+K\cdot M_{ni_j}$ is connected. But every point of $L'+K\cdot M_{ni_j}$ is at a distance from P less than $\eta/2$ and $L'+K\cdot M_{ni_j}$ contains both P and Q. If Q is not a point of K, it lies on an arc α_Q of L which contains a point e of $K\cdot M_{ni_j}$ such that the subarc eQ of α_Q is of diameter less than $\eta/2$. Then $\alpha_Q+L'+K\cdot M_{ni_j}$ is a connected subset of $L+K$ containing P and Q and such that every point is at a distance from P less than η. Therefore $K+L$ is connected im kleinen at every point P.

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* If S_γ and S_d denote hyperspheres with center P and radii γ and d respectively, then only a finite number of arcs of L have points in $I(S_d)$ for any $d < \gamma$ and contain no point of $I(S_\gamma) \cdot K$ since any such arc is at least of diameter $\gamma - d$. There is a number $d_1 > 0$ such that for $d \leqq d_1$ there is no such arc. Also there is a number $d_2 > 0$ such that no arc of L of diameter greater than $\eta/2$ contains a point whose distance from P is less than d_2 unless the arc contains P. On each of the finite set of arcs of L of diameter greater than $\eta/2$ that contain P there is a point q such that the subarc qP of the arc is of diameter less than $\eta/2$. Let d_3 be the smallest of the finite set of distances from P to the points q. Let ρ be the smallest of the numbers d_1, d_2 and d_3.

Let P be any point of K at which K is non-dense. Then if S_1 is any hypersphere with center P, the set $I(S_1)$ contains a hypersphere S_2 such that $S_2+I(S_2)$ contains no point of K. Since only a finite number of the arcs of L are of diameter greater than a given positive number, there are only a finite number of arcs of L that have points in $I(S_2)$. Then there is a hypersphere S_3 lying in $I(S_2)$ such that $I(S_3)$ contains no point of L. Then the interior of every hypersphere S_1 with center at P contains a hypersphere S_3 such that $I(S_3)$ contains no point of $K+L$. Hence $K+L$ is non-dense at the point P.

THEOREM 2. *In order that a continuum M lying in euclidean space of n dimensions be a continuous curve it is necessary and sufficient that for every two distinct points A and B of M there should exist a subset of M which consists of a finite number of continuous curves and which separates A and B in M.*

PROOF. The condition is necessary. Let d be the distance from A to B. Let S_1 and S_2 be hyperspheres with center A and radii $d/2$ and $d/4$ respectively. Let $H=S_1+I(S_1)-I(S_2)$. The set $M\cdot H$ is closed and it is easy to see that there is at least one component of $M\cdot H$ containing points on both S_1 and S_2. As M is a continuous curve there cannot be more than a finite number of such components. Let K_1, K_2, $K_3, \cdots, K_m$ denote the set of all components of $M\cdot H$ which contain a point on S_1 and a point on S_2. By Theorem 1, for each i, $1\leqq i\leqq m$, there is a continuous curve M_i which contains

K_i, is a subset of M and such that every point of M_i is within a distance $d/8$ of some point of K_i. Suppose that A and B lie in a connected subset of $M - \sum_{i=1}^{m} M_i$. Then there is an arc with end points A and B lying in $M - \sum_{i=1}^{m} M_i$.* This arc contains a subarc α which is a subset of H and has one end point on S_1 and the other on S_2. Then α must belong to some set K_i and thus to $\sum_{i=1}^{m} M_i$. But this is impossible, for $M - \sum_{i=1}^{m} M_i$ contains α. Therefore $\sum_{i=1}^{m} M_i$ separates A and B in M.

The condition is sufficient. If M is not a continuous curve there exist two concentric hyperspheres S_1 and S_2 and an infinite set of subcontinua M_∞, M_1, M_2, M_3, $\cdots$ of M satisfying the conditions of the Moore-Wilder lemma.† Let S_3 and S_4 be distinct hyperspheres concentric with S_1 and lying between S_1 and S_2. Each continuum M_i contains a subcontinuum K_i which contains a point P_i on S_3 and a point Q_i on S_4 and is a subset of the set G consisting of S_3 and S_4 and all points which lie between S_3 and S_4. There exists a sequence of integers n_1, n_2, $\cdots$, such that $[P_{n_i}]$ has a sequential limit point A and $[Q_{n_i}]$ has a sequential limit point B. By hypothesis there exists a finite set of continuous curves C_1, C_2, C_3, $\cdots$, C_m which are subsets of M and separate A and B in M.

CASE I. Suppose infinitely many of the continua K_{n_i} contain a point of $\sum_{k=1}^{m} C_k$. As there are but a finite number of the curves C_k, one curve $C_{k'}$ must contain a point p_{n_i} of K_{n_i} for infinitely many values of i. The set $[p_{n_i}]$ has a limit point P, which must belong to M_∞ and to G. Let ϵ be a positive number such that no point of $S_1 + S_2$ is within a distance ϵ of P. As $C_{k'}$ is a continuous curve, the point P

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* R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), pp. 254–260. Moore's theorem is stated for two dimensions, but the extension to n dimensions is obvious.

† R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, this Bulletin, vol. 29 (1923), p. 296; R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), p. 371. The lemma holds equally well for n dimensions and for unbounded continua.

belongs to $C_{k'}$ and there is a number $\delta_\epsilon > 0$ such that any point of $C_{k'}$ whose distance from P is less than δ_ϵ can be joined to P by an arc of $C_{k'}$ of diameter less than ϵ. There is a point p_{n_r} of $[p_{n_i}]$ whose distance from P is less than δ_ϵ. Let α denote an arc of $C_{k'}$ with end points P and p_{n_r} and of diameter less than ϵ. The arc α contains a point of M_{n_r} and a point of M_∞ and lies entirely between S_1 and S_2. By the Moore-Wilder lemma, M_{n_r} is a component of the common part of M and the set composed of S_1 and S_2 and all points lying between S_1 and S_2. Hence M_{n_r} contains the arc α. But this contradicts the condition of the lemma that M_{n_r} and M_∞ have no common points.

Case II. Suppose only a finite number of the continua K_{n_i} contain points of $\sum_{k=1}^{m} C_k$. The set $M - \sum_1^m C_k$ is the sum of two mutually separated sets M_A and M_B containing A and B respectively. Every set K_{n_i} which contains no point of $\sum_1^m C_k$ lies wholly in M_A or wholly in M_B. There is an integer j such that for $i \geqq j$, the continuum K_{n_i} contains no point of $\sum_1^m C_k$. Both A and B are limit points of the set $\sum_{i=j}^{\infty} K_{n_i}$. Either infinitely many of the sets K_{n_i} $(i \geqq j)$ belong to M_A or infinitely many belong to M_B. If the first holds then B is a limit point of M_A; under the second possibility the point A is a limit point of the set M_B. In either possibility we have a contradiction since M_A and M_B are mutually separated.

The assumption that M is not a continuous curve leads to a contradiction with the assumed condition in either case. Therefore the condition is sufficient.

It is to be noticed that in the proof of the necessity of the condition in Theorem 2 we showed that the separating continuous curves were bounded. Hence we have the following corollary and theorem.

Corollary. *If A and B are points of a continuous curve M lying in euclidean space of n dimensions, there exists a subset of M which consists of a finite number of bounded continuous curves and which separates A and B in M.*

THEOREM 3. *If K_1 and K_2 are any two mutually exclusive and closed point sets, one of which is bounded, then K_1 lies wholly in a finite number of the complementary domains of K_2.*

PROOF. Suppose the contrary is true. Then there exists an infinite sequence $D_1, D_2, D_3, \cdots$ of distinct complementary domains of K_2 each of which contains at least one point of K_1. For each positive integer i, let P_i denote a point of K_1 belonging to D_i. Let H denote the set of points $P_1+P_2+P_3+\cdots$. By hypothesis either K_1 or K_2 is bounded. If K_1 is bounded, then H is bounded because H is a subset of K_1; and if K_2 is bounded, then since H contains at most one point in the unbounded complementary domain of K_2, it readily follows that H is bounded. Hence, in any case, H is bounded; and since it is infinite, it must have at least one limit point P. Since K_1 is closed and contains H, it must contain the point P; and since K_1 and K_2 are mutually exclusive, P must belong to some complementary domain D of K_2. Clearly this is impossible, since P is a limit point of H, and not more than one point of H can belong to D. Thus the supposition that Theorem 3 is not true leads to a contradiction.

THEOREM 4. *In order that a continuum M in a euclidean space E_n of n dimensions should be a continuous curve it is necessary and sufficient that every two mutually exclusive, closed, and bounded subsets of M should be separated in M by the sum of a finite number of subcontinua of M.*

PROOF.* The condition is sufficient. For suppose a continuum M satisfies the condition but is not a continuous curve. Then by the Moore-Wilder lemma† it follows that there exist two different concentric hyperspheres C_1 and C_2 and a countable infinity of mutually exclusive subcontinua of M: $W, M_1, M_2, M_3, \cdots$ such that (1) if D denotes the

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* Compare this proof with that given by R. L. Moore for Theorem 1 of his paper, *A characterization of a continuous curve*, loc. cit.

† See reference to the Moore-Wilder lemma above.

n-dimensional domain whose boundary is C_1+C_2, then each of these continua contains at least one point on each of the hyperspheres C_1 and C_2, and each of them, save possibly W, is a component of the set of points $M\cdot(D+C_1+C_2)$, and (2) W is the sequential limiting set of the sequence of continua $M_1, M_2, M_3, \cdots$. Let A and B denote the sets of points $W\ C_1$ and $W\cdot C_2$ respectively and, for each positive integer i, let a_i denote the set of points $M_i\cdot C_1$ and b_i the set $M_i\ C_2$. Since A and B are mutually exclusive, closed, and bounded subsets of M, by hypothesis there exists a subset L of M such that (1) $M-L$ is the sum of two mutually separated point sets M_a and M_b containing A and B respectively, and (2) L is the sum of a finite number of continua $L_1, L_2, L_3, \cdots, L_m$. Since neither A nor B has a point in common with L, and A contains no point of M_b and B contains no point of M_a, therefore there exist open sets C_a and C_b, containing A and B respectively, such that C_a contains no point of $L+M_b$ and C_b contains no point of $L+M_a$. There exists an integer δ such that, for every j greater than δ, the point set a_j lies wholly in C_a and the point set b_j lies wholly in C_b. Thus, for every j greater than δ, M_j contains a point of M_a and also a point of M_b. But M_j is a subcontinuum of M, and every subcontinuum of M which contains a point of each of the sets M_a and M_b must contain at least one point of L. Hence, for every j greater than δ, M_j contains a point of L, and therefore of some one of the sets $L_1, L_2, \cdots, L_m$. It follows that there exists an integer g and an infinite sequence of distinct positive integers $t_1, t_2, t_3, \cdots$ such that, for every j, L_g contains at least one point in common with M_{t_j}. Since, for every j, the subcontinuum L_g of M contains a point of M_{t_j} and a point of $M_{t_{j+1}}$ it follows by a lemma of R. L. Moore's* that L_g must contain a point either of a_{t_j} or of b_{t_j}. Thus there exists an infinite sequence of distinct integers $j_1, j_2, j_3, \cdots$, such that either L_g has a point in common with each point set of the sequence $a_{j_1}, a_{j_2}, a_{j_3}, \cdots$, or it has at least one point in common with each point set of the sequence $b_{j_1}, b_{j_2}, b_{j_3}, \cdots$. In the first case it readily follows that A contains at least one point of L_g, and in the second case that B contains at least one point of L_g. But $A+B$ is a subset of $M-L$. Thus the supposition that M is not a continuous curve leads to a contradiction.

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* *A characterization of a continuous curve*, loc. cit., Lemma 2.

The condition is also necessary. For let M be any continuous curve in E_n, and let K_1 and K_2 be any two mutually exclusive, closed, and bounded subsets of M. It follows by Theorem 3 that there exists a finite number $D_1, D_2, D_3, \cdots, D_m$ of the complementary domains of K_2 whose sum contains the point set K_1. For each positive integer $i \leqq m$, let B_i denote the boundary of D_i, let H_i be the set of points common to K_1 and D_i, and let $4d_i$ be the minimum distance between the closed sets of points H_i and B_i. For each point P of H_i+B_i, let C_p denote a hypersphere with P as center and radius d_i, and let G_i' be the collection of all the hyperspheres $[C_p]$ for all points P of H_i+B_i. Since K_1+K_2, and hence also H_i+B_i, is bounded, then by the Borel theorem there exists a finite subcollection G_i of the hyperspheres of G_i' such that H_i+B_i is a subset of the sum I_i of the interiors of the collection G_i. Let T_i denote the point set $(D_i+B_i) - I_i \cdot (D_i+B_i)$. Then T_i is closed. Let F_i denote the sum of all the hyperspheres (not including their interiors) of the collection G_i which enclose at least one point of H_i, and let N_i be the sum of all those which enclose at least one point of B_i. Since the least distance between H_i and B_i is $4d_i$, and since the radius of each hypersphere of G_i is d_i, it follows that F_i and N_i are mutually exclusive closed sets whose least distance apart is $>d_i$. Let Q_i denote the collection of all those maximal connected subsets of M which lie wholly in T_i and contain at least one point of each of the sets F_i and N_i. Each element of Q_i is a continuum, and since M is a continuous curve, it follows by the Moore-Wilder lemma that Q_i has just a finite number of elements. Hence Q_i is a finite collection of mutually exclusive continua L_{1i}, L_{2i}, L_{3i}, $\cdots$, $L_{n_i i}$ which belong to M.

Now let L denote the point set $\sum_{i=1}^{m}\sum_{j=1}^{n_i}L_{ji}$. Then L is the sum of a finite number of mutually exclusive subcontinua of M. Let M_a denote the sum of all those components of $M-L$ which contain at least one point of A, and let M_b denote the point set $M-(M_a+L)$. No point of B belongs to M_a. For if a point X of B belonged to M_a, then* X could be joined in M to some point Y of A by an arc which contains no point of L, and this arc would contain a subarc t which is a subset of some set T_i and which has its end points on F_i and N_i respectively; and the arc t would necessarily be a subset of some continuum of the collection Q_i, contrary to the fact that t contains no point of L. Therefore B must be a subset of M_b. Since M is connected im kleinen and L is closed, it readily follows that M_a and M_b are mutually separated. Hence $M-L$ is the sum of two mutually separated sets M_a and M_b containing A and B respectively, and therefore L separates A and B in M.

THEOREM 5. *In order that a continuum M in a space of n dimensions should be a Menger regular curve*† *it is necessary and sufficient that every two points of M should be separated in M by some finite subset of M.*

PROOF. The condition is sufficient. Let P be any point of M and ϵ any positive number. Let C_1 and C_2 be two distinct hyperspheres each of which has P as center and is of radius less than $\epsilon/4$. Let D denote the domain between C_1 and C_2, and let K denote the set of points common to $D+C_1+C_2$ and to M. Then K is closed. Now by Theorem 2 it follows that M is a continuous curve. By hypothesis, for each point X of K there exists a finite subset N_x of M which separates X and P in M. For each such point X, the maximal connected subset H_x of $M-N_x$ which contains X is an open subset of M which does not contain P and whose M-boundary is finite (a subset of N_x). Let G_0 denote the collection of sets $[H_x]$ for all points X of K. Since K is closed and bounded, then by the Borel theorem the collection G_0 contains a finite subcollection G which covers K. Let R denote the sum of all the point sets of the collection G. Then K is a subset of R, and R is an open subset of M. Furthermore B, the M-boundary of R, is finite, for R is the sum of a finite number of the sets H_x. Now, supposing that C_1 is within C_2, let A denote the set of all those points of B which lie on or within C_1. Now $R+A$ does not contain P, for P belongs to no set H_x and to no N_x. Let T denote the maximal connected subset of $M-A$ which contains P. It is readily seen that T must lie within C_1. Hence the diameter of T is less than ϵ. The M-boundary of T is finite, because it is a subset of A. Then, since T is an open subset of M, it follows that P is a regular point of M and that M is a Menger regular curve.

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* R. L. Moore, *Concerning continuous curves in the plane*, loc. cit.

† A continuum M is said to be a Menger regular curve provided that for each point P of M and each positive number ϵ there exists an open subset T of M of diameter less than ϵ which contains P and whose M-boundary is finite. The M-boundary of an open subset T of a continuum M is the set of all those points of $M-T$ that are limit points of T. See K. Menger, *Grundzuge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925–1926), pp. 276–306.

That the condition is necessary follows at once from the definition of a Menger regular curve.

THEOREM 6. *If every two points of a continuum M are separated in M by some finite subset of M, then every two mutually exclusive, closed, and bounded subsets of M are separated in M by some finite subset of M.*

PROOF. It follows by Theorem 5 that M is a Menger regular curve. Then by a theorem of Menger's,* it follows that every two mutually exclusive, closed, and bounded subsets of M can be separated in M by some finite subset of M.

THE UNIVERSITY OF TEXAS

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* Loc. cit., Theorem 12.

THE POLAR CURVES OF PLANE ALGEBRAIC CURVES IN THE GALOIS FIELDS*

BY A. D. CAMPBELL

By imitating the proofs in Fine's *College Algebra* (pp. 460–462) and Veblen and Young's *Projective Geometry* (vol. I, pp. 255–256) we can readily show that also in the Galois fields of order p^n (p a prime integer) we have Taylor's expansion

$$\begin{aligned} f(x + \lambda X, y + \lambda Y, z + \lambda Z) & \\ \equiv f(x, y, z) + \frac{\lambda}{1!}(f_x' X + f_y' Y + f_z' Z) & \\ + \frac{\lambda^2}{2!}(f_x' X + f_y' Y + f_z' Z)^{(2)} + \cdots & \\ + \frac{\lambda^r}{r!}(f_x' X + f_y' Y + f_z' Z)^{(r)} + \cdots + f(X, Y, Z) = 0, & \end{aligned}$$

where $(f_x' X+f_y' Y+f_z' Z)^{(i)}$ is symbolic for an expression containing derivatives of the ith order, and $f(x, y, z)=0$ is an algebraic curve of order n. In the above expansion we must take all the derivatives as though p were not a modulus, cancel out common factors from numerators and denominators, and then set $p=0$.

The rth polar of (X, Y, Z) with respect to $f(x, y, z)=0$ is

$$\frac{1}{r!}(f_x' X + f_y' Y + f_z' Z)^{(r)} = 0.$$

In particular the rth polar of $(1, 0, 0)$ is $(1/r!)\partial^r f(x, y, z)/\partial x^r=0$. We suppose first of all that n has the value

$$n = \alpha p^m + \beta p^{m-1} + \cdots + \gamma p^2 + \delta p + \epsilon,$$
$$\epsilon \neq 0, \quad p = \epsilon + \zeta, \quad \zeta \neq 0.$$

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* Presented to the Society, December 28, 1927.

We can write the polars of (1, 0, 0) by a sort of detached coefficients, underlining the coefficients that have p as a factor, as follows:

$$\begin{aligned}
&(1/1!)[n, n-1, n-2, \cdots, \underline{n-\epsilon}, n-\epsilon-1,\\
&\qquad \cdots, \underline{n-p-\epsilon}, \cdots, \underline{n-p^2-\epsilon}, \cdots, 3,2,1] = 0,\\
&(1/2!)[n(n-1), (n-1)(n-2), \cdots, (n-\epsilon+1)\underline{(n-\epsilon)},\\
&\qquad \underline{(n-\epsilon)}(n-\epsilon-1), \cdots, (n-p-\epsilon+1)\underline{(n-p-\epsilon)},\\
&\qquad \underline{(n-p-\epsilon)}(n-p-\epsilon-1), \cdots, (n-p^2-\epsilon+1)\\
&\qquad \cdot\underline{(n-p^2-\epsilon)}, \underline{(n-p^2-\epsilon)}(n-p^2-\epsilon-1), \cdots, 3\cdot 2, 2\cdot 1] = 0,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&[1/(\epsilon+1)!][n(n-1)(n-2)\cdots\underline{(n-\epsilon)}, (n-1)(n-2)\cdots\\
&\qquad \cdot\underline{(n-\epsilon)}(n-\epsilon-1), \cdots, \underline{(n-\epsilon)}(n-\epsilon-1)\cdots(n-2\epsilon),\\
&\qquad (n-\epsilon-1)(n-\epsilon-2)\cdots(n-2\epsilon)(n-2\epsilon-1), \cdots,\\
&\qquad (\epsilon+1)!] = 0,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&(1/p!)[n(n-1)\cdots\underline{(n-\epsilon)}\cdots(n-p+1)\cdots, p!] = 0,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot
\end{aligned}$$

where $(n-\lambda)$ $(n-\lambda-1)\cdots(n-\lambda-i)$ stands for all the terms of the same $(n-\lambda-i-1)$ power, which then have this common factor in their coefficients. From the above polars we see that the ϵth polar has at (1, 0, 0) a tangent having $(\epsilon+1)$-point contact if (1, 0, 0) is not on $f(x, y, z)=0$, otherwise a multiple point of order $\epsilon+1$. The $(\epsilon+1)$th polar, $(\epsilon+2)$th, $\cdots$, $(p-1)$th polar all have multiple points of order $\epsilon+1$ at (1, 0, 0). Similarly the $(p+\epsilon+1)$th polar, $(p+\epsilon+2)$th, $\cdots$, $(2p-1)$th have at (1, 0, 0) multiple points of order $\epsilon+1$; also the $(2p+\epsilon+1)$th polar points of order $(3p-1)$th, $\cdots$, the $(\theta p^i+\cdots+\phi p+\epsilon+1)$th polar points of order $(\theta p^i+\cdots+\phi p+p-1)$th, etc. Moreover we note that if any one of the polar curves that have multiple points at (1, 0, 0) is a curve of degree $\epsilon+1$, then this polar curve is degenerate. Thus for $p=2$, $n=2^2+1$, $\epsilon=1$, we find the 2d polar is degenerate; for $p=3$, $n=3+1$, $\epsilon=1$, we find again the 2d polar is degenerate.

If $n=\alpha p^m+\beta p^{m-1}+\cdots+\gamma p^2+\delta p$, i.e. $\epsilon=0$ in n, then all the polars of (1, 0, 0) pass through (1, 0, 0) whether or not this point lies on $f(x, y, z)=0$.

If $n<p$ we find no peculiarities like the above.

SYRACUSE UNIVERSITY

THE CHARACTERISTIC EQUATION OF A MATRIX*

BY E. T. BROWNE

1. *Introduction.* Consider any square matrix A, real or complex, of order n. If I is the unit matrix, $A-\lambda I$ is called the *characteristic matrix* of A; the determinant of the characteristic matrix is called the *characteristic determinant* of A; the equation obtained by equating this determinant to zero is called the *characteristic equation* of A; and the roots of this equation are called the *characteristic roots* of A. If A happens to be a matrix of a particular type certain definite statements may be made as to the nature of its characteristic roots. For example, if A is Hermitian its characteristic roots are all real; if A is real and skew-symmetric, its characteristic roots are all pure imaginary or zero; if A is a real orthogonal matrix, its characteristic roots are of modulus unity. However, if A is not a matrix of some special type, no general statement can be made as to the nature of its characteristic roots. In 1900 Bendixson† proved that if $\alpha+i\beta$ is a characteristic root of a real matrix A, and if $\rho_1\geqq\rho_2\geqq\cdots\geqq\rho_n$ are the characteristic roots (all real) of the symmetric matrix $\frac{1}{2}(A+A')$, then $\rho_1\geqq\alpha\geqq\rho_n$. The extension to the case where the elements of A are com-

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* Presented to the Society, December 28, 1927.

† Bendixson, *Sur les racines d'une équation fondamentale*, Acta Mathematica, vol. 25 (1902), pp. 359–365.

plex was made by Hirsch* in 1902. In 1904 Bromwich† further extended the theorem as follows: If $\alpha+i\beta$ is a characteristic root of a matrix A whose elements are real or complex, and if $\rho_1, \rho_2, \cdots, \rho_n$ are the characteristic roots (all real) of $\frac{1}{2}(A+\bar{A}')$ and $i\mu_1, \cdots, i\mu_n$ are the characteristic roots of $\frac{1}{2}(A-\bar{A}')$, then α lies between the greatest and the least of $\rho_1, \cdots, \rho_n$, and $|\beta|$ does not exceed the greatest of $|\mu_1|, \cdots, |\mu_n|$.

In some cases the theorems just cited give very good limits for the characteristic roots of a matrix, while in other cases the limits are not so restricted. Thus in the case of a real orthogonal matrix these theorems may merely state that the characteristic roots lie in the square $x=\pm 1$, $y=\pm 1$. In this paper we shall give a criterion which in some cases, notably in the case of a real orthogonal matrix, give more restricted limits than the theorems above.

2. *Reduction of a Matrix to a Semi-Unitary Form.* Let A be any square matrix of order n. Then $A\bar{A}'$ is Hermitian and there exists a unitary matrix κ (that is, $\kappa\bar{\kappa}'=I$) such that

$$\kappa A\bar{A}'\bar{\kappa}' = M,$$

where M‡ is zero except in the diagonal, and the elements in the diagonal are the (real) characteristic roots $\rho_1, \rho_2, \cdots, \rho_n$ of $A\bar{A}'$. We may write

$$M = \kappa A\bar{\kappa}'\kappa\bar{A}'\bar{\kappa}' = B\bar{B}', \tag{1}$$

where

$$B = \kappa A\bar{\kappa}'. \tag{2}$$

From (1) the elements b_{ij} of B evidently satisfy the conditions

$$\sum_{t}^{1,\ \ ,n} b_{it}\bar{b}_{jt} = \rho_i\delta_{ij}, \quad (i, j = 1, \cdots, n), \tag{3}$$

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* Hirsch, Acta Mathematica, vol. 25 (1902), p. 367.

† Bromwich, *On the roots of the characteristic equation of a linear substitution*, Acta Mathematica, vol. 30 (1906), pp. 295–304.

‡ Hilton, *Homogeneous Linear Substitutions*, Oxford, 1914, p. 41.

where δ_{ij} is the Kronecker symbol, and equals 1 if $i=j$; 0 if $i\neq j$. In view of the conditions (3) we shall say that B is in a *semi-unitary* (*semi-orthogonal*, if B is *real*) form. If $\rho_i=1$, $(i=1,\cdots,n)$, B is unitary. We may then state the following theorem.

THEOREM I. *If A is any square matrix of order n there exists a unitary matrix κ such that $\kappa A\bar{\kappa}'=B$, where B is in a semi-unitary form.*

If M is of rank r, κ may be so chosen that $\rho_i>0$, $(i=1,\cdots,r)$; $\rho_i=0$, $(i=r+1,\cdots,n)$. Since $\rho_i=\sum_t b_{it}\bar{b}_{it}=0$, $(i=r+1,\cdots,n)$, evidently $b_{it}=0$, $(i=r+1,\cdots,n;\ t=1,\cdots,n)$; that is, the last $n-r$ rows of B consist entirely of zeros, so that B is of rank at most r. Hence, B must be of rank exactly r. Since the rank of A equals the rank of B, and the rank of $A\bar{A}'$ equals the rank of M, incidentally we have given a proof of the following well known theorem.

THEOREM. *If A is any square matrix of order n, the ranks of A and $A\bar{A}'$ are the same.*[*]

3. *The Characteristic Roots of $A\bar{A}'$.* Referring to the matrix B defined as in (1) and (2), let us form a non-singular matrix $C=(c_{ij})$ by replacing the zeros in the last $n-r$ rows of B by elements $(x_{s1}, x_{s2},\cdots,x_{sn})\neq(0,0,\cdots,0)$, such that

$$(4)\qquad \sum_t^{1,\cdots,n} b_{it}\bar{x}_{st}=0,\qquad (i=1,\cdots,r;\ s=1,\cdots,n-r),$$

and, moreover, such that

$$\sum_t^{1,\cdots,n} x_{it}\bar{x}_{jt}=0,\qquad (i,j=1,\cdots,n-r;\ i\neq j).$$

Thus, we may find $(\bar{x}_{11},\bar{x}_{12},\cdots,\bar{x}_{1n})$ by determining a non-zero solution of the $n-r$ linear homogeneous equations (4). Having obtained $(x_{11},\cdots,x_{1n})$ we may proceed to find

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[*] Hilton, *Homogeneous Linear Substitutions*, Exercise 4, p. 51.

$(\bar{x}_{21}, \bar{x}_{22}, \cdots, \bar{x}_{2n})$ by adjoining to the system (4) the additional linear homogeneous equation

$$\sum_{t}^{1,\cdots,n} x_{1t}\bar{x}_{2t} = 0\,;$$

and so on. If $\sum_t^{1,\cdots,n} c_{it}\bar{c}_{it} = \rho_i$, $(i = 1, \cdots, n)$, then $\rho_i > 0$ and if we write

$$\chi_{ij} = \frac{c_{ij}}{(\rho_i)^{1/2}}, \qquad (i, j = 1, \cdots, n),$$

the matrix χ thus obtained is a unitary matrix. It is evident from the manner in which χ was built up that $B\bar{\chi}'$ is zero except in the diagonal. The elements in the last $n-r$ places in the diagonal are also zero, while those in the first places are $(\rho_i)^{1/2}$, the square roots of the characteristic roots of $A\bar{A}'$. Since $B\bar{\chi}'$ is real and symmetric, the characteristic roots of

$$N = \chi\bar{B}'B\bar{\chi}' = (B\bar{\chi}')^2$$

are the squares of the characteristic roots of $B\bar{\chi}'$, and are therefore the characteristic roots of $A\bar{A}'$. But

$$N = \chi\bar{B}'B\bar{\chi}' = \chi\kappa\bar{A}'\bar{\kappa}'\kappa A\bar{\kappa}'\bar{\chi}' = \chi\kappa\bar{A}'A\bar{\kappa}'\bar{\chi}' = \psi\bar{A}'A\bar{\psi}',$$

where ψ is the unitary matrix $\chi\kappa$. Thus it follows* that the characteristic roots of $\bar{A}'A$ are the same as those of N and therefore of $A\bar{A}'$. Hence we have the following theorem.

THEOREM II. *If A is any square matrix of order n the characteristic roots of $A\bar{A}'$ are the same as the characteristic roots of $\bar{A}'A$.*

Since the unitary matrices κ, χ above are such that

$$\kappa A\bar{\kappa}' = B, \text{ and } B\bar{\chi}' = \chi\bar{B}',$$

it follows at once that

$$\kappa A\bar{\kappa}'\bar{\chi}' = B\bar{\chi}' = \chi\bar{B}' = \chi\kappa\bar{A}'\bar{\kappa}'.$$

Hence

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* Hilton, *Homogeneous Linear Substitutions*, p. 20.

$$\bar{\kappa}'\bar{\chi}'\kappa A\bar{\kappa}'\bar{\chi}'\kappa = \bar{A}'.$$

Writing $\bar{\kappa}'\bar{\chi}'\kappa=\phi$, we have the following theorem.

THEOREM III. *If A is any square matrix of order n there exists a unitary matrix ϕ such that*

$$\phi A\phi = \bar{A}'. \tag{5}$$

In this connection compare Hilton, *Homogeneous Linear Substitutions*, Ex. 6, p. 124.

Since from (5)

$$A\phi = \bar{\phi}'\bar{A}' = (\bar{A}\bar{\phi})',$$

$A\phi$ is Hermitian, so that we have the following theorem.

THEOREM IV. *If A is any square matrix of order n, there exists a unitary matrix ϕ such that $A\phi$ is Hermitian.*

4. *The Characteristic Roots of A.* From (2) the characteristic roots of A are evidently the same as the characteristic roots of B. Suppose then that λ is a characteristic root of B so that there exists a set $(x_1, x_2,\cdots, x_n)\neq(0, 0,\cdots, 0)$ such that

$$\sum_{t}^{1,\cdots,n} b_{ti}x_t = \lambda x_i, \qquad (i = 1, \cdots, n). \tag{6}$$

Taking the conjugates of both members of each of these equations, we have

$$\sum_{s}^{1,\cdots,n} \bar{b}_{si}\bar{x}_s = \bar{\lambda}\bar{x}_i, \qquad (i = 1, \cdots, n). \tag{7}$$

Multiplying corresponding equations in (6) and (7), member for member, and summing as to i, we find

$$\sum_{s,t}^{1,\cdots,n}\left[\sum_{i}^{1,\cdots,n} b_{ti}\bar{b}_{si}\right]x_t\bar{x}_s = \lambda\bar{\lambda}\sum_{i}^{1,\cdots,n} x_i\bar{x}_i\,;$$

that is

$$\sum_{i}^{1,\cdots,n} \rho_i x_i\bar{x}_i = \lambda\bar{\lambda}\sum_{i}^{1,\cdots,n} x_i\bar{x}_i.$$

Let G be the largest and s the smallest of the characteristic roots of $A\bar{A}'$. Then

$$\lambda\bar{\lambda} \sum x_i\bar{x}_i \leqq G \sum x_i\bar{x}_i,$$

so that $\lambda\bar{\lambda} \leqq G$. Similarly, $\lambda\bar{\lambda} \geqq s$; i. e.,

$$s \leqq \lambda\bar{\lambda} \leqq G.$$

In particular, if A is unitary so that $A\bar{A}' = I$, then $G = s = 1$, so that $1 \leqq \lambda\bar{\lambda} \leqq 1$; i.e., $\lambda\bar{\lambda} = 1$, as is well known. Hence we have the following theorem.

THEOREM V. *If* λ *is a characteristic root of a square matrix* A *and if* G *and* s *are respectively the largest and the smallest characteristic roots of* $A\bar{A}'$, *then*

$$s \leqq \lambda\bar{\lambda} \leqq G.$$

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LIE'S WORKS, VOLUME VI

Sophus Lie: Gesammelte Abhandlungen. Edited by Friedrich Engel and Poul Heegaard. Volume VI, edited by Friedrich Engel. Leipzig, Teubner and Oslo, Aschehoug, 1927. xxiv+940 pp.

In this edition of the works of Sophus Lie his published memoirs are to appear in six volumes while a seventh is to be devoted to supplementary matter. Volumes I and II are to contain those of his memoirs in which the geometrical elements predominate, while volumes III and IV are given to differential equations and volumes V and VI to transformation groups. Volumes III and V have previously appeared; and brief reviews of them are to be found in this Bulletin (vol. 29 (1923), pp. 367–369, and vol. 31 (1925), pp. 559–560, respectively). Volume VI is the third to be published, and it is to be followed by volume IV, on which work is already in progress.

In each of the three divisions the arrangement of the material is mainly chronological. The matter in the first of two related volumes is that which was first published at Christiania while the second is composed principally of the memoirs which appeared in Mathematische Annalen and the publications of the Leipzig Akademie. In this way one avoids having in a single volume two memoirs one of which is mainly a reworking of the other.

The editor's preface of sixteen pages gives a brief analysis of the contents of the present volume VI. In volumes V and VI together one has all the memoirs of Lie on both finite and infinite continuous transformation groups. In these volumes have also been included many works having to do with the applications of the theory of transformation groups to the theory of differential equations. In these Lie sets forth the point of view, with a development of its consequences, from which he approached the theory of transformation groups and their use for the treatment of differential equations. His remaining memoirs on differential equations are assigned to volumes III and IV.

The memoirs on finite continuous transformation groups have for the most part been worked into Lie's large three-volume treatise on the theory of transformation groups (1888–1893), where a nearly complete development has been given in a systematic treatment. The memoirs on infinite continuous groups have not been incorporated in a systematic development, but here the treatment in the memoirs themselves is sufficiently systematic largely to offset this deficiency.

The editor's carefully prepared analysis (pp. vii–xxii) of the contents of this volume relieves the reviewer of the necessity of such a discussion, particularly since that analysis is more adequate than anything which could be said here in the space which might appropriately be used for such a purpose.

The extreme care with which the editorial work has been done—evidently a labor of love—is again shown by the extent of the notes (pp. 755–940); these clear up many points of difficulty and sometimes give important matter relating to the development of the theories in Lie's mind. Especially useful in this respect is the account given (pp. 777–793) of the antecedents of memoirs II and III on differential invariants and the differential equations which admit a finite continuous group. This account is mainly in the form of excerpts from letters from Lie to Mayer and Klein. Noteworthy is Lie's insistence (p. 781) on the importance of the synthetic element in the origin of his discoveries. Attention may also be called to the account which Lie gives (pp. 781–782), in a letter to Mayer, of the order in which his memoirs may best be studied.

A free translation of a passage (pp. xxi–xxii) from the editor's introduction will not be without interest:

"If one should go through the whole history of mathematics, I believe that he will not find a second case where, from a few general thoughts, which at first sight do not appear promising, has been developed so extensive and wide-reaching a theory. Considered as an edifice of thought Lie's theory is a work of art which must stir up admiration and astonishment in every mathematician who penetrates it deeply. This work of art appears to me to be a production in every way comparable with that . . . of a Beethoven . . . . It is therefore entirely comprehensible if Lie . . . was embittered that 'deren Wesen, ja Existenz, den Mathematikern fortwährend unbekannt zu sein scheint' (p. 680). This deplorable situation, which Lie himself felt so keenly, exists no longer, at least in Germany. In order to do whatever lies in my power to improve the situation still further, ... I have sought to clarify all the individual matters (Einzelheiten) and all the brief suggestions in these memoirs."

R. D. CARMICHAEL

BORN ON ATOMIC DYNAMICS

Problems of Atomic Dynamics. I. *The Structure of the Atom* (20 lectures). II. *The Lattice Theory of Rigid Bodies* (10 lectures), by Max Born.

(Delivered at the Massachusetts Institute of Technology in 1925–26, and published by the Institute.) 8vo., 200 pp.

The name of the author of this volume is alone sufficient to insure an authoritative presentation of the subjects which it treats. As a synopsis—followed rather closely—of a course of thirty lectures, it has the faults and the merits which such a method of development involves. Each lecture is, to a certain extent, devoted to some special phase of the subject, and this allows the reader to find out easily what the author has to say on any topic. For anyone who is familiar with the general lines which atomic physics has taken during the last ten years this has decided advantages. On the other hand, it produces some feeling of a lack of connectedness. A mathematician who wishes to learn of the developments which have taken place up to a couple of years back is advised to read first some general descriptive account and then to turn to Professor Born's volume. If he has some previous acquaintance with the mathematical methods of which use is made, he should have a sufficient basis to find out what developments the mathematical side needs in order to assist in the interpretation of the observed phenomena.

The first attempts to explain the structure of the atom were naturally made with the use of the classical theory of dynamics which involved the continuity of space and time and of all phenomena deduced from them. It is true that mass was discontinuous but its motions were always continuous. As observation became more searching, it was found that classical dynamics was no longer sufficient to correlate the phenomena and the idea of the "quantum," or change by finite as opposed to infinitely small steps, was introduced. Certain relations of classical dynamics which involved energy and momentum were retained; others involving rates of change had to be abandoned. Hence Professor Born gives a brief summary of the Hamilton-Jacobi presentation of classical dynamics and follows it with the limitations required by the quantum hypothesis.

Then follow several lectures on Bohr's atomic work which has had such great success in correlating the numerical values of the wave lengths of the spectral lines of the elements and particularly those of hydrogen. The numerous observers, however, soon overtook the mathematicians, and modifications and limitations had to be introduced, even to division of the quantum. The reply was the introduction of the harmonic oscillator and the mathematical theory of matrices and still later, wave mechanics. The physical definition of the first is left conveniently vague—mathematically it is a coordinate which is a simple harmonic function of the time. This has had a certain degree of success and several lectures are devoted to it. But that it is by no means final appears in the last lecture where the application of general observational calculus, introduced by Wien and the author, is briefly outlined. In the same lecture the latter gives his views of the trend of future developments, stressing particularly the necessity for careful consideration of the observed phenomena in the mathematical developments.

The second series is largely devoted to attempts to deduce the various observed properties of rigid bodies and particularly those of crystals, from the theories of atomic structure set forth in the first series. The reading and comprehension of this series constitutes a liberal education in general physics. The general geometrical basis of the lattices is mathematically quite simple; the complications are introduced in the attempts to apply it to various crystals.

The volume is lacking on the personal side. While the names of those who have contributed are freely mentioned throughout, the occurrence of these names appears to be quite incidental. No references are given, and while there is no index, the brief table of contents gives a fairly good idea of the subjects treated. These statements are not made as criticisms but simply as descriptive of the subject matter and manner. There can be no question concerning its value as a summary of the status of the subject at the time of the delivery of the lectures.

E. W. Brown

BURGESS ON STATISTICS

Introduction to the Mathematics of Statistics. By Robert Wilbur Burgess, Senior Statistician, Western Electric Company. Boston, Houghton Mifflin, 1927. viii+304 pp. $2.50.

The addition of another textbook on statistics to a field already rather crowded is not justified unless the new book has worthwhile advantages. The reviewer feels that the new text by Burgess meets this test. Up to the present time textbooks on statistics have fallen for the most part into two distinct groups: fairly condensed texts which were intended to appeal mainly to students with considerable mathematical training, and the voluminous texts to be used in connection with work in economics. The text under review combines the two types in a relatively small volume.

Although no two trained statisticians would probably agree in all particulars upon a definition of statistics, it is all to the good that the author undertakes a simple and direct explanation of the meaning and need of statistics from the very start, which not only serves to orient the student but also paves the way well for what follows. The summaries at the ends of the chapters also add to the effectiveness of the text.

The author has succeeded remarkably well in covering so much ground without employing the principles of the calculus. However, the reviewer feels that the author should have been consistent and referred elsewhere for some of the lengthy algebraic work (pp. 101–2, 142, 165, 225, 240–42, 245, 250–51) which will tend to distract the attention of the student from the main discussion unless, as will probably happen, the student skips that work. Such work might well have been relegated to the appendix.

There are several places where the author does not say exactly what he surely meant. For example, we are told (p. 33) to use the ratio chart when we wish "to compare *changes* (my italics) rather than absolute magnitudes." Also the author uses "percentages" in several places where "rates" are surely meant; thus, on this same page (p. 33) he says that "equal vertical distances represent equal percentages of change." In another place (p. 34) we are referred to "the downward curve" of a graph which has a positive slope throughout; the reference is, of course, to a decrease in the slope. Again, we are told (p. 133) that "the number of series (to be used to determine an index number) should be large enough . . . but not so large as to cause undue difficulty . . . ."; we know only too well how some students would interpret this advice. It is rather cruel to say (p. 258) "In order to help the student to understand the *fundamental idea* (my italics) of the probable error (of which little has been said so far), let us follow through the proof of the formula for the probable error of the mean." However, most of such slips are not serious, especially in the light of the text just preceding or just following.

Although one of the claims of the author is generality of treatment, rather condensed explanations of the usual work with index numbers and time series are included. The text carries one finally to the theory of correlation of three and four variables.

The space devoted above to criticism of individual excerpts is relatively much greater than they deserve, and is apt to give an erroneous effect. The main impression made upon the reviewer is that the book is very readable and even interesting (rarely true of textbooks on statistics), except possibly for the algebraic passages referred to above. Teachers of courses which cannot, or do not, require any knowledge of the calculus, will probably find in this book the text they have long sought.

C. H. FORSYTH

HUDSON ON CREMONA TRANSFORMATIONS

Cremona Transformations in Plane and Space. By Hilda P. Hudson. Cambridge University Press, 1927. xx+454 pp.

The appearance of a first exhaustive treatise in any field of mathematics is a matter of concern to those who pursue the particular subject. Books of that type frequently determine the trend of mathematical thought and progress for a considerable period. Perhaps no topic in algebraic geometry has been in greater need of such exposition than Cremona transformations. The earlier presentations are either elementary or incidental to some immediate geometric need. Existing encyclopedic accounts are rather cursory. Thus a large body of researches on the subject, widely distributed in the journals, has been either inaccessible or unknown to those who might wish to become acquainted with the field.

To digest and to unify this mass of material was a task which demanded not merely a mastery of the subject but also an uncommon capacity for detail. This task Miss Hudson has accomplished in a most admirable manner in the book under review.

The book itself gives an impression of unity which is rather remarkable in view of the diversity of the transformations of which it treats. This doubtless is due to the wisdom of the author in selecting from the field a naturally related group of topics. Only transformations in the plane and in space are considered. For each case these are discussed first with reference to the properties common to all and secondly with reference to their division into various types. The single application considered is to the resolution of singularities of curves and surfaces and a treatment of this is practically inevitable since such singularities are present in the transformations. This is the division of the subject to which most of the author's own contributions have been made. No geometric applications are given except as they may be involved in the construction of a type, or as they may be inherent in the general class such as the isologues of a transformation in superposed planes or the complex associated with a transformation in space. Applications to other fields of mathematics are omitted. No account of Cremona groups, finite or infinite, appears.

The nomenclature of the author is on the whole well chosen even though individual contributors to the subject must generally expect to find that their own notations have not, in all cases, been adopted. Miss Hudson adopts the term *F-system* (*fundamental system*) for the aggregate of points in S whose correspondents in S' are indeterminate; and *P-system* (*principal system*) for the aggregate of points in S whose correspondents in S' are adjacent to (that is, directions about) the points of the F-system in S'. It is to be hoped that this usage may become general. No confusion can arise in the plane, by calling a P-curve an F-curve and this is frequently done but in space a distinction must be made.

About one-third of the text is devoted to the planar transformations, one-half to the spatial transformations, and the remainder to a historical sketch, a bibliography of 425 titles, a series of tables of types, and an index. In the plane, Chapters I and II are devoted to the general theory; Chapter IV to the determination of the types of transformation; Chapter III to the quadratic transformation, and Chapter VI to other special types; and Chapter VII to the resolution of the singularities of plane curves. In Chapter V we find the geometry which arises when the planes S, S' are brought into coincidence; and in Chapter VIII proofs of Noether's theorem that every transformation is a product of quadratic transformations. For space we find in Chapters IX and XIII the general theory; and in Chapters X and XIV the quadratic and other particular transformations. The intervening Chapters XI and XII are devoted respectively to postulation and equivalence, and to contact conditions. In Chapter XV the cubo-quartic transformation, which presents examples of most of the phenomena previously discussed, is minutely examined. Chapter XVI contains an account of the work up to the present time on the resolution of the singularities of surfaces. The final Chapter XVII contains the historical sketch and the bibliography.

In general the exposition is clear, brief, and effective. One may differ at times from the author with respect to methods of proof and emphasis on aspects of the theory. Thus there is associated with a given Cremona transformation T a linear transformation L, with integer coefficients, first freely used by S. Kantor, which expresses the effect of T upon the curves of order n in the plane. L and its invariant linear and quadratic forms are introduced rather incidentally (p. 26). But from the obvious invariance under T of the genus of one curve, and of the free intersections of two curves, there follows the invariance of the two forms under L; and therefore as an immediate consequence the incidence relations of p. 18. Also the nature of the product T_1T_2 (p. 56) is immediately read off from the coefficients of the product L_1L_2. In the determination of types carried out in Chapter IV the emphasis is placed on a division of types with respect to the degree n of T. It is the reviewer's opinion that a division with respect to the number σ of F-points of T is more fundamental. Nevertheless on these two matters, and indeed throughout, the presentation follows the historical order of development.

It is in connection with the space theory that the book will be of greatest service to the advanced reader. The original articles, in widely different notations, with many obscurities and not a few errors, are here combined into an organic whole whose lapses from perfection are as a rule clearly indicated. The content is brought up to present date with, for example, a proof of the recently announced theorem of Tummarello on the equality of the sums of the genera of F-curves in S, S'; and with certain novelties such as the construction of a T with an arbitrarily given curve as an F-curve of the first kind.

In the chapter on the resolution of singularities of surfaces the author does not give complete proofs but refers on essential points to the sources. There is indeed difference of opinion as to whether such resolution has actually been accomplished. Thus the present exposition of progress to date is opportune.

References to the bibliography are made in the text by number. This convenient method would have been much improved if the authors cited were also given.

A forthcoming report on topics in algebraic geometry contains chapters on Cremona transformations which will cover as well the matters not discussed by Miss Hudson. Fortunately the prior appearance of this volume has enabled the authors concerned both to make their general list of references more complete, and to refer for a broader account to the book itself.

A. B. COBLE

JORDAN ON STATISTICS

Statistique Mathématique. By Charles Jordan. Paris, Gauthier-Villars, 1927. xvii+344 pp.

This book is a scholarly treatment of the subject of mathematical statistics. It is in many respects the book which mathematicians have been waiting for. It performs for statistics much the same service as a standard *Cours d'Analyse* performs for analysis, and with the same typical French clearness; it would seem to be also of about the same degree of difficulty. Moreover, in point of difficulty, it is, considering the variety of subjects treated, of remarkably uniform grade throughout. The development is carefully planned in advance, and is carried out logically, and with uniform notation and nomenclature, producing maximum clearness of exposition in minimum space. There are no topics of dominant importance which the author has not considered, although there are some minor matters—this is true of almost every book—which he has omitted and the reviewer would like to have had included. Apparently the author has also drawn on all the important sources, but though many times it is possible to recognize in his exposition the essential arguments of others, never are these arguments copied bodily; they always bear the impress of his own method of thinking. The only possible criticism in this connection is that, in acknowledging his indebtedness to others, Jordan rather frequently mentions names only, without giving references to the publications—a flaw which could easily be remedied in a second edition.

Although the various chapters are almost uniformly good, the reviewer was particularly impressed with the first three introductory chapters and with those on frequency curves (9 and 10) and less well pleased with the final chapter, on sampling. The first three chapters contain an excellent account of the preliminary notions; the material is well chosen and the sequence carefully planned. Much is left unproved here; the account is rather descriptive with only an occasional demonstration. Chapter 4, on probability, contains some stimulating material, including a rather elegant account of the series of Lexis and Poisson, along with much that is rather ordinary. The author is not always careful here to distinguish between what is really a definition and what is a theorem. The fundamental theorems which are hardest to demonstrate rigorously from his definition of probability, he does not prove at all, possibly because he makes reference to an account of probability which he has published elsewhere. His "empirical principle" is so vaguely stated that it would seem impossible to use it rigorously in a demonstration: (the italics are mine) "Si l'on répète une expérience *un grand nombre* de fois, chaque évènement se produit *à peu près* en un nombre de fois proportionnel à sa probabilité." On the next page (77) *évènements indépendents* are not defined at all. Assuming the author's definition of probability on page 28, his statement of Bayes' Theorem on page 79 is not really a theorem, but an extension of the meaning of the earlier definition. It is rather unfortunate to say that (p. 86) J. Bernoulli proved his celebrated theorem when his so-called demonstration was really so very bad. Jordan lays a good deal of stress on the usefulness of the normal law as an approximation, but omits the case where the approximation is specially good, viz.: when it is a question of summing an equal number of terms of a point binomial on both sides of the mean. The reviewer gravely doubts that (p. 120) "experience proves that the (normal) law of errors is always valid when the errors are small relative to the magnitudes measured." Jordan's account of the method of least squares is too compressed to be of much service to the reader. It follows classic lines, as do his directions for computation, and so he implies that he is not expecting the computer to use a machine!

Chapter 5, on *Classification Statistique*, is quite similar to Yule's treatment, but enjoys distinction from Yule's in not being verbose. Averages and moments are treated in the sixth chapter. Again there is a refreshing absence of that elaboration of the obvious which is characteristic of so many books at this point. Some of the illustrative material in this chapter is very interesting. Chapter 7 is on the point binomial. Chapters 9 and 10 are on frequency curves, and as has been indicated above, they seem to the reviewer excellent. He does not entirely agree with what appears to be the author's point of view (p. 207) as to the purpose of obtaining these analytical curves, namely, that it is solely to find a background of probability theory which one can then claim determines the data; but, adopting it for the moment, it would appear unfortunate not to have the "*différentes considérations*" (p. 241) which underlay Pearson's differential equation explained. For it is that very background of probability which has given Pearson's curves a theoretical advantage over certain other methods of representation. Jordan is, however, consistent when he criticizes Charlier's series and some others because they give negative frequencies. But the very small negative frequencies which he condemns are nevertheless very good approximations to the very small positive frequencies that they are supposed to equal, and so if one will admit that an important purpose of such series is to substitute for unwieldy expressions simple and manageable approximative formulas, that particular criticism is answered. Chapter 10, on certain series fitted to frequency distributions by least squares, contains valuable suggestive material. Graduation, in Chapter 11, is treated rather curtly, but it was clearly necessary to avoid long incursions into special fields of application, and therefore it was probably necessary for Jordan to stop where he did or else so to extend this chapter that it would become out of harmony with the others.

Finally, we have the theories of correlation in Chapter 12, and of sampling in Chapter 13. This last subject involves matters of extreme delicacy, and they cannot be adequately set forth in a brief ten pages, even though all the statements be technically accurate. It is too easy here to confound approximations with exact statements, and indeed it is difficult to be sure that so-called approximations are not really wide of the mark. One such approximation which the author fails to indicate as such is "$\delta\mu_2 = 2\zeta\delta\zeta m$," used to find the probable error of the standard deviation when the probable error of the square of the standard deviation is given Formula (15), which precedes the above equation, is also open to suspicion without the extra terms which have just been dropped. It is of course quite true if one admits the hypothesis that "$\eta = 0$," but this condition is not usually satisfied in the many cases to which the formula and its derivatives are usually applied. Jordan seems to be aware of this himself but does not properly warn the reader.

The book is bound only in paper, but the printing and materials are good. The proof reading has been almost perfect. Only a few misprints have been noted, and they are listed below; no one of them happens to be of such a character as to mislead the reader, except perhaps the one on page 49.

Page 3, line 12, for *Chieff* read Chief. Page 13, first line of footnote, for $n(u)_n$ read $nh(u)_n$. Page 36, last line, the omitted numerator of the first fraction is s. Page 49, in the equation between equation (16) and equation (17), replace 2 by -2. Page 83, line 6 from bottom, for $(\frac{6}{1})$ read $\binom{6}{1}$. Page 92, line 11, for 3, read 13. Page 241, line 12 from bottom: the omitted first letter should probably be L. Page 266: the upper limit is omitted in the integral of (1); it should be plus infinity. Page 288, first line of footnote, for *adjustement*, read *adjustment*. The page following page 298 should be numbered 299, instead of 29.

B. H. CAMP

SHORTER NOTICES

The Theory of Integration. By L. C. Young. Cambridge University Press, 1927. vii+52 pp.

This is one of the well known Cambridge Tracts in Mathematics and Mathematical Physics. A detailed discussion of the contents in therefore superfluous, as the chief duty of the author is to present the material intelligibly and accurately; in this case these requirements are satisfied. On the other hand it seems to the reviewer desirable to call attention to two defects of another kind. As the author points out in his preface, "the object of small treatises of this kind is to enable the general student to gain rapid access to the various branches of Modern Mathematics." Viewed in this light the title is somewhat misleading, as the book deals almost exclusively with W. H. Young's theory of integration. Although this is stated in the preface, it is to be feared that the general student would not realize that he had read *a* theory of integration and was still unacquainted with the one most widely known. The other defect is the total absence of references to the literature. Surely a small, well chosen bibliography should be included in any book of this kind.

W. A. Wilson

Integral Bases. By W. E. H. Berwick. Cambridge University Press, 1927. 95 pp.

This new Cambridge tract contains the author's investigations of the problem: *To determine an integral base for an algebraic field given by an arbitrary equation* $f(x)=0$. This problem can be reduced to the determination of a partial base for all primes p dividing the discriminant of $f(x)$. For this case the author has, as he asserts in the preface, evolved various new methods, which can also be used for finding the decomposition of p in prime-ideals. "Failing cases exist, but the approximations given are sufficient to cover nearly any numerical equation not specially constructed to defy them."

The method is, in brief, the following: Let

$$f(x) \equiv \phi_1(x)^{a_1} \cdots \phi_r(x)^{a_r} \pmod{p}$$

be the decomposition of $f(x)$ in prime-functions (mod p); then $p = A_1 \cdots A_r$, where all ideals A are relatively prime. To find the further decomposition of A, we write $f(x)$ in the form

$$f(x) = \sum Q_i(x) p^{\alpha_i} \phi(x)^i,$$

where $Q_i(x) \not\equiv 0$ (mod $p, \phi(x)$) and construct the Newton polygon to the lattice-points (i, α_i). Then $A = B_1^{\lambda_1} \cdots B_u^{\lambda_u}$, where k is the number of sides and λ_i certain constants. Under certain conditions the prime-ideal decomposition of p and the corresponding partial bases can be found in this way.

The author seems to be unaware of the fact that his new method and his results have already been completely evolved in my paper *Zur Theorie* *der algebraischen Korper*, Acta Mathematica, vol 44 (1923), pp. 219–314, and that this method has been used for the solution of the same problems and various other problems in the theory of algebraic fields (Acta Mathematica, vols. 45–46). I also mention that it is possible to generalize the method so that it can be applied in every case.

The book is difficult to read, but contains various good numerical illustrations of the method. The final chapter is dedicated to the study of fields defined by a binomial equation $x^n - a = 0$ and gives an elaborate study of the different possibilities arising in this case.

OYSTEIN ORE

Einfuhrung in die Wahrscheinlichkeitsrechnung. By J. L. Coolidge. German translation by Dr. Friedrich M. Urban. Leipzig, 1927. ix+212 pp. 10 R. M.

In the one-page preface of the translator the reason is given for publishing a German edition of this American work. There are several German texts of a very high order on this subject, but none of them appears to meet the needs of the beginner in such an attractive and elementary manner. The application of statistics to new fields is constantly increasing and many of the methods of the theory of probability must be used by investigators whose principal interest is not in mathematics. They have not had the training necessary for profitable study of the advanced treatises. This justifiable demand ought to be met by a book which has few preliminary mathematical requirements.

To the English student who is beginning to extend his study of probability to the great German works this translation will be the Rosetta stone that furnishes him the required technical vocabulary. This is important when we lack an interlingual mathematical dictionary. Many of the terms used in probability have fine distinctions in our own tongue, such as dispersion, error, deviation and discrepancy. When we meet such words in another language for the first time, it sometimes requires an immoderate amount of reading and re-reading to establish the proper vocabulary.

The translation is quite accurate in the technical statements but some of Coolidge's delightful expressions were not adequately turned. "This unlovely formula" (p. 170) becomes "diese hassliche Formel" (p. 168). With a little more care this might have been rendered by a more appropriate adjective, yet one could hardly expect to find a single word in German carrying the same double meanings as Coolidge's "unlovely." "Unliebenswurdige" would translate one meaning, the other might be given by "unzierliche," which is as unusual in German as "unlovely" is in English.

Most of the errors mentioned in the review by H. L. Rietz of the English edition in this Bulletin (vol. 32, No. 1) have been corrected, but in formula (11), Chapter III, $1/\sqrt{2npq}$ has become $1/\sqrt{2n\pi q}$ where it should be $1/\sqrt{2n\pi pq}$. A few others have also been corrected.

F. A. FORAKER

Conductibilité Électrique des Métaux et Problèmes Connexes. Rapports et Discussions du Quatrième Conseil de Physique, Institut International de Physique Solvay. Paris, Gauthier-Villars, 1927. 368 pp.

This is the proceedings of the Fourth Solvay Conference on conduction in metals held in Brussels in April 1924. It contains nine more or less comprehensive papers presented by leading physicists interested in that subject, and also the discussions by those attending the conference. The papers run in the following order: H. A. Lorentz, *On the application of electron theory to the properties of metals;* P. W, Bridgman, *On the phenomena of metallic conduction in general and some possible theoretical explanations;* O. W. Richardson, *On another proposed theory of metallic conduction;* W. Rosenhain, *On the structure of alloys;* W. Broniewski, *On electrical resistance and thermal expansion;* A. Joffé, *On the conductivity of crystals;* Kamerlingh Onnes, *On some recent researches on superconductivity;* E. H. Hall, *On metallic conduction and the transverse effects in a magnetic field;* A. Joffé and N. Dobronravoff, *On some experimental tests of the idea of light quanta.*

S. C. Wang

Lehrbuch der Variationsrechnung. By Adolph Kneser. Second edition. Braunschweig, F. Vieweg & Sohn, 1925. 397 pp.

In the second edition of his Lehrbuch der Variationsrechnung Kneser has followed, in general, the order of development of the first edition. There is, however, much difference in the two texts, and the second edition is certainly an improvement over the first.

Those familiar with the older edition will recall that while chapter headings were given there were no section headings. Consequently when one wished to know what notion was to be discussed in a given section he had either to read it through or else turn back to the table of contents. It is, therefore, gratifying to turn to the first page and find listed the topic discussed in the first section. It is much more gratifying to find that the method of presentation is improved. The reader need only study any chapter of the new edition and then work through the corresponding chapter of the first edition in order to be impressed by this change.

This fact so impressed itself upon the present reviewer that he decided to look up the comments of the reviewer of the first edition (this Bulletin, vol. 12, p. 172). It was found that while that reviewer classed the work as one of monumental importance, he also asserted that it was not lucid. Most of us who have had occasion to use the first edition agree with this comment. There is no doubt, however, that this criticism has been overcome to a very great extent in the new edition. Here the notation has been explained in greater detail, statements of needed existence theorems inserted at the proper places and the illustrative examples explained more fully. While some of the theoretical developments which have taken place since 1900 have been inserted the work has not been complicated.

This second edition will continue to be recognized as one of the most important texts on the calculus of variations.

E. J. Miles

Examen des Différentes Méthodes Employées pour Résoudre les Problèmes de Géométrie. By G. Lamé. Paris, Hermann (no date). xii+124 pp. 2 plates.

This rather unusual little book will be read eagerly and profitably by the student whose grasp of the rudiments of elementary geometry, algebra, analytic geometry, calculus, and descriptive geometry is sufficient to enable him to appreciate comparisons of their methods, aims, limitations, and applicability. The teacher will be indebted to the author not only for the occasional paragraphs of suggestions on teaching, but also for the help gained indirectly from a book which is something more vital than a mere store-house of information. In the author's opinion, the most noteworthy parts of the book are: the analytic expression of the common intersection of geometric loci; the complete determination of curves and surfaces of the second degree by descriptive geometry, when a sufficient number of points are given, and the theory of curves and surfaces represented by the equations $x^\alpha/a^\alpha+y^\alpha/b^\alpha=1$ and $x^\alpha/a^\alpha+y^\alpha/b^\alpha+z^\alpha/c^\alpha=1$.

E. B. COWLEY

Problems of Modern Physics. By H. A. Lorentz. Boston, Ginn and Co. vi+312 pp.

That a course of lectures delivered at the California Institute of Technology in 1922 by Lorentz now appears in print edited by Bateman is cause for rejoicing. Lorentz is always good reading because he is so sound and so honest. He gives both sides of a question and in our present state of perplexity in physics we need both sides. Take, for instance, his §45 (pp. 134–140) entitled "How Far Can Molecular Forces be Explained by the Electromagnetic Actions?" in which he presents and comments on work by Born and Landé. Or (p. 95) the discussion of the reality of variations of length due to translation. And (p. 47 et seq.) the question of whether moving electrons do radiate. Again §42 (p. 125) commenting on the structure of the electron and (in numerous places) on our troubles with quanta. Throughout, the book is illuminating.

Many years ago Lorentz adopted the plan of composition which develops the main line of thought with relatively simple formulas in the main text and relegates to an appendix the detailed mathematical analysis. This plan is followed here and adds to the readability of the book. Moreover the author has the faculty of getting directly and simply at the fundamental problems he wishes to discuss and thus covers a deal of physics in a brief space and time. He gets into general relativity, including Levi-Civita's parallelism, with no great mathematical splurge. Why cannot more of us write that way? Probably because we cannot think hard enough. It would be desirable for Lorentz to come again and cover the many new ideas which have sprung up since 1922 and which in his brief preface he says might form the subject of a treatise.*

E. B. WILSON

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* Since this was written, the scientific world has been saddened by the news of the death of this great man.

Geometria Proiettiva Differenziale. By G. Fubini and E. Čech. Bologna, Nicola Zanichelli, 1927. Vol. II, 406 pp.

The second volume of the Projective Differential Geometry of Fubini and Čech is a direct continuation of the first. Chapters, sections, and pages are numbered consecutively, the second volume beginning with Chapter 8, §71, page 389; and an index is provided at the end of the second volume for the entire work. Moreover, the continuity is not altogether mechanical. In the last chapter of the first volume it was proved that if a non-ruled surface in ordinary space is projectively deformable at all, then it is projectively deformable in ∞^3, ∞^2, or ∞^1 ways, and the concluding sections of that volume were devoted to a discussion of the first case. The opening chapter of the second volume completes the discussion of the other two cases.

Chapters 9 and 10 are devoted to a consideration of various portions of the projective differential geometry of a surface in the neighborhood of one of its points. Among the topics treated are the quadric of Moutard; Čech's transformation Σ between points in a tangent plane of a surface and planes through the contact point, with various special cases one of which is connected with a two-dimensional metric of Weyl; the canonical pencil of lines; and the cone of Segre. Arbitrary curvilinear coordinates are used in these chapters.

Complexes and congruences of lines are studied in Chapter 11. Here the closest possible analogy with the theory of surfaces is preserved. The theory proceeds from certain fundamental differential forms to the differential equations which determine the configuration except for a projective transformation. Much emphasis is placed on the notion of projective applicability of complexes and of congruences.

The projective differential geometry of hyperspace appears for the first time in Chapter 12. The now familiar transition is made from the fundamental differential forms for a hypersurface in S_n to the differential equations, which might themselves have been used as fundamental. Non-parabolic surfaces in S_4 are studied, and ruled surfaces in spaces of even and odd dimensions are briefly considered.

It seems appropriate to remark in this connection that a general projective differential theory of a V_k in an S_n, with $n>4$ and $1<k<n-1$, remains yet to be constructed. Two methods seem to be available for constructing such a theory, namely, the method of Wilczynski who starts with a system of differential equations and employs the Lie theory of continuous groups, and the method of Fubini who starts with a system of differential forms and employs the absolute calculus of Ricci. But the labor involved in the former method seems at present to be prohibitive, and the latter method has so far failed because of the lack of a suitable fundamental quadratic form, or perhaps because of the lack of a suitable absolute calculus for an n-ary p-adic differential form.

Nearly the last third of the second volume is occupied by four appendices written by three different authors. In the first appendix Tzitzeica treats in French a special problem in the deformation of surfaces. In the second Bompiani gives a systematic exposition of his own contributions to the projective differential geometry of curves and surfaces in ordinary space. Bompiani has been especially interested in geometrical definitions of the fundamental forms of Fubini. In the third appendix Terracini gives an exposition of the most interesting results in the projective differential geometry of hyperspace, accompanied by a valuable bibliography, and in the fourth the same author gives a brief account of a special problem concerning surfaces whose asymptotics belong to linear complexes.

These volumes constitute the only existing comprehensive treatise on the modern theories of projective differential geometry. Every one interested in this subject will need to have a copy of this work available, as references will be made to it for some time to come. There still seems to be room, however, for a well organized treatment in English of the same material. And most of all there is need for a text which can be put into the hands of graduate students who are being introduced to the subject, because the monumental work of Fubini and Čech, valuable as it is for the expert, does not seem to be suited to the needs of a beginner in this field.

E. P. LANE

Les Mathêmatiques du Chimiste. By L. Gay. Paris, J. Hermann, 1926 vii+208 pp.

As indicated by the title, this book is not a mathematics text, but rather a monograph embodying only those principles which the author believes to be of particular utility to the chemist; more especially to the physical chemist. Hence he has omitted all reference to trigonometric functions, polar coordinates, integrals of surfaces and volumes, the mathematics used in crystallography (believing this to belong to the realm of physics), and he has only touched on the subjects of limits, series, imaginaries, etc.

The scope of the book may be judged from the following consecutive chapter headings: powers and roots, logarithms, functions of a single variable, functions of more than one variable, differentials, derivatives of functions of more than one variable, integrals of functions of a single variable, and integrals of differentials which depend upon more than one variable.

It seems to the reviewer that Chapter one might have been omitted; surely every chemist must be familiar with its contents. Much of Chapter two, dealing with logarithms, might also have been deleted, especially the discussion of their use in multiplying, dividing, raising to powers, and extracting roots. However, the reviewer is pleased to find included in the chapter an explanation of the numerical values of logarithms of numbers less than one, and a discussion of the errors which may result from the use of logarithms in making computations. In Chapter three it seems scarcely necessary to explain how to plot using rectangular coordinates, or to point out that the graph of a first order equation is a straight line. Methods of interpolation and extrapolation are given the attention they well deserve. In Chapter four we find a good exposition of the use of the triangle in representing systems of three components. In Chapter five are given the rules for differentiating the more common algebraic equations. Although the author has defined the use of *d* and Δ in the usual way, he seems at times to confuse the two conventions, especially in his discussion of the differential of the product of two variables.

Chapters six and seven contain a brief treatment of first and second derivatives, series, the characteristics of maxima and minima, points of inflection, and first and second partial derivatives. In Chapter six it seems unfortunate that the author chose to discuss van der Waal's equation especially as applied to gases in the neighborhood of their critical points which is just the region in which the equation gives the most unsatisfactory results. The last two chapters deal with integration. Here we find that it is quite simple to integrate an irregular area by cutting out the plotted area, weighing the paper, and comparing the weight with that of a paper of the same thickness and quality and of known area. Although this method has been repeatedly recommended, it is the experience of the reviewer that it may lead to considerable error, and that it should be used with considerable skepticism. The characteristics of exact differentials and their physical significance are also discussed.

The book contains some errors, presumably typographical. On page 42, in discussing the velocity of a first order reaction, we read that the temperature, instead of the time, is a logarithmic function of the concentration. On page 59, heat of fusion should read heat of vaporization. On page 68, the corrected chemical equation is $2BaO + O_2 \rightleftarrows 2BaO_2$. In the footnote on page 78 the name epicycloid is given to the curve usually considered as the cycloid. In the footnote on page 128 lead is, erroneously, said to crystallize; this should read bismuth. On page 199, in the integration between the limits *A* and *B* it would seem more logical to use the subscripts *A* and *B* for the variables in the final equation. On page 205,

$$\frac{-l}{T_2} \text{ should be } \frac{-l}{T^2}.$$

In the opinion of the reviewer, the value of the book would have been enhanced if the author had included equations for the common curves, more about empirical equations, something on the method of least squares, and if he had laid more emphasis on the errors in results calculated from experimental data.

The book is evidently intended primarily for French consumption. The author in his preface states that there is no other book of its scope in the French language. This, of course, cannot be said for the English language. However, the English student of chemistry will find much of value in this little book. Its contents is common knowledge to the research worker in physical chemistry.

BLAIR SAXTON

NOTES

The opening number of volume 30 of the Transactions of this Society (January, 1928) contains the following papers: *Simpler proofs of Waring's theorem on cubes, with various generalizations*, by L. E. Dickson; *Cubic curves and desmic surfaces*, second paper, by R. M. Mathews; *Possible orders of two generators of the alternating and of the symmetric group*, by G. A. Miller; *Optics in hyperbolic space*, by James Pierpont; *Geodesics on surfaces of genus zero with knobs*, by D. E. Richmond; *Concerning end points of continuous curves and other continua*, by H. M. Gehman; *The apportionment of representatives in Congress*, by E. V. Huntington; *Conditions for associativity of division algebras connected with non-abelian groups*, by John Williamson; *A generalization of Taylor's series*, by D. V. Widder; *A problem in the calculus of variations with an infinite number of auxiliary conditions*, by R. G. D. Richardson; *A contribution to the theory of fundamental transformations of surfaces*, by M. M. Slotnick.

The opening number of volume 50 of the American Journal of Mathematics contains: *Additive number theory for all quadratic functions*, by L. E. Dickson; *Démonstration de quelques propriétés des ensembles abstraits*, by M. Fréchet; *On certain points in the theory of Dirichlet series*, by J. F. Ritt; *On a theorem of Severi*, by O. Zariski; *On Gierster's classnumber relations*, by J. V. Uspensky; *Note on a theorem of Bôcher*, by G. C. Evans; *Generalized Neumann problems for the sphere*, by G. C. Evans; *On Taylor's series admitting the circle of convergence as a singular curve*, by J. J. Gergen and D. V. Widder; *A mathematical theory of depreciation and replacement*, by C. F. Roos.

The opening number of volume 29, series 2, of the Annals of Mathematics (December, 1927) contains: *Developments in Hermite polynomials*, by M. H. Stone; *A generalization of the calculus of finite differences to include the differential calculus*, by J. P. Ballantine; *Relations satisfied by coefficients of periodic solutions*, by W. J. Trjitzinsky; *An integral equation with an associated integral condition*, by L. Guggenbuhl; *Partition polynomials*, by E. T. Bell; *A new formulation for general algebra*, by J. W. Young; *A determination of the groups of order p^5*, by H. A. Bender; *Representation of functions determined by their initial values*, by W. J. Trjitzinsky; *An integration method of summing series*, by G. James; *On groups of order p^m, p being an odd prime number, which contain an abelian subgroup of order p^{m-1}*, by H. A. Bender; *Osculating derivative of a ruled surface*, by D. Sun; *Chains of congruences for the numerators and denominators of the Bernoulli numbers*, by J. L. Bell.

A new journal, the Boletin Matematico, has been founded at Buenos Aires, under the editorship of Professor B. I. Baidaff. The first number appeared in January, 1928.

A committee of this Society, consisting of Professors T. S. Fiske (chairman), R. C. Archibald, J. L. Coolidge, L. E. Dickson, E. R. Hedrick, Dunham Jackson, James Pierpont, M. I. Pupin, R. G. D. Richardson, and Oswald Veblen, has been appointed to arrange for fitting activities to mark the fiftieth anniversary of the founding of the Society in 1888.

The Calcutta Mathematical Society will celebrate the twentieth anniversary of its foundation by publishing a Commemoration Volume in the fall of 1928.

At the Nashville meeting of the American Association for the Advancement of Science, the following officers (among others) were elected: president, Professor H. F. Osborn; vice-president of Section A, Professor R. C. Archibald; secretary of Section A, Professor C. N. Moore. Professor L. E. Dickson was elected a member of the Council, and Professors D. R. Curtiss, F. R. Moulton, M. I. Pupin, and Edwin B. Wilson members of the Executive Committee. Professor Oswald Veblen is a member of the Committee on Grants for Research.

Owing to the favorable reception which has been given to the Mémorial des Sciences Mathématiques, a similar series to be known as the Mémorial des Sciences Physiques is to be published under the auspices of the French Academy and several other Academies, under the direction of Professors Henri Villat and Jean Villey. A number of titles of forthcoming volumes have been announced.

The Carnegie Foundation has recently presented the sum of $10,000 to the American Philosophical Association as a revolving fund for the publication of a series of source books in the history of the sciences, under the general editorship of Professor G. D. Walcott of Hamline University. It is proposed that each volume of about 600 pages, shall contain extracts from the most important contributions to science from about 1500 to 1900. At least one of these volumes will relate to mathematics, and will be under the immediate charge of a committee consisting of Professors R. C. Archibald, Florian Cajori, and David Eugene Smith (chairman). The articles will be about 6–20 pages each, including photographs of the original text, a translation, and comments. Numerous suggestions of articles for reprinting have been made already, but further suggestions with precise references to the original articles will be welcomed. Volunteers to assist in preparing and editing specific articles are especially desired. Suggestions and offers of assistance may be sent to any member of the committee.

An informal conference on analysis situs was held at Princeton University on April 4–5, 1928, in order to provide a close contact and a thorough exchange of ideas among those particularly interested, in advance of the symposium on the same topic at the meeting of the Society in New York on April 6–7. Among those participating were Professors Alexander, Lefschetz, and Veblen, and Dr. Alexandroff and Dr. Hopf, of Princeton, Professor H. M. Morse of Harvard, Professor E. W. Chittenden of Iowa, and Professor Kline of Pennsylvania. On the first day, Professor Veblen spoke at length on the development of the subject; on the second day, Professor Morse spoke at length on critical points of functions. There were numerous shorter talks and informal discussions.

The gold medal of the Royal Astronomical Society has been awarded to Professor R. A. Semple, for his theory of the four great satellites of Jupiter.

The Royal Society of London has awarded a royal medal to Professor J. C. McLennan, for his work in spectroscopy and atomic physics.

The Messel medal of the British Society of Chemical Industry has been awarded to Dr. R. A. Millikan, of the California Institute of Technology, in recognition of his achievement in measuring the electrostatic charge of the electron.

Professor M. I. Pupin has been given the Washington award of the Western Society of Engineers, in recognition of his work in long distance telephony and radio communication.

The Deutsche Gesellschaft für Wissenschaft und Kunst of Brünn has founded an endowment in memory of Professor Emil Waelsch, of the German Technical School at Brünn, for the establishment of scholarships in geometry at that Technical School.

The University of the Northern Caucasus (Rostov on the Don) and the Polytechnic Institute of the Don, with their associated scientific societies, held a meeting on April 22, 1928, in honor of Professor D. D. Mordoukhay-Boltovskoy, to celebrate his completion of thirty years of teaching.

M. Joseph Auclair has been elected a correspondent of the Paris Academy of Sciences in the section of mechanics.

The annual William Lowell Putnam Lecture was delivered on Wednesday, March 21, 1928, at Harvard University, by Professor Constantin Carathéodory, of the University of Munich. His subject was *Selected problems of the calculus of variations*.

Professor E. W. Brown, of Yale University, has been elected an associate of the Royal Academy of Belgium.

On the occasion of its semi-centennial celebration, the University of Colorado conferred a doctorate of laws on Professor R. A. Millikan, of the California Institute of Technology.

Dr. George Sauté, of Harvard University, has been awarded one of the fellowships of the Commission for Relief in Belgium Educational Foundation, for the study of mathematics in Belgium.

The academic jubilee of Professor Ch. J. de la Vallee Poussin was celebrated on May 13, 1928, at Louvain, Belgium, under the auspices of an international committee, by fitting academic exercises and by the presentation of a bust.

It is announced that Guggenheim fellowships in astronomy, mathematics, and physics have been awarded to the following persons: Professor Perry Byerly, of the University of California, for the study of mathematical geophysics; Professor Olive C. Hazlett, of the University of Illinois, for the study of the arithmetics of linear associative algebras together with their application and interpretation in other lines of mathematics, especially the theory of numbers; Professor J. J. Hopfield, of the University of California, to study the Zeemann effect of the infra-red spectrum of oxygen and nitrogen; Dr. R. J. Kennedy, of the California Institute of Technology, for research in the theory of radiation; Professor N. C. Little, of Bowdoin College, for the study of thermo-magnetic properties of gaseous molecules; Professor F. W. Loomis, of New York University, for the study of quantum mechanics; Professor L. E. Reukema, of the University of California, for the theoretical and experimental study of electric discharge in gases at high frequencies; Professor Otto Struve, of the University of Chicago, for a theoretical study of the distribution and physical properties of diffuse matter in interstellar space; Professor W. W. Watson, of the University of Chicago, for the study of molecular spectra.

The University of Göttingen announces the following courses in mathematics and mathematical physics for its summer session of 1928 (July 9 to August 4): Professor Hilbert, Foundations of mathematics; Professor Landau, Mapping with smooth functions; Professor Prandtl, Problems of aerodynamics; Professor Courant, Development of mathematics in the 19th century; methods of analysis; Professor Betz, Theory of hydro- and aerodynamics (with demonstrations); Dr. Schuler, The top in astronomy, physics, and technology (with demonstrations); Dr. Walther, New methods in applied mathematics; Dr. Grandjot, Infinite series in number theory; Dr. Neugebauer, Mathematics in antiquity. Lectures and courses in experimental and theoretical physics will be given by Professors Pohl, Reich, Born, Franck, Kienle, Coehn, and Oldenberg, and Drs. Sponer, Rupp, and Heitler. The subjects include electrical and optical phenomena, physical optics, electrons, band spectra, astronomy, physics of the atom, photo-chemistry, and electro-acoustics. These courses are intended especially for foreign students. Further particulars may be obtained from the Secretary of the University, Wilhelmsplatz (Aula), Göttingen, Germany.

Dr. Reinhold Fürth has been promoted to an associate professorship of theoretical physics at the German University of Prague.

Associate Professor Lothar Koschmieder, of the University of Breslau, has been appointed professor of mathematics at the German Technical School at Brünn.

Dr. Wolfgang Pauli, of the University of Hamburg, has been appointed professor of theoretical physics at the Zurich Technical School.

Dr. Hans Lewy has been admitted as privat docent in mathematics at the University of Göttingen.

Professor H. Mohrmann, of Basel, has been called to the professorship of descriptive geometry at the Darmstadt Technical School.

Dr. S. Mandelbrojt has been appointed "chargé de cours" in differential and integral calculus at the University of Lille.

Professor G. A. Gibson, of the University of Glasgow, has retired.

The following have been admitted as docents in European universities: Axel Schur, for mathematics, at Bonn; B. Segre, for analytic geometry, and G. Krall, for rational mechanics, at Rome; C. Poli, for rational mechanics, at Turin; O. Nikodym at Warsaw.

The following 41 doctorates with mathematics or mathematical physics as major subject were conferred by American universities during 1927; the university, month in which the degree was conferred, minor subject (other than mathematics), and title of dissertation are given in each case if available.

R. G. Archibald, Chicago, June, *Diophantine equations in division algebras.*

W. L. Ayres, Pennsylvania, June, *Concerning continuous curves and correspondences.*

F. R. Bamforth, Chicago, December, *A classification of boundary value problems for a system of ordinary differential equations of the second order.*

L. M. Blumenthal, Johns Hopkins, June, *Lagrange resolvents in euclidean geometry.*

W. F. Cheney, Jr., Massachusetts Institute of Technology, June, physics, *Infinitesimal deformation of surfaces in riemannian space.*

Alonzo Church, Princeton, June, *Alternatives to Zermelo's assumption.*

E. U. Condon, California, December, Physics, *Theory of intensity distribution in band systems.*

T. F. Cope, Chicago, December, *An analogue of Jacobi's condition for the problem of Mayer with variable end points.*

C. C. Craig, Michigan, June, *An application of Thiele's semi-invariants to the sampling problem.*

D. R. Davis, Chicago, June, *The inverse problem of the calculus of variations in higher space.*

D. A. Flanders, Pennsylvania, June, *Double elliptic geometry in terms of point, order, and congruence.*

P. A. Fraleigh, Cornell, June, physics, *Regular bilinear transformations of sequences.*

H. H. Germond, Wisconsin, June, physics, *The effect of space charge on a three element vacuum tube.*

Lois W. Griffiths, Chicago, June, *Certain quaternary quadratic forms and diophantine equations by generalized quaternion algebras.*

Laura Guggenbühl, Bryn Mawr, June, education, *An integral equation with an associated integral condition.*

D. C. Harkin, Chicago, September, *The duality and abstract identity of the theories of modular systems and ideals.*

Ruby U. Hightower, Missouri, June, physics and astronomy, *On the classification of the elements of a ring.*

Charles Hopkins, Illinois, May, physics, *Non-abelian groups whose groups of isomorphisms are abelian.*

C. G. Jaeger, Missouri, June, physics, *A character symbol for primes relative to a cubic field.*

Vern James, Stanford, June, statistics, *Primitive linear homogeneous groups of variety two or three.*

A. R. Jerbert, University of Washington, December, physics, *Projective differential geometry of a system of linear differential equations of the first order.*

G. W. Kenrick, Massachusetts Institute of Technology, March, electrical engineering, *A new method of periodogram analysis with illustrative applications.*

James McGiffert, Columbia, May, *Particular solutions in closed form of types of linear differential equations of second order.*

E. L. Mackie, Chicago, September, *The Jacobi condition for a problem of Mayer with variable end points.*

Florence M. Mears, Cornell, June, physics, *Riesz summability for double series.*

W. M. Miller, Illinois, May, astronomy, *On the variance of interpolated observations.*

T. W. Moore, Yale, June, *The complete system of invariants of the rational space quintic curve.*

J. S. Morrell, Illinois, May, physics, *Expansion of functions in series of functions generalizing the gamma function.*

J. H. Neelley, Yale, June, *Compound singularities of the rational plane quartic curve.*

F. C. Ogg, Illinois, May, physics, *Certain configurations on cubics.*

Hillel Poritsky, Cornell, June, physics, *Topics in potential theory.*

T. H. Rawles, Yale, June, *The invariant integral and the inverse problem in the calculus of variations.*

E. H. Reimer, California, May, astronomy, *A new Eddingtonian geometry, with applications to differential geometry.*

C. H. Richardson, Michigan, June, *The statistics of sampling.*

Edward Saible, Massachusetts Institute of Technology, October, physics, *Analytic and topological transformations of closed surfaces.*

Hazel E. Schoonmaker, Cornell, June, education, *Non-monoidal involutions having a congruence of invariant conics.*

H. C. Shaub, Cornell, September, astronomy, *Rational involutorial transformations in* S_4 *which leave invariant* ∞^4 *quadratic varieties.*

A. A. Shaw, California, December, analytic mechanics, *Solution of homogeneous linear difference equations by means of infinite determinants.*

I. M. Sheffer, Harvard, June, *On the theory of linear differential equations of infinite order.*

C. H. Smiley, California, May, astronomy, *On the number of solutions of Leuschner's direct method for determining parabolic orbits.*

E. P. Starke, Columbia, July, *Certain uniform functions of rational functions.*

W. J. Trjitzinsky, California, August, celestial mechanics, *The elliptic cylinder differential equation.*

H. S. Wall, Wisconsin, June, mathematical physics, *On the Padé approximants associated with the continued fraction and series of Stieltjes.*

A. K. Waltz, Cornell, June, physics, *The steady flow of liquid through a circular hole in an infinite plane.*

G. T. Whyburn, Texas, June, organic chemistry, *Concerning continua in the plane.*

W. M. Whyburn, Texas, June, physical chemistry, *Linear boundary value problems for ordinary differential equations and their associated difference equations.*

John Williamson, Chicago, June, *Conditions for associativity of division algebras connected with non-abelian groups.*

A. S. Winsor, Johns Hopkins, June, *The composition of homologies and of homographies.*

The following graduate courses in mathematics are announced for the summer of 1928:

UNIVERSITY OF CHICAGO, first term, June 18 to July 25; second term, July 26 to August 31.—By Professor L. E. Dickson: Topics in the theory of numbers.—By Professor E. T. Bell: Theory of functions of a complex variable; Applications of analysis to the arithmetic of quadratic forms.—By

Professor E. P. Lane: Synthetic projective geometry; Projective differential geometry I.—By Professor F. D. Murnaghan: Modern hydrodynamical theory.—By Professor M. I. Logsdon: Higher plane curves.—By Professor L. M. Graves: Modern theories of integration.—By Professor R. W. Barnard: Advanced calculus; Introduction to higher algebra.—By Professor Walter Bartky: Introduction to celestial mechanics II.—By Dr. F. R. Bamforth: Differential equations.

UNIVERSITY OF COLORADO, first term, June 18 to July 21; second term, July 23, to August 24.—By Professor Light: Teacher's course in mathematics; History of mathematics; Calculus of variations.—By Professor Kempner: Advanced teacher's course in mathematics; Differential equations; Group theory and introduction to Galois theory. Second term:—By Professor Light: Statistics; Theory of equations; Calculus of variations (continued).—By Professor Kempner: Advanced teacher's course (repeated); Differential equations (continued); Group theory and introduction to Galois theory (continued).

COLUMBIA UNIVERSITY, July 9 to August 17.—By Professor W. B. Fite: Differential equations; Theory of infinite series.—By Professor J. F. Ritt: Theory of functions of a complex variable.—By Professor K. W. Lamson: Differential geometry.—By Dr. B. O. Koopman: Introduction to modern geometry.

UNIVERSITY OF ILLINOIS, June 18 to August 11.—By Professor G. A. Miller: Introduction to higher algebra, based on Bôcher; Critical study of the history of mathematics.—By Professor J. B. Shaw: Functions of a real variable.—By Professor H. R. Brahana: Projective geometry.—By Dr. L. L. Steimley: Differential equations.—By Dr. H. W. Bailey: Teachers' courses in elementary mathematics.—By Mr. G. L. Edgett: Mathematical statistics for teachers.

UNIVERSITY OF IOWA, first term, June 7 to July 20.—By Dr. M. A. Nordgaard: Subject matter and teaching of mathematics.—By Dr. Conkwright: Ordinary differential equations; Theory of functions of a complex variable.—By Professor Wylie: Elementary mechanics; Mathematics of finance; Descriptive astronomy.—By Professor Woods: Advanced coordinate geometry; Elliptic integrals.—By Professor Reilly: Algebra for high school teachers; Disability theory applied to life insurance.—By Professor Chittenden: Advanced calculus; Functions of infinitely many variables. Second term, July 23 to August 24.—By Dr. Nordgaard: The history of mathematics.—By Dr. Ward: Elementary mechanics; Theory of functions of a complex variable.—By Professor Baker: Differential equations; Geometry of forces.—By Professor Reilly: The numerical solution of equations; Frequency distributions and correlation.

UNIVERSITY OF MICHIGAN, June 25 to August 17.—By Professor W. B. Ford; Advanced calculus; Infinite series with special reference to Fourier series.—By Professor L. C. Karpinski: Teaching of algebra; History of mathematics.—By Professor Peter Field: Vector analysis.—By Professor

T. R. Running: Graphical methods.—By Professor J. W. Bradshaw: Figures of solid geometry; Selected topics in projective geometry.—By Professor T. H. Hildebrandt: Theory of functions of a complex variable; Theory of the potential.—By Professor H. C. Carver: Advanced mathematical theory of statistics.—By Professor L. A. Hopkins: Elements of mechanics.—By Professor C. J. Coe: Integral equations.—By Professor Norman Anning: Differential equations; Determinants and theory of equations.—By Professor J. A. Nyswander: Theory of probability; Finite differences.—By Mr. O. J. Peterson: Solid analytic geometry.

UNIVERSITY OF MINNESOTA, first term, June 16 to July 28; second term, July 30 to September 1. First term:—By Assistant Professor Gibbens: Differential equations; Synthetic metric geometry.—By Professor Brink: Advanced calculus; Reading in advanced mathematics. Second term:—By Professor Underhill: Reading in advanced mathematics.

UNIVERSITY OF NEBRASKA, June 4 to July 13.—By Professor Brenke: Differential equations.—By Professor Gaba: Advanced euclidean geometry; Foundations of algebra and geometry.—By Professor Camp: Advanced algebra.—By Professor Pierce: Theory of equations.—By Mr. Harper: Elements of the mathematical theory of statistics.

OHIO STATE UNIVERSITY, June 18 to August 31.—By Professor C. L. Arnold: The teaching of mathematics; Differential equations.—By Professor C. C. Morris: Theory of equations; Mathematical statistics.—By Professor J. H. Weaver: Differential geometry; Introduction to higher algebra.

UNIVERSITY OF PENNSYLVANIA, July 2 to August 11.—By Professor H. H. Mitchell: Probability.—By Professor M. J. Babb: College geometry.—By Professor Arnold Dresden, of Swarthmore College: Elliptic functions; Differential equations.—By Professor J. R. Kline: Theories of integration.—By Professor J. M. Thomas: Introduction to tensor analysis.

STANFORD UNIVERSITY, June 21 to August 31.—By Dr. Alfred Errera (Belgium): Topics selected from the foundations of mathematics; Geometry and the elements of axiomatics.—By Professor E. Hille (Princeton): Differential equations; Theory of functions of a complex variable.

UNIVERSITY OF TEXAS, first term, June 5 to July 16.—By Professor R. L. Moore: Functions of real variables; Foundations of geometry.—By Professor E. L. Dodd: Infinite processes; Mathematical statistics.—By Professor H. J. Ettlinger: Boundary value problems; Ruler and compass constructions.—By Professor C. D. Rice: Differential geometry; Advanced calculus.—By Professor P. M. Batchelder: Teaching problems in mathematics.—By Professor C. M. Cleveland: Advanced calculus.—By Professor Goldie Horton: Theory of equations. Second term, July 16 to August 27.—By Professor H. J. Ettlinger: Boundary value problems; Definite integrals.—By Professor P. M. Batchelder: Teaching problems in mathematics.—By Professor C. M. Cleveland: Advanced calculus.—By Professor R. G. Lubben: Non-euclidean geometry.—By Professor G. T. Whyburn: Functions of real variables.

UNIVERSITY OF WISCONSIN, six weeks session, June 25 to August 3—By Professor J. H. Taylor: Differential equations; Differential geometry.—By Professor E. B. Van Vleck: Modern geometrical concepts; The location of the roots of an algebraic equation. Special nine weeks course for graduates, June 25 to August 24.—By Professor R. E. Langer: Partial differential equations; Calculus of variations.—By Professor Warren Weaver: Theory of potential; Introduction to statistical mechanics.

Dr. Ethel L. Anderton, of Smith College, has been appointed assistant professor of mathematics at Mount Holyoke College.

Professor Léon Brillouin, of the Collège de France, has been appointed acting professor of theoretical physics at the University of Wisconsin, for the academic year 1927–28.

Mr. Alexander Cook, lecturer at the University of Alberta, has been promoted to an assistant professorship of mathematics.

Dr. H. N. Davis, professor of mechanical engineering at Harvard University, has been elected president of the Stevens Institute of Technology.

Assistant Professor Øystein Ore, of Yale University, has been made associate professor of mathematics.

Lieutenant Colonel A. J. C. Cunningham, known for his work in number theory and the preparation of factor tables, is dead. Colonel Cunningham had been a member of the American Mathematical Society since 1905.

Professor Krishna Prasad Dé, of University College, Rangoon, is dead. Professor Dé was a member of the American Mathematical Society.

Professor R. W. Genese, of the University College of Wales, Aberystwyth, died January 21, 1928, at the age of seventy-nine. Professor Genese was a member of the American Mathematical Society.

Professor J. L. Heiberg, of Copenhagen, known for his work in the history of Greek mathematics, died January 4, 1928, at the age of seventy-two.

Professor Hendrik Antoon Lorentz, of the University of Leyden, the distinguished physicist, died February 4, 1928, at the age of seventy-four.

Dr. W. C. Eglin, vice-president and chief engineer of the Philadelphia Electric Company and president of the Franklin Institute, died February 7, 1928.

William Wallace Payne, of the observatory of the National Watch Company, Elgin, Ill., the founder of Popular Astronomy, died January 29, 1928, at the age of ninety-one.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

BALL (W. W. R.). Histoire des mathématiques. Paris, Hermann, 1927. 336 pp.

BARBARIN (P.). La géométrie non-euclidienne. 3e édition suivie de notes sur la géométrie non-euclidienne dans ses rapports avec la physique mathématique par A. Buhl. (Scientia, No. 15.) Paris, Gauthier-Villars, 1928. 176 pp.

BENNY (L. B.). Mathematics for students of technology. Senior course. Part I. Oxford, University Press, 1927. 415+27 pp.

BERTINI (E.). Complementi di geometria proiettiva. Bologna, Zanichelli, 1928. 354 pp.

BOHDE (H.). Mathematisch-technische Zahlentafeln. 5te vermehrte Auflage. Berlin, Springer, 1927.

BUHL (A.). See BARBARIN (P.).

CASSINIS (G.). Calcolo numerici, grafici e meccanici. Pisa, Mariotti-Paccini, 1928. 6+672 pp.

COHN-VOSSEN (S.). See KLEIN (F.).

COURANT (R.). See KLEIN (F.).

DE DONDER (T.). Théorie des invariants intégraux. Paris, Gauthier-Villars, 1927.

DUARTE (F. J.). Nouvelles tables de log $n!$ à 33 décimales depuis $n=1$ jusqu'à $n=3000$. Genève, Albert Kundig, et Paris, Index Generalis, 1927. 24+136 pp.

EISENHART (L. P.). Non-Riemannian geometry. (American Mathematical Society Colloquium Publications, volume VIII.) New York, American Mathematical Society, 1927. 8+184 pp.

ENCYKLOPÄDIE DER MATHEMATISCHEN WISSENSCHAFTEN. 2ter Band: Analysis. 3ter Teil, 2te Hälfte. Leipzig, Teubner, 1923–27. 14+675+1648 pp.

ENCYKLOPÄDIE DER MATHEMATISCHEN WISSENSCHAFTEN. 3ter Band: Geometrie. 3ter Teil. Leipzig, Teubner, 1902–27. 606+206 pp.

ENRIQUES (F.). L'évolution des idées géométriques dans la pensée grecque. Point, ligne, surface. Traduit sur la 3e édition italienne par Maurice Solovine. Paris, Gauthier-Villars, 1927. 48 pp.

GILLAIN (O.). La science égyptienne. L'arithmétique au moyen empire. Bruxelles, Edition de la Fondation Egyptologique Reine Elisabeth, 1927. 16+326 pp.

GODFREY (C.) and SIDDONS (A. W.). Four-figure tables. Cambridge, University Press, 1927. 40 pp.

HAAG (J.). Formulaire de mathématiques élémentaires. Paris, Gauthier-Villars, 1928. 60 pp.

HARDY (G. H.). See RAMANUJAN (S.).

KLEIN (F.). Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert. Teil 2: Die Grundbegriffe der Invariantentheorie und ihr Eindringen in die mathematische Physik. Für den Druck bearbeitet von R. Courant und S. Cohn-Vossen. (Die Grundlehren der mathematischen Wissenschaften, Band 25.) Berlin, de Gruyter, 1927. 10+208 pp.

KNOPP (K.). Aufgabensammlung zur Funktionentheorie. Teil 2: Aufgaben zur höheren Funktionentheorie. (Sammlung Goschen.) Berlin, de Gruyter, 1928. 143 pp.

LAURÈS (C.). Les bases de la géométrie et de la physique. L'invariance de l'espace euclidien. Paris, Blanchard, 1928. 125 pp.

LIETZMANN (W.). Aufbau und Grundlage der Mathematik. Leipzig, Teubner, 1927.

MACROBERT (T. M.). Spherical harmonics, An elementary treatise on harmonic functions with applications. New York, Dutton, 1928. 12 +302 pp.

MALSCH (F.). Geschichte der Mathematik. Leipzig, Quelle und Meyer, 1928. 115 pp.

NÖRLUND (N. E.). Sur la "somme" d'une fonction. (Mémorial des Sciences Mathématiques, No. 24.) Paris, Gauthier-Villars, 1927. 53 pp.

PARSONS (G. L.). Elementary differential calculus. Cambridge, University Press, 1927. 10+204 pp.

DU PASQUIER (L. G.). Leonard Euler et ses amis. Paris, Hermann, 1927. 126 pp.

RAMANUJAN (S.). Collected papers. Edited by G. H. Hardy, P. V. Seshu Aiyar, and B. M. Wilson. Cambridge, University Press, 1927. 36+355 pp.

REMOUNDOS (G.). Extension aux fonctions algébroides multiformes du théoreme de M. Picard et de ses généralisations. (Mémorial des Sciences Mathématiques, No. 23.) Paris, Gauthier-Villars, 1927. 62 pp.

REY PASTOR (J.). Los matematicos espanoles des siglo XVI. Madrid, Biblioteca Scientia, 1926. 163 pp.

VON SANDEN (H.). Mathematisches Praktikum. Leipzig, Teubner, 1927.

SESHU AIYAR (P. V.). See RAMANUJAN (S.).

SIDDONS (A. W.). See GODFREY (C.).

SMOLUCHOWSKI (M.). Oeuvres. Tome 2. Cracovie, Académie Polonaise des Sciences et des Lettres, et Paris, Béranger, 1927. 656 pp.

SOLOVINE (M.). See ENRIQUES (F.).

STEINBERG (W.). Der Erkenntnisbegriff in Physik und Geometrie. Kirchhain (Nieder-Lausitz), Zahn und Baendel, 1927. 87 pp.

WATSON (G. N.). See WHITTAKER (E. T.).

WHITTAKER (E. T.) and WATSON (G. N.). A course of modern analysis. 4th edition. Cambridge, University Press, 1927. 6+608 pp.

WIELEITNER (H.). Der Begriff der Zahl in seiner logischen und historischen Entwicklung. 3te Auflage. Leipzig, Teubner, 1927.

WILSON (B. M.). See RAMANUJAN (S.).

YANO (T.). Tables de calcul. Tokyo, The Actuary, 1927. 3+1816 pp.

PART II. APPLIED MATHEMATICS

d'Abro (A.). The evolution of scientific thought from Newton to Einstein. New York, Boni and Liveright, 1927. 544 pp.

Bachelard (G.). Etude sur l'évolution d'un problème de physique: la thermique dans les solides. Paris, Vrin, 1928. 183 pp.

Baly (E. C. C.). Spectroscopy. 3d edition. Volume 3. London, Longmans, 1927. 532 pp.

Becker (W.). See de Broglie (L.).

Bennett (T. A.). See Youngson (P.).

Bloch (E.). Thermionic phenomena. Translated by J. R. Clarke. London, Methuen, 1927. 8+145 pp.

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THE APRIL MEETING IN CHICAGO

The two hundred sixty-first meeting, twenty-ninth regular Western meeting of the Society was held at the University of Chicago on Friday and Saturday, April 6–7, 1928. About one hundred twenty persons attended the meeting, among whom were the following ninety-six members of the Society:

A. A. Albert, N. L. Anderson, R. W. Babcock, Bamforth, Bardell, Barnett, Bartky, Bear, Bibb, G. A. Bliss, Bussey, G. N. Carmichael, R. D. Carmichael, Carver, Chaimovitch, Conkwright, Cook, Cope, H. B. Curtis, Curtiss, H. T. Davis, Davisson, Dickson, Douglas, Earl, Everett, A. L. Foster, C. A. Garabedian, H. L. Garabedian, Garver, Georges, Gladys Gibbens, J. W. Glover, Gouwens, L. M. Graves, Griffiths, V. G. Grove, Hartung, Hassler, Hyden, Louis Ingold, Ingraham, Dunham Jackson, B. W. Jones, Kinney, E. P. Lane, Langer, LaPaz, Laun, Logsdon, C. E. Love, Lunn, J. V. McKelvey, W. D. MacMillan, W. A. Manning, March, William Marshall, Marquis, T. E. Mason, Mickelson, C. N. Moore, E. H. Moore, E. J. Moulton, A. L. Nelson, Olson, Oppenheim, Palmié, Parkinson, Pettit, H. L. Rietz, Roever, Roth, Rupp, Sanger, Schottenfels, Sherer, Shohat, W. G. Simon, Skinner, Slaught, E. R. Smith, Virgil Snyder, Stouffer, J. H. Taylor, E. L. Thompson, Thornton, Trjitzinsky, J. S. Turner, Van Vleck, Wall, L. E. Ward, Warren Weaver, Wiggin, Roscoe Woods, Wyant, Yeaton.

On Friday afternoon Professors E. B. Stouffer and E. P. Lane gave symposium addresses on *Recent Developments in Projective Differential Geometry.* On a motion by Professor Carver, a rising vote of thanks was given the speakers. These addresses appear in full in the present issue of this Bulletin

On Friday evening members and their guests attended dinner in the Del Prado Hotel. Professor Jackson acted as toastmaster. At the dinner President Snyder announced the first award of the Frank Nelson Cole prize to Professor L. E. Dickson for his book *Algebren und ihre Zahlentheorie* and other work. President Snyder presented the prize to Professor Dickson, who thanked the President. Both speakers paid tributes to the memory of Professor Cole. The toastmaster also called upon Professor Slaught, Dr.

Griffiths and Professors Manning and C. N. Moore for talks. The dinner was attended by eighty-five persons.

The Council of the Society met Friday afternoon.

The following sixteen persons were elected to ordinary membership:

Mr. Charles Trotter Bunnell, University of Rochester;
Professor Andrew G. Clark, Colorado Agricultural College;
Professor Howard Adams DoBell, New York State College for Teachers;
Mr. Harold L. Dorwart, Yale University;
Miss Allegra Eckles, Territorial Normal School, Honolulu;
Mr. Edwin Harold Hadlock, Cornell University;
Professor Bruce Vickroy Hill, Phillips University;
Sister Mary Columba, Villa Maria College;
Sister Mary St. Bernard, Villa Maria College;
Mr. James Lewis Noecker, Thiel College;
Mr. Oscar John Peterson, University of Michigan;
Mr. Harry Ward Seward, Utica, N. Y.;
Professor Clarence DeWitt Smith, Louisiana College;
Professor Elva Elizabeth Starr, Alfred University;
Mr. John Russell Vatnsdal, State College of Washington;
Mr. Clement Weinstein, University of Pennsylvania.

The following were elected to membership as nominees of Members of Department of Mathematics, University of Wisconsin:

Mr. C. W. Dancer, University of Wisconsin;
Mr. G. N. Carmichael, University of Wisconsin;
Mr. R. O. Gilmore, University of Wisconsin.

The President announced as the Committee on Arrangements for the semi-centennial celebration of the Society: Professors T. S. Fiske (Chairman), R. C. Archibald, J. L. Coolidge, L. E. Dickson, E. R. Hedrick, Dunham Jackson, James Pierpont, M. I. Pupin, R. G. D. Richardson, and Oswald Veblen.

The following were appointed as a Committee for the nomination of officers to be elected at the next annual meeting: Professors L. E. Dickson (Chairman), E. T. Bell, D. R. Curtiss, H. M. Morse, J. K. Whittemore.

It was decided to hold a fall meeting at Cincinnati this year. It is proposed, if this meeting shows the need, to hold such a meeting each year in the Ohio Valley Region. This meeting will probably take place on the Friday and Saturday after Thanksgiving. The President appointed as a Committee on Arrangements to act with the Associate Secretary: Professors C. N. Moore (Chairman), Henry Blumberg, P. P. Boyd, Louis Brand, and H. T. Davis.

The following dates for meetings of the Society in the Middle West were determined upon: December 31-January 1, 1928-9, University of Chicago; March 28-29, 1929, University of Chicago. As a Committee to arrange for a symposium at the latter of these meetings, the President appointed Professors E. P. Lane (Chairman), M. H. Ingraham, C. C. MacDuffee.

It was determined to hold the Western winter meeting of the Society for 1929-30 at Des Moines in connection with with the meeting of the American Association for the Advancement of Science.

The following were appointed as a Committee to study the question of how to increase the value of the summer meetings: Professors Arnold Dresden (Chairman), C. R. Adams, G. C. Evans, L. M. Graves, A. J. Kempner, and Warren Weaver.

The Council unanimously voted to thank the Carnegie Corporation for its recent gift of $5,000 to the endowment of the Society.

At the Saturday morning meeting of the Society the following resolution expressing the Society's appreciation of the action of the Mathematical Association of America in becoming a sustaining member was unanimously passed:

The American Mathematical Society wishes to express its particular appreciation of the constant and strong support that the Mathematical Association of America has given it, and especially of its gracious action in becoming a sustaining member of the Society It hopes that this unity may strengthen the efforts of the two organizations to further mathematics in America.

The papers whose abstracts appear below were read in three sections: (1) Geometry and Applied Mathematics—

Friday morning; (2) Algebra and Number Theory—Friday morning; (3) General, mostly Analysis—Saturday morning. Professor Roever presided at the first section, Professor C. N. Moore at the second, President Snyder and Professor Stouffer at the third. President Snyder presided at the symposium on Friday afternoon. Papers numbered 1–20 were read in the section on Geometry and Applied Mathematics, 21–30 in the section on Algebra and Number Theory, and 31–49 Saturday morning. Papers numbered 3, 8–10, 12, 14–19, 42–49 were read by title. Professor Hufford was introduced by Professor Davis, and Mr. Shook, Mr. Yank, and Mr. Morrow were introduced by Professor Dickson.

1. Professor I. A. Barnett: *The finite transformations generated by an infinitesimal projective transformation in function space.*

In a paper entitled *Projective transformations in function space*, Transactions of this Society, 1919, Professor L. L. Dines gave a method for determining the one-parameter family of projective transformations generated by a regular infinitesimal projective transformation. The method, however, made use of certain identities involving the inverse of a projective transformation in function space. In this paper, it is shown that by the introduction of homogeneous coordinates in the infinitesimal transformation, it is possible to avoid the formalities which Dines uses and obtain the result much more briefly and directly. For the particular case of the sub-group which leaves the unit sphere in function space invariant, the problem is reduced to integrating two systems of two simultaneous integro-differential equations.

2. Professor J. O. Hassler: *Plane nets whose first and minus first Laplacian transforms each degenerate into a straight line.*

This paper will appear in full in an early issue of this Bulletin.

3. Professor Arnold Emch: *Multiple systems, their geometric representation and groups.*

This paper considers triple and multiple systems with particular reference to their geometric representations and group properties. Two triple systems each invariant under a G_{168} on 7 elements without a common triple may be mapped on a torus. The configuration of triangles thus obtained is connected with the 7 hexagons on the torus invariant under a G_{42}. There exist triple systems such that all possible triples may be arranged in a complete set of triple systems, in which every triple occurs just once. This fact as shown by the author in case of 9 elements seems to be new. Two triple systems on 9 elements without a common triple may be mapped on a torus. The group leaving a triple system on 9 elements invariant is a G_{432}. The configuration on the torus is connected with a polyhedron on the torus bounded by 6 octagons and 6 squares and is invariant in a non-cyclic G_6. A very interesting configuration is obtained by the mapping of a triple system on 6 elements in which every couple occurs twice. We obtain a polyhedron on the projective plane bounded by 6 pentagons which is invariant in an icosahedral G_{60}. The lowest non-trivial (1234) quadruple system, containing every triple one, is on 8 elements. Such a quadruple system is invariant under a G_{1344}. Two such quadruple systems without a common triple may be mapped into a hyperpolyhedron in space of four dimensions, which is invariant in a G_{336}. The quadruple system on 16 elements containing every element 6 times contains 24 quadruples and is invariant under a G_{192}. Its map is the four-dimensional cube.

4. Professor Virgil Snyder: *Quadratic involutions in multiple linear complexes.*

It is shown that space involutorial birational transformations exist having the three properties: (a) Lines joining a pair PP' of conjugate points belong to a non-singular linear complex; (b) each line of the complex contains k such pairs of conjugate points, k any positive integer; (c) such transformations exist which have no surface of invariant points. Every plane contains a self-conjugate curve of order $2k+1$, and the correspondence PP' on it is always singular for $k>1$.

5. Dr. Jesse Douglas (National Research Fellow): *Reduction of the problem of Plateau to the minimization of a certain functional.*

In a previous paper (abstract in this Bulletin, March-April, 1927) the author shows how to reduce the Plateau problem to the integral equation (1)$^*\int_0^1 K(t, \tau) \operatorname{ctn} \frac{1}{2}(\phi(\tau)-\phi(t))d\tau=0$, or to the equivalent integral equation $^*\int_0^1 K(t, \tau)d\tau/(\psi(\tau)-\psi(t))=0$, where $\psi(t)=\operatorname{ctn} \frac{1}{2}\phi(t)$. The function $K(t, \tau)$ is known as soon as the contour is given. The star indicates that the improper integral is to be taken in its Cauchy principal value The function $\phi(t)$ is required to be always increasing and continuous as t increases from 0 to 1, and $\phi(0)=0$, $\phi(1)=2\pi$. The present paper points out that (1) is the Euler-Lagrange equation for the minimization of the functional $A(\phi)=-\int_0^1\int_0^1 K(t, \tau) \log \sin \frac{1}{2}\left|\phi(\tau)-\phi(t)\right| dt\, d\tau$; that is, (1) expresses the vanishing of the Volterra derivative $A'(\phi, t)$. An existence theorem is established for this minimum problem in case $K(t, \tau)$ is positive for all t, τ on the basis of the ideas in Fréchet's thesis (Palermo Rendiconti, vol. 22 (1906)), relative to functions on closed, compact sets.

6. Dr. Jesse Douglas: *A characterization of Riemann spaces of constant non-zero curvature and $n\geqq 3$ dimensions.*

Consider a projective space P_n of $n\geqq 3$ dimensions, referred to coordinates $x^i(i=1, 2, \cdots, n)$. A metric for angles may be imposed on P_n by means of a relative quadratic form $A=\nu^2 g_{ij}dx^i dx^j$, where the g_{ij} are fixed functions and ν^2 is an arbitrarily variable function of the x's; hence only the ratios of the g's are important. A metric for areas may be imposed on P_n by defining the area of the infinitesimal parallelogram determined by the elements dx, δx, to be the square root of $F=A_{ij,kl}(dx^i\delta x^j-dx^j\delta x^i)(dx^k\delta x^l-dx^l\delta x^k)$, where $A_{ij,\lambda l}$ are arbitrarily assigned functions of the x's. An angle-area space is identifiable with a Riemann space when and only when the $A_{ij,kl}$ are proportional to the two-rowed determinants $g_{ik}g_{jl}-g_{il}g_{jk}$. The present paper shows that if $n\geqq 3$, the only way to impose upon P_n an angle-area metric which shall be such that in every rectilinear triangle the angular excess is equal to a non-zero constant times the area, is to define A, F as in the Cayley metric based on any fixed quadric in P_n.

7. Dr. Jesse Douglas: *A new special form for the linear element of a surface.*

This paper deals with the case $n=2$ of the problem of the preceding paper. If $n=2$, any A, F metric is identifiable with a Riemann metric by a proper choice of the factor ν^2. Thus we have to find what the linear element of a surface must be in order that there may exist upon it a system of paths having the following two properties: (1) the system shall be linear (equivalent by point transformation to the straight lines of a plane); (2) the angular excess of any triangle of paths shall be proportional to its area, with a constant of proportionality $k\neq 0$. This problem is solved, a formula being obtained for ds^2 which involves four arbitrary functions: U_1, U_2 of u alone, and V_1, V_2 of v alone. Only one of these functions is essential, if we assume the surface to be real. Many surfaces not of constant Gaussian curvature are included.

8. Dr. Jesse Douglas: *Necessary and sufficient conditions for the equivalence of general affine and descriptive spaces of paths.*

This paper is a generalization of the work of Christoffel on the equivalence of quadratic differential forms, and of Veblen and J. M. Thomas on the equivalence of "restricted" spaces of paths, that is, systems of paths definable by $d^2x^i/dt^2=\Gamma_{jk}^i(dx^j/dt)(dx^k/dt)$, where the Γ's are any functions of of the x's. For references, see Chapter V of O. Veblen, *Invariants of Quadratic Differential Forms*, Cambridge, 1927. General affine and descriptive spaces of paths have been defined by the writer in his paper *The general geometry of paths*, soon to appear in the Annals of Mathematics. The analytic basis here is a system of differential equations of the form $d^2x^i/dt^2=H_2^i(x, dx/dt)$, where H_2^i denotes any function (generally transcendental) homogeneous of the second degree in the arguments dx/dt. The present paper develops criteria for the affine and for the descriptive equivalence of two such general spaces of paths.

9. Dr. Jesse Douglas: *Normal coordinates for a space of K-spreads.*

Normal coordinates for the "restricted" geometry of paths: $d^2x^i/dt^2 = \Gamma_{jk}^i(x)(dx^j/dt)(dx^k/dt)$, generalizing Riemann's normal coordinates, have been defined by O. Veblen, (Proceedings of the National Academy, vol. 8 (1922)). Normal coordinates for the general geometry of paths: $d^2x^i/dt^2 = H_2^i(x, dx/dt)$, have been defined by the writer in his paper cited in the preceding abstract. In the present paper, normal coordinates are defined for a space of K-spreads (abstract, this Bulletin, Jan.-Feb. 1928). They are relative to a coordinate system (x), a fixed point x_0, and a fixed K-element at x_0. It is proved that these normal coordinates have the essential property that when the coordinate system (x) undergoes arbitrary analytic transformation, the related normal coordinates y undergo linear homogeneous transformation. This property enables us to define the process of *extension* of a tensor by means of partial differentiation with respect to the normal coordinates.

10. Dr. Jesse Douglas: *Differentiation with respect to a parameter of the Cauchy principal value of an improper integral.*

It is found that formal differentiation under the integral sign does not apply to the Cauchy principal value of an improper integral, even when the functions involved are quite regular. Thus: $(d/dt^*(\int_0^1 K(t, \tau)(\tau-t)^{-1}dt = {}^*\int_0^1 (\partial K/\partial t + \partial K/\partial\tau)(\tau-t)^{-1}d\tau$. This result is generalized to the case ${}^*\int_0^1 K(t, \tau)F(\phi(\tau)-\phi(t))d\tau$, where F has an infinity of order one at the zero value of its argument.

11. Professors H. T. Davis and M. E. Hufford: *Diffraction and the wave theory of light.*

The purpose of this paper is to check the energy in a diffraction pattern of 66 rings secured by passing monochromatic light through a circular orifice. The assumptions made are those of the classical wave theory of light and the mathematical development follows that given in the celebrated papers of E. Lommel. (See Gray, Mathews and MacRobert, *Bessel Functions*, 1922, Chapter 14.) The mathematical interest centers in the discovery of divergent series asymptotic to the Lommel functions $U_1(x)$ and $U_2(x)$. The striking agreement of the calculated energy with that found in the diffraction pattern, particularly as shown in the broadening and darkening of the rings toward the outer part of the plate, gives renewed confidence in the classical theory of light as applied to diffraction phenomena.

12. Dr. C. D. Smith: *On generalized Tchebycheff inequalities in mathematical statistics.*

It is the main purpose of the paper to give a further development of the theory and properties of what may be appropriately called generalized Tchebycheff inequalities. The inequality is discussed as a proposition of geometry. The Markoff Lemma is generalized in such a manner as to give rise to several important forms of the inequality in a very simple way. Some closer inequalities are derived. It is shown that the closeness of the inequality may be improved in certain cases by moving the origin. Certain types of distributions are discussed to show that rather simple types of distributions exist within which the difference between the members of the inequality attains a minimum. Types of functions are discussed for which the difference between the members of the inequality may be made arbitrarily small.

13. Professor H. W. March: *The torsion problem for prisms of non-isotropic material.*

It is shown by means of a linear transformation that the soap film method can be used to solve the torsion problem for a prism composed of non-isotropic material having three mutually perpendicular planes of elastic symmetry, two of which are parallel to the length of the prism. The torsional rigidity of the given prism is equal to that of a transformed prism of isotropic material whose modulus of rigidity is expressed in terms of the moduli of the material of the non-isotropic prism. The lines of shearing stress in the cross section of the transformed prism become, after transformation to the cross section of the original prism, the lines of shearing stress in that section.

14. Professor E. L. Dodd: *A test for periods when certain measurements are of two variates combined.*

The usual tests in periodogram analysis are decidedly untrustworthy in dealing with data where some measurements probably represent two variates erroneously united. Such an error may occur in measuring the thickness of layers not always distinctly separated. For such data, an "intensity" should be defined (1) not greatly affected by the above slipping of phase, (2) not too difficult to compute, (3) with expected value zero, and (4) with mean error unity for every trial period. The following definitions for the intensity I meets requirements (1), (3), and (4), and perhaps (2). Let the data be divided into sets of s variates, for example, $s=60$ to test for a period of k. Suppose $m=s/k$ is an integer. Take $t=2\pi/k$. Then for each of the m sub-sets of k variates X_r, compute, with $r=0, 1, \cdots, k-1$, $C=\sum X_r \cos rt$, $S=\sum X_r \sin rt$, $U=C^2+S^2$. Let a be the second moment of the s variates about the mean, b the fourth moment. With $k>2$, take $A^2=kb+(k^2-3k)a$, $I=(\sum U-sa)/(Am^{1/2})$. Suitable modifications are possible for $k=2$, and for fractional m; and the I's may be combined for successive sets of s variates.

15. Professor C. C. Camp: *Note on unexpected errors in Barlow's tables and others.*

One should not expect to find an error in a list of errata. M. J. Perrott in writing of Steinhauser's twenty-place table says "log 1088 doit terminer par 26 et non par 23." Actually, 226 occurs after the nineteenth decimal and Perrott must have miscounted. In the 30th edition of Schron's table an inversion of the correct digits 42 occurs in his extended form of log 102238. Barlow's Tables as revised in 1839 were supposed to have errors of unity in the last digit of square and cube roots of some numbers beyond 1250. For numbers under 1250 De Morgan says that "an error of a unit in the last place can . . . . hardly exist . . . . , though it is barely possible in numbers not much below 1250." An error is found in the cube root of 80, also it was found that the cube roots of 12, 20 were too large and those of 9, 14, 52 too small by unity in the last digit as given, the actual errors being greater than that in three cases. Moreover, errors exist in nine related cube roots. Such tables, which are widely copied, should be made entirely free of error.

16. Mr. J. H. Roberts: *Certain types of atriodic continua.*

R. L. Moore has introduced the notion *triodic continuum* and shown that if G is an uncountable set of triodic continua in the plane, then G contains an uncountable subset G_1, such that every two continua of G_1 have a point in common. A continuum M is said to be triodic if it contains four distinct continua a, b, c and k, such that k is the common part of each two of the continua a, b and c. Professor Moore has raised the following question: Is it true that if M is a plane atriodic continuum, then there exists in the plane an uncountable set G of mutually exclusive continua such that each continuum of G can be thrown into M by a continuous one-to-one transformation of the plane into itself. This is shown to be true for the special case where, given any positive number ϵ, M can be covered by a simple chain of regions of diameter less than ϵ. An example is given of an *unbounded* atriodic continuum M for which no such set G exists.

17. Professor G.T.Whyburn: *A theorem on plane continua.*

In this paper the following theorem is proved. Suppose M is a plane continuum having the property that each of its subcontinua contains a continuum N such that there exists a positive number d such that every subcontinuum of N contains at least one point which is accessible from at least two complementary domains of M which are of diameter $>d$; then M is a continuous curve. A number of interesting corollaries of this theorem are given, among which are the following: (1) Under the same hypothesis as above, (a) every subcontinuum of M contains an arc which belongs to the common boundary of two of the complementary domains of M, and (b) M is a Menger regular curve. (2) If every subcontinuum of M contains a point which is accessible from two unbounded complementary domains of M, then every point of M is a cut point of M.

18. Professor G. T. Whyburn: *Some theorems on connected point sets.*

In this paper the following generalizations of theorems due to R. L. Wilder (Fundamenta Mathematicae, vol. 7, p 371), R. L. Moore (Proceedings of the National Academy, vol. 9, p. 102, Theorem B+), and Kuratowski and Zarankiewicz (this Bulletin, vol. 33, p. 574, Remark B) are given. (1) Condition (4) in the Moore-Wilder lemma characterizing continua which are not continuous curves and proved by Wilder only for bounded continua is shown to hold for unbounded continua. (2) No connected subset K of a connected point set M contains an uncountable collection of mutually exclusive point sets each of which cuts M but not K. (3) If S is any connected point set and Z is any collection of mutually exclusive connected subsets of S such that for each element X of Z, $M-X$ is neither connected nor the sum of two connected point sets, then Z is countable. (Kuratowski and Zarankiewicz imposed the unnecessary condition that each set of the collection Z is closed relative to S.) In proving (1) use was made of the following lemma: If N is a connected subset of a connected point set M and K is any component of $M-N$, then $M-K$ is connected.

19. Professor G. T. Whyburn: *Concerning collections of the cuttings of continua.*

In this paper the following results are established. Let M denote any plane continuum. (1) No plane continuum M contains an uncountable collection of mutually exclusive connected point sets each of which contains a proper subset which cuts M; hence if G is any collection of mutually exclusive connected subsets of M each of which contains a cutting of M, then all save possibly a countable number of the elements of G must be continua and must themselves be irreducible cuttings of M. (2) If M is bounded and G is any collection of mutually exclusive componentwise irreducible cuttings of M such that for each element g of G, $M-g$ is not the sum of two connected point sets, then G is countable. (3) If G is any collection of mutually exclusive bounded subcontinua of M each of which contains a cutting of M, then G contains a subcollection G^* which contains all but a countable number of the elements of G and which is an upper semi-continuous collection.

20. Professor C. A. Rupp: *On an extension of Pascal's theorem to a space of n dimensions.*

The edges of a simplex F meet a hyperquadric V in S_r in $r(r+1)$ points, which may be grouped to define a second simplex P. The coordinates of the intersections of corresponding faces of the simplexes are linearly dependent. This is shown to be equivalent to saying that the pair of simplexes F and P are a Schlafli pair, or that lines in the position of Schlafli are linearly dependent. Applying the theorem to circles gives a new construction for a Stephanos pentacycle, and shows that the circles of a pentacycle are linearly dependent. In line geometry there appears the concept of linear congruences in involution, and by means of six linear complexes in involution are defined six linear congruences such that any linear congruence in involution with five of them is also in involution with the sixth.

21. Mr. A. A. Albert: *A determination of all normal division algebras in sixteen units.*

It is known that every normal division algebra in 16 units is of rank 4 and contains an element satisfying an irreducible quartic in the reference field F. In this paper the quartic is taken in the reduced form and it is shown that either all of its roots are rational functions of one of them or the algebra contains an element satisfying the equation $\phi(\omega)=\omega^4+\alpha\omega^2$

$+\beta$, α, β in F, which is irreducible in F. It is then shown that either this equation has the property that all of its roots are rational functions of one of them or its group is G_8. The algebras containing an element satisfying an equation of this type with group G_8 are then determined, and hence all division algebras in sixteen units, since those algebras which contain an element satisfying an irreducible equation having the property that all of its roots are rational functions of one of them are the algebras of type Γ of L. E. Dickson and are known.

22. Professor W. A. Manning: *A theorem on simply transitive primitive groups.*

Let G_1 be a subgroup that fixes one letter of a simply transitive primitive permutation-group. Beginning in 1917, the author published a series of theorems concerning the structure of G_1 when one of its transitive constituents is doubly transitive. The most recently published theorem (1921) states that if G_1 has a doubly transitive constituent of degree m, either G_1 is a simple isomorphism between doubly transitive groups of degree m, or G_1 has a transitive constituent whose degree is a divisor ($>m$) of $m(m-1)$. His present theorem asserts that if G_1 has a doubly transitive constituent of degree m, G_1 has also a transitive constituent whose degree is a divisor ($>m$) of $m(m-1)$.

23. Professor L. E. Dickson: *The general Waring problem for quadratic functions.*

This paper reports on an elaborate investigation of the following difficult problem. Let the quadratic function $q(x)$ have a positive coefficient of x^2, be an integer for every integer $x\geqq 0$, and be negative for one or more integers $x\geqq 0$. There is found the least integer E such that every integer $\geqq L$ is a sum of E numbers chosen from 0 to 1 and four values $\geqq 0$ of $q(x)$ for integers $x\geqq 0$, where L denotes the least sum of four such values. In previous papers, the writer treated the case in which $q(x)$ is never negative when x is positive.

24. Mr. R. C. Shook: *Two extended Waring problems.*

The first problem considers the conditions that a form $f=a_1x_1^6+\cdots+a_nx_n^6$, $x_i\geqq 0$, shall satisfy in order that it may represent all integers. A minimum order $n=10$ is established and twenty-two forms of order 10 are shown to represent all integers up to 4,096. The second problem discusses the maximum gaps in a table of sums by four of (i) values of quadratic functions $f(x)$, having integral values $\geqq 0$ for $x\geqq 0$, and (ii) the positive values of quadratic functions $q(x)$ having integral values not all equal to or greater than zero for $x\geqq 0$.

25. Professor L. E. Dickson: *Positive binary quadratic forms B such that every positive integer is a sum of s values of B.*

Since B must represent unity, we may transform B into $f=x^2+gxy+hy^2$ where $g=0$ or 1, and $h>0$. Hence the case $s>3$ is trivial. For $s=2$ or 3, it is shown that every positive integer is a sum of s values of f if and only if $h=1$, 2, or 3 if $s=2$, and $h=1, \cdots, 7$ if $s=3$. The paper will appear in Journal de Mathématiques.

26. Professor L. E. Dickson: *Generalization of Waring's theorem on cubes.*

This paper contains a further consideration of the theorems announced in the American Mathematical Monthy, April, 1927.

27. Mr. K. C. Yang: *All positive integers are sums of* 9 *pyramidal numbers* $(x^3-x)/6$, $x=0, 1, 2, \cdots$.

E. Maillet, in the Bulletin de la Société de France, vol. 23, (1895), pp. 40-49, proved that every integer$\geqq 19272$ is a sum of at most 12 pyramidal numbers. By an elementary proof, it is here shown that nine suffice.

28. Dr. Loïs W. Griffiths: *Representations of integers in the form* $x^2+2y^2+3z^2+6w^2$.

The number of representations $2T(N)$ of a positive odd integer N in the form $x^2+2y^2+3z^2+6w^2$ is not known exactly. In this paper new conditions on, and new limits for, $T(N)$ are obtained in terms of the divisors of N. The determination of $T(N)$ for N odd and composite is reduced to that of $T(P)$, where P is in turn the distinct prime factors of N. The paper is to appear in the American Journal of Mathematics.

29. Mr. B. W. Jones: *Representation of integers by positive ternary quadratic forms.*

L. E. Dickson (Annals of Mathematics, (2), vol. 28 (1927), p. 333) made the following definition: "All the integers not represented by a *regular form* $f[=ax^2+by^2+cz^2$, where a, b and c are positive integers] coincide with all the positive integers given by certain arithmetical progressions," and proved that not more than seventeen forms f are regular, where $a=1$, b and c relatively prime and less than certain large integers. Applying his methods and extension of them, this paper proves first that not more than 103 forms f are regular when 1 is the greatest common divisor of a, b and c. Then, using methods of Dirichlet and Dickson, and certain extensions of these, together with elementary transformations and previously known results for certain ternaries, all but six of these forms are proved regular. (Partial results are known for these six.) Many quadratic ternary forms with cross products are similarly proved regular. Also numerous semi-regular forms are considered.

30. Mr. D. C. Morrow: *All quaternary quadratic forms which represent every positive integer.*

It is proved that all positive quaternary quadratic forms are equivalent to certain reduced forms. For forms which represent all positive integers, it is shown that the number of reduced forms is finite. The conditions on the coefficients, which are necessary in order that the reduced form shall represent every positive integer, are derived, and certain equivalent cases are eliminated. The paper then proceeds to determine all those reduced forms which represent every positive integer.

31. Dr. F. R. Bamforth: *A classification of boundary value problems for a system of ordinary differential equations of the second order.*

Boundary value problems of the form $y'(x)=(A(x)+\lambda B(x))y(x)$, $My(a)+Ny(b)=0$, are considered where λ is a parameter, capital letters denote square matrices of order two, and small letters vectors. It is shown that every boundary value problem of this form, where the matrix $B(x)$ satisfies certain hypotheses, can be reduced to one and only one of a set of four normal forms. This reduction can be made by means of a non-singular transformation of the form $y=U_n$. Necessary and sufficient conditions are found for a boundary value problem in one of these normal forms to be self-adjoint according to the definition of G. A. Bliss. It is pointed out which of the normal forms have been studied and a boundary value problem in one of these forms, hitherto not studied, is solved completely. Finally, a general boundary value problem is deduced from an isoperimetric problem in the calculus of variations.

32. Dr. L. E. Ward: *On third order boundary value and expansion problems.*

A classification of boundary value problems of the system consisting of the differential equation $u'''+\rho^3u=0$ and three boundary conditions linear and homogeneous in the values of u and its first two derivatives at two real points is given. The forms of the boundary conditions in one of the irregular cases are considered, and some of the properties of an infinite series of the characteristic functions arising from this case are derived.

33. Dr. L. E. Ward: *On the uniqueness of the coefficients in a certain expansion problem.*

In a paper published in the Transactions of this Society, vol. 29 (1927), the author showed that a certain irregular boundary value problem leads to characteristic functions in an infinite series in which any analytic function can be expanded, and that the formal series for the function converges uniformly to the function in the interior of a certain triangle. In this paper it is shown that zero can be expanded in such a series, uniformly convergent in the interior of the same triangle, and that therefore the coefficients in this type of expansion are not unique.

34. Mr. T. F. Cope: *An analogue of the Jacobi condition for the problem of Mayer with variable end points.*

In this paper the generalized Mayer problem formulated by G. A. Bliss (Transactions of this Society, vol. 19 (1918), p. 305) is considered. The second variation for this problem is computed and reduced to a particularly simple form. It is then shown that there is a boundary value problem associated with the second variation, from which a necessary condition for a solution of the original problem is deduced. This condition is, essentially, that for a minimizing arc for the original problem, the boundary value problem can have no solution for negative values of its parameter. Finally, the boundary value problem associated with the second variation is transformed into an equivalent problem, which is shown to be definitely self-adjoint according to the definition of G. A. Bliss (Transactions of this Society, vol. 28 (1926), p. 570).

35. Dr. W. J. Trjitzinsky: *A class of quasi-analytic functions.*

In the field of quasi-analytic functions (they are non-analytic, but determined by their initial values at a point) some of the unsolved questions are: (1) construction of such functions, (2) determination of conditions under which an assigned set of initial values defines such a function. In this paper it is shown that functions of the form $\sum a_n/(x-c_n)$ are quasi-analytic in the real interval $(-1, +1)$, if $\sum_n |a_n| \log_k{}^{n+1} n$ $(k \geqq 2)$ converges and c_n is an enumerable set of points whose limiting points are everywhere dense on $(-1, +1)$ and whose distances from the real axis approach zero not faster than $1/\log_k n$. Using this result, sufficient conditions are found for the existence of quasi-analytic functions when a set of initial values is assigned.

36. Professor C. N. Moore: *On Gibbs' phenomenon for the developments in Bessel's functions.*

In a paper of 1911 (Transactions of this Society, vol. 12, pp. 181-206) the author developed an asymptotic formula for the coefficients of the developments in Bessel's functions which made it possible to infer facts concerning the convergence of these developments in the neighborhood of the origin which have not as yet been obtained in any other manner. In the present paper it is shown that the nature of the Gibbs' phenomenon in the neighborhood of the origin for these developments can be readily deduced from the same asymptotic formula.

37. Professor Dunham Jackson: *A note on closest approximation.*

This paper is concerned with a variation of the problem of closest approximation, which may be formulated as follows: Let $f(x)$ be a given continuous function of period 2π. Let h be a given positive number—less than or greater than 2π. (The value $h=2\pi$ is not excluded of necessity, but because the results in this case are already known.) For a given trigonometric sum $t_n(x)$, of the nth order, let $I(t_n)$ be the maximum of the integral of $[f(x)-t_n(x)]^2$ over an interval of length h, as the initial point of the interval is allowed to vary. Among all trigonometric sums of the nth order, let $T_n(x)$ be the one for which I has the smallest possible value. The minimum problem has a unique solution, and the minimizing sum $T_n(x)$ converges uniformly toward $f(x)$ as n becomes infinite, if $f(x)$ satisfies suitable hypotheses (for example, if $f(x)$ satisfies a Lipschitz condition). The problem can be greatly generalized without increasing the difficulty of the treatment.

38. Professor J. A. Shohat: *On the convergence of mechanical quadratures on an infinite interval.*

A function $\Psi(x)$ defined on an interval (a, b), finite or infinite non-decreasing, with infinitely many points of increase, gives rise to a system of orthogonal Tchebycheff polynomials $\{\phi_n(x)\}$ of degree n $(=0, 1, 2, \cdots)$ having each roots (x_i) real, distinct and between a, b. We then construct the formula of "Mechanical Quadratures"

$$\int_a^b f(x)d\Psi(x) = \sum_{i=1}^{n} H_i f(x_i) + R_n(f), \tag{1}$$

where $R_n(f) \equiv 0$, if $f(x)$ is a polynomial of degree $< 2n$. In the present paper the author investigates the convergence of formula (1), that is, for what classes of functions $f(x)$ is $\lim_{n\to\infty} R_n(f) = 0$. A complete solution is given in the most important case: $\Psi(x)$ is continuous in (a, b), which includes that of $\int_a^b f(x)p(x)dx$, as a particular case. The case (a, b) finite is herein included, thus generalizing a result due to Stieltjes. Applications are given to Tchebycheff polynomials, also to the theory of probability. Part of these results appeared in a note in the Comptes Rendus, Feb., 1928.

39. Professor J. A. Shohat: *On the convergence of the expansion of continuous functions in series of Sturm-Liouville functions.*

Consider the expansion $f(x) = \sum_{n=1}^{\infty} A_n U_n(x)$, $\left(A_n = \int_a^b f(x)\, U_n(x)dx\right)$, where $U_n(x)$ denotes the solution of the reduced Sturm-Liouville differential equation with the known boundary conditions. The author shows that the the convergence of the expansion (1) can be settled immediately, by means of the most elementary propositions of integral calculus, in case $f(x)$ satisfies a Lipschitz condition. In many cases we thus readily derive the order (with respect to n) of A_n and of the remainder in (1). The method is applicable to differential equations of higher order, also to many other expansions in series of orthogonal functions.

40. Mr. W. C. Risselman: *On the approximate representation of a given function by means of polynomials in another given function.*

The problem is that of approximating to a given function $f(x)$ by means of a polynomial P_n of given degree in a function $\phi(x)$ so as to mimimize $\int |f(x) - P_n[\phi(x)]|^m dx$, the integral being taken between finite limits (a, b). On the assumption that the inverse of $\phi(x)$ is single-valued and under certain further hypotheses, which are less restrictive in case $m = 2$ than for other values of m, uniform convergence of the sequence $\{P_n[\phi(x)]\}$ to $f(x)$ is proved. When the inverse of $\phi(x)$ is multiple-valued, there is convergence to a suitably defined mean of the values of $f(x)$ corresponding to a single value of $\phi(x)$. The method of proof is the same in principle as one which has been applied recently in discussing the particular case $\phi(x)\equiv x$.

41. Mr. J. M. Earl: *On the convergence of polynomials of approximation on an infinite interval.*

This paper deals with the approximation to a given function $f(x)$ by means of the polynomial $P_n(x)$ of degree n or less determined so as to minimize $\gamma(n)$, the integral from $-\infty$ to $+\infty$ of the mth power of the absolute value of the error multipled by a non-negative weight function $\rho(x)$. It represents therefore a broad generalization of the problem of Hermite series, corresponding to the special case in which $\rho(x)=e^{-x^2}$ and $m=2$. Under suitable restrictions on $f(x)$ and $\rho(x)$, it is shown that $P_n(x)$ converges uniformly to $f(x)$ on any finite interval on which $\rho(x)$ has a positive lower bound. The proof is deduced from theorems on polynomial approximation leading to the calculation of an upper bound for the integral $\gamma(n)$. A similar discussion for the interval 0 to ∞ can be made to depend on the above.

42. Dr. Elizabeth Carlson: *A simplified proof for the extension of Bernstein's theorem to Sturm-Liouville sums.*

By using a method similar to one used by de la Vallée Poussin in proving the corresponding theorem for trigonometric sums, the author has proved (Transactions of this Society, vol. 26) the following theorem: The maximum of the absolute value of the derivative of a Sturm-Liouville sum of order $n(n\geqq 1)$ can not exceed npM, where M is the maximum of the absolute value of the sum itself, and p is independent of n and of the coefficients in the sum. In this paper, a simpler proof is given by using a well known asymptotic expression for the characteristic solution $v_k(x)$ of a Sturm-Liouville system, in the form $v_k(x)=\cos kx+(1/k)\beta(x)\sin kx+(1/k^2)\propto(xk)$.

43. Mr. T. S. Peterson: *A class of invariant functionals of quadratic functional forms.*

In his paper *Invariant functionals of functional forms* (abstract in this Bulletin, vol. 34 (1928), pp. 8–9), A. D. Michal considered the law of transformation of a few important invariant functionals of a quadratic functional form. In particular, there were considered the Fredholm determinant and the resolvent kernel of the functional coefficients of the form. The present paper makes a detailed study of the functional invariants arising out of the Fredholm minors of the Fredholm determinant. Use is made of a theorem of Platrier to obtain the law of transformation of these invariants.

44. Professor R. L. Moore: *Concerning upper semi-continuous collections.*

A bounded continuous curve M will be called a *cactoid* (*opuntioid*) if every non-degenerate maximal cyclic subset (see. G. T. Whyburn, Proceedings of the National Academy, vol. 13, pp. 31-38) of M is a simple closed surface and no point of M lies in a bounded complementary domain of any subset of M. A cactoid M is said to be *aspiculate* if it does not contain any arc t such that no point of t except its end points is a limit point of $M-t$. An aspiculate cactoid M is *simple* if no two cut points of M are separated in M by infinitely many different points. If G is an upper semi-continuous collection of mutually exclusive continua filling up a spherical surface S then the space whose elements are the continua of G is, in the sense indicated in the author's paper *Concerning upper semi-continuous collections of continua* (Transactions of this Society, vol. 27, pp. 416–428), topologically equivalent to a cactoid; and, conversely, every cactoid bears this relationship to some G and S. This theorem remains true if "cactoid" is replaced by "simple aspiculate cactoid" and the condition is imposed that the non-degenerate continua of G form a contracting sequence.

45. Professor K. P. Williams: *The symbolic development of the disturbing function.*

The paper gives a modification of Newcomb's symbolic development of the disturbing function. A set of fundamental operators are isolated and lead directly to Newcomb's operators. The paper will appear in the American Journal of Mathematics.

46. Professor K. P. Williams: *A comment on certain equations in the theory of radiative equilibrium.*

In Eddington's discussion of the internal constitution of a star a set of equations occurs, which determine the coefficients in an expansion in terms of Legendre polynomials. In this paper it is shown that the equations do not warrant the conclusion that Eddington draws from a hasty examination. An integrable case is assumed, and the results confirm the conclusion of the general discussion. The paper appeared in the May number of the Astrophysical Journal.

47. Professor L. M. Graves: *The second variation for discontinuous solutions in the calculus of variations.*

In a paper presented to the Society (Sept., 1927) the author treated the extension of the Jacobi-Carathéodory conditions for discontinuous solutions to space problems, and gave a new formulation and new proof of the Carathéodory conditions which must hold at the corners. The present paper treats the same condition by means of the second variation. A new definition of conjugate point is formulated to include all cases, and the Jacobi-Carathéodory condition is that no minimizing arc can contain a pair of conjugate points. This method takes care of the exceptional cases of the previous methods, and proves also that no minimizing arc can contain a cusp. The equivalence of this formulation of the condition with the earlier one is shown.

48. Mr. E. R. C. Miles: *Fredholm solution of a generalized Neumann problem in the plane.*

Consider the class of functions expressible as potentials of a single layer, $V(M)=\int\log(1/MP)d\mu(s_p)$ where $\mu(s_p)$ is of limited variation on the curve C. The normal derivative of $u(m)$ is discontinuous across C, but wherever μ' exists, formulas analogous to the usual ones, but in terms of Stieltjes integrals, are obtained, if C has a curvature at the point. The generalized Neumann problem states that $(1+\lambda)/(2\lambda)$ times the total flux across a curve which approaches an arc AB of C uniformly from inside minus $(1-\lambda)2\lambda$ times the corresponding outside limit shall be a given function $g(B)-g(A)$, of limited variation and with regular discontinuities on C. This problem has a unique solution provided λ is not one of a set of characteristic values. The value $\lambda=-1$ yields a Neumann problem for the exterior region, and is not a characteristic value; $\lambda=+1$ yields a Neumann problem for the interior region, and is a characteristic value. In the latter case a necessary and sufficient condition for a solution is that $\int_C dg(s)=g(C)=0$, and the solution is determined except for an arbitrary constant.

49. Mr. E. R. C. Miles: *Fredholm solution of a generalized Dirichlet problem in the plane.*

Analogous results to those of the previous paper, with reference to the generalized Dirichlet problem, are obtained for the class of functions expressible as potentials of a double layer, $u(m)=\int[(\cos nr)/r]dv(s_p)$. Both problems are handled by solving Stieltjes integral equations, and include the usual treatments as particular cases. One interest in these problems is the fact that the methods used are extensible to the three-dimensional situation, as will be shown in a paper which the author is writing in conjunction with Professor Evans.

M. H. INGRAHAM,
Associate Secretary.

THE APRIL MEETING IN NEW YORK

The two hundred sixtieth regular meeting of the Society was held at Columbia University on Friday and Saturday, April 6 and 7, 1928. The attendance included several distinguished guests (Professors Paul Alexandroff, Moscow; C. Carathéodory, Munich; and Heinz Hopf, Berlin), as well as the following one hundred twenty-four members of the Society:

C. R. Adams, R. B. Adams, Alexander, R. G. Archibald, J. W. Arnold, A. A. Bennett, Benton, William James Berry, William Johnston Berry, Blumberg, Blumenthal, Bowden, B. H. Camp, W. B. Campbell, Caris, Carrié, E. T. Carroll, Cheney, Chittenden, Alonzo Church, Coble, Abraham Cohen, L. W. Cohen, L. P. Copeland, Currier, Curry, Dantzig, Derby, Dorweiler, Dresden, Esty, Fiske, Fite, Fithian, D. A. Flanders, Frink, Gehman, Gill, Gronwall, Hall, Robert Henderson, Hickson, L. S, Hill, Hille, Himwich, Hofmann, Hollcroft, Hotelling, Huber, Jeffery. Joffe, M. I. Johnson, R. A. Johnson, Johnston, Kasner, Keller, Kline, Knebelman, Mark Kormes, Langman, Lefschetz, Lepeshkin, Littauer, Lubin, MacColl, MacInnes, H. A. Merrill, Merriman, H. H. Mitchell, T. W. Moore, Richard Morris, D. S. Morse, Marston Morse, Moyle, Murnaghan, F. H. Murray, Musselman, Neelley, C. A. Nelson, Olds, Ore, F. W. Owens, H. B. Owens, Pepper, Pfeiffer, Phalen, Pixley, Poritsky, Post, R. G. Putnam, Raudenbush, Reddick, C. N. Reynolds, R. G. D. Richardson, Ritt, Roos, Rutt, Seely, Serghiesco, Sheffer, Siceloff, Simons, Singer, Slotnick, Smail, C. E. Smith, P. A. Smith, W. M. Smith, Sosnow, Swingle, Tamarkin, J. M. Thomas, T. Y. Thomas, Veblen, Weisner, H. S. White, W. M. Whyburn, Wiener, R. L. Wilder, W. A. Wilson, Mabel M. Young, Zariski, Zippin.

On recommendation of the Council, the Society adopted amendments to the By-Laws by which the editorial committee of the Colloquium Publications become members of the Council of the Society, on the same basis as the editorial committees of the Bulletin and the Transactions, and as the Society's representatives on the editorial committee of the American Journal of Mathematics.

The afternoon sessions on Friday and Saturday were devoted to a Symposium on Analysis Situs; they were presided over by Ex-President Oswald Veblen, relieved by Professor H. H. Mitchell at the opening of the Friday session.

The following papers were presented: Friday, April 6: *Three-dimensional manifolds; generalized Riemann surfaces and knots*, by Professor J. W. Alexander; *The critical points of functions and the calculus of variations in the large*, by Professor Marston Morse; *Applications of analysis situs to algebraic surfaces, and the classification of curves on surfaces*, by Professor Solomon Lefschetz; Saturday, April 7: *Abstract sets from the point of view of analysis situs*, by Professor E. W. Chittenden; *The notion of curve and the problems that it presents*, by Professor J. R. Kline; *Application of analysis situs to the general theory of topological spaces*, by Professor P. S. Alexandroff. The discussion was opened on Friday by Professor Heinz Hopf, and on Saturday by Professor H. M. Gehman; on both days it was carried on for a considerable time and in a lively manner by several participants, among them Professor Carathéodory. The success of this symposium fully justifies the belief that there is a gain in devoting a part of each meeting to a more concentrated field of interest than can be supplied by the usual programs of short papers.

The Friday and Saturday morning sessions were sectional: Friday morning, Section A, Analysis, Professor A. A. Bennett presiding; and Section B, Geometry, Professors T. R. Hollcroft and F. W. Owens presiding; Saturday morning, Section A, Analysis Situs and Point Sets, Professor J. R. Kline presiding, and Section B, Algebra and Applied Mathematics, Professor A. B. Coble presiding. Titles and abstracts of the papers read at the sectional sessions follow below. The papers numbered 1 to 11 were read before the Section of Analysis; Nos. 12–19 before the Section of Geometry; Nos. 20–34 before the Section of Analysis Situs and Point Sets; Nos. 35–44 before the Section of Algebra and Applied Mathematics. The papers of Adams (second paper), Ayres, Blumberg (second paper), Garver, Graustein, Gronwall, Hurwitz, Miller, Moore (second paper), Stetson, Swingle, and Whyburn were read by title. Mr. Kaplan was introduced by Professor F. D. Murnaghan, and Mr. Miller by Professor R. L. Wilder,

1. Mr. A. O. Hickson: *An application of the calculus of variations to boundary value problems.*

This paper considers the boundary value problem which arises when one attempts to minimize the second variation $I_2(\eta)$ of the problem of the calculus of variations for a space of $n+1$ dimensions. A certain succession of classes of admissible arcs is chosen and the problems of minimizing $I_2(\eta)$ in these classes are considered. For each minimizing problem there is obtained the Euler-Lagrange system of ordinary linear differential equations of the second order containing a parameter λ as a multiplier. In each class of arcs $I_2(\eta)$ has a minimum value, and these values form an increasing sequence $\lambda_0, \lambda_1, \cdots, \lambda_i, \cdots$. Also in each class there exists a set of linearly independent arcs, which give $I_2(\eta)$ its minimum value in the class considered, and which satisfy the corresponding boundary value problem with the parameter λ equal to this minimum of $I_2(\eta)$. The results are obtained by means of well known theorems from the calculus of variations and the theory of ordinary linear differential equations.

2. Dr. G. M. Merriman (National Research Fellow): *On the expansion of functions in terms of orthogonal polynomials.* Preliminary report.

It is shown here, as an initial result in the field indicated in the title, that a function $f(x,y)$, harmonic in the interior of a closed, rectifiable Jordan curve and continuous in the closed region consisting of the curve and its interior, can be expanded in terms of a set of harmonic polynomials, normalized and orthogonalized with respect to the curve, the expansion in series being uniformly convergent interior to the curve. A set of such polynomials, which depend only on the curve, is exhibited.

3. Dr. C. F. Roos (National Research Fellow): *Solutions of limited variation of Volterra integral equations of the first kind.*

Volterra has found a continuous solution of the Volterra integral equation of the first kind, $v(x)=\int_a^x K(x,s)u(s)ds$, by reducing the equation to one of the second kind by a differentiation. W. H. Young has extended this theory to include Lebesgue integrals and has obtained a bounded solution. Both Volterra and Young have employed a differentiation to reduce the equation to a Volterra equation of the second kind. In the present paper the author shows that an integration by parts can be used to reduce the equation to the type $v(x)=u(x)\int_a^x K(x,t)dt-\int_a^x[\int_a^x K(x,t)dt]du(s)$; this integration by parts only requires that the kernel $K(x,s)$ be integrable in s and of limited variation in x for all s and that $u(s)$ be of limited variation. Under these hypotheses and the added hypothesis that $v(x)$ be of limited variation, it is possible to obtain a solution of limited variation in the form resulting from the integration by parts.

4. Mr. H. H. Pixley: *Discontinuous solutions in the problem of depreciation and replacement.*

In a recent paper on the calculus of variations, Dr. C. F. Roos has treated a Lagrange problem with variable end points and discontinuous integrand to enable him to develop his new dynamical theory of depreciation and replacement. He assumes that the price-time and rate of production-time curves are continuous at the time of replacement. In an actual problem this would probably not be true. In the present paper a general theory corresponding to that of Roos without the assumption of continuity at the time of replacement is developed. In particular a special problem is discussed in which the second machine is started at a time and rate different from those at which the first machine stops.

5. Professor C. R. Adams: *On the linear ordinary q-difference equation.*

The equation considered is (1): $\sum_{i=0}^{n} a_i(x) f(q^{n-i}x)$, in which the known functions $a_i(x)$ are analytic or have poles at $x=0$, q is a constant $\neq 0$, and $f(x)$ is to be determined. Without loss of generality we may assume $a_i(x)=a_{i0}+a_{i1}x+\cdots$ (for $|x|<R$), with at least one of the a_{i0} not zero. These a_{i0} are the coefficients of the characteristic equations of (1). We exhibit formal power series satisfying (1) in all possible cases. When the roots of the characteristic equation are finite and not zero, and $|g|\neq 1$, we prove directly that these solutions have positive radii of convergence; their existence and convergence can be inferred from the functional equations derived by Grévy (Paris thesis, 1894; also Annales de l'Ecole Normale, (3), vol. 11 (1894), pp. 249-323), but the present treatment possesses advantages of directness, simplicity, and brevity. When the characteristic equation has infinite or zero roots or both and $|g|\neq 1$, some (never all) of the formal series may converge only for $x=0$. At $x=\infty$ the situation is similar if the $a_i(x)$ are analytic or have poles there. The non-homogeneous equation is also treated, with similar results.

6. Professor C. R. Adams: *Note on factorial series in two variables.*

This paper appears in full in the present issue of this Bulletin.

7. Dr. Stefan Serghiesco: *On a certain double integral.*

In an earlier paper, the author has introduced a new general formula giving the number of roots of a system of equations which contains differential invariants and is represented, in the case of two equations, by a difference of two integrals, a simple and a double integral. An abstract of this paper has appeared in the Comptes Rendus for January 23, 1928. In the present paper the author makes a further study of this problem and introduces a double integral which by itself gives the number of roots of the system. This integral is then extended to n equations. The treatment, through differential invariants, of this problem is thus widely generalized for the case of simple common roots of a system of equations.

8. Professors Einar Hille and J. D. Tamarkin: *Contributions to the theory of integral equations.* Preliminary communication.

Assuming the kernel to be integrable together with its square, the authors apply the theory of linear equations in infinitely many unknowns to a study of the characteristic values of a linear integral equation. These values are obtained as the zeros of an infinite determinant which is frequently easier to handle than the (modified) Fredholm determinant. Estimates are obtained for the rate of growth of the determinant and of the characteristic values in the general case as well as in various special cases, e.g., when the kernel is analytic in one or both variables or possesses a certain number of partial derivatives or satisfies a Lipschitz condition, etc. Extensive use is made of the theory of trigonometric and polynomial approximation in two dimensions.

9. Professor Harold Hotelling: *Stability in competition.*

Cournot's solution of the problem of duopoly was criticized by J. Bertrand and F. Y. Edgeworth on the ground that his "equilibrium" is unstable. It does not seen to have been recognized, however, by any writer on the subject that this instability is due to the tacit assumption that the competitors are equally accessible and their products are of equal utility, not merely to the market as a whole but to every consumer. Discarding the last condition, which is artificial, a simple example typical of actual conditions is set up and its stability is demonstrated. By means of the same example there become evident a number of qualitative properties of competitive situations generally overlooked. Among these are are some which have a bearing on the "cement trust" controversy.

10. Professor Norbert Wiener: *Coherency matrices.*

The theory of harmonic analysis already developed by the author is extended to a set of several simultaneous functions, and yields an Hermitian function of the frequency, the coherency matrix. The problem, for example, of characterizing light with respect to its state of polarization may be reduced to that of transforming a "coherency matrix" to diagonal form by a Weyl "Unitärtransformation." The theory has important applications to quantum theory and the Heisenberg matrices.

11. Dr. T. H. Gronwall: *On the convergence region of a power series in several variables.*

Fabry has shown that the region of convergence of a power series in x and y becomes a convex region when $\log|x|$ and $\log|x|$ are used as coordinates, and that this necessary condition on a convergence region is also sufficient. Hartogs and Faber have given simplified proofs, and the latter has extended the theorem to n variables. The present note gives a proof (for n variables) which is considerably simpler than the previous ones.

12. Professor J. H. Neelley: *Concerning covariants of the rational plane quartic curve with compound singularities.* First paper.

This paper discusses covariant forms of the rational plane quartic curve with ramphoid cusp. Some covariant curves degenerate, and groups have a common characteristic. The curve R_2^4 is taken parametrically as well as in ternary form. Some forms, which vanish identically for R_2^4 with this compound singularity and with other singularities, are considered. Some facts previously derived from invariants are verified by covariants.

13. Professor T. R. Hollcroft: *The postulation of multiple curves and surfaces on varieties in i dimensions.*

Numerical formulas are obtained for the postulation of a multiple line and of a multiple plane on a variety of given order in i dimensions. That for a multiple line is a generalization of Noether's formula for the postulation of a multiple line on a surface. There is also found the reduction in postulation due to an intersection of two multiple lines (planes) when both lines (planes) are of the same or different multiplicities on a variety of given order in i dimensions. Numerical formulas for the postulation of any given curve or surface of given multiplicity on a variety of given order in i dimensions are derived from the above. The postulation of any given one- or two-dimensional manifold of given multiplicity on a variety of given order is similarly obtained.

14. Professor H. R. Phalen: *Metric properties of the quadric of Moutard.*

Under the assumption that a certain surface S is non-developable. this paper defines a quadric of Moutard, derives its equation, and obtains the discriminant and the coefficients of the discriminating cubic, the coordinates of the center, and the mean and total curvature of S and of the Moutard quadric at the origin, and enunciates the following theorems: (a) At a given point P upon S the Moutard quadric and the surface S have the same total curvature, the same mean curvature, and the same surface normal. (b) The center of every Moutard quadric at a point P upon S lies outside the tangent plane to S at P. (c) At a point P upon S, twelve tangents may be drawn such that the corresponding Moutard quadric is non-central. (d) If the surface S be defined by an equation of the form $z=f(x,y)$ in which no term of the right member is of degree less than the third, the Moutard quadric degenerates into the tangent plane counted twice. (e) If in the equation of the Moutard quadric the term in z^2 is absent, the quadric is central. In conclusion, criteria are deduced for classifying the various quadrics.

15. Professor Edward Kasner: *Characterization of geodesic families.*

It is shown that just as natural families may be characterized by integrability with reference to orthogonality (the author's converse of the Thomson-Tait theorem, 1909), so geodesic families of curves may be characterized geometrically by integrability with reference to any polarity (the base cone of the polarity may vary arbitrarily from point to point). Polar transversalities are the only projective correspondences which can serve as transversality relations (see these Transactions, July, 1928). The results are valid in space of three or more dimensions.

16. Professor Edward Kasner: *The second derivative of a polygenic function.*

The second derivative $\sigma = d^2w/dz^2$ of an arbitrary polygenic function $w = \phi(x,y) + i\psi(x,y)$ depends not only on the point $z = x + iy$ but also on the slope and curvature of the path of approach, that is, $\sigma = \Omega(x,y,y',y'')$. For a given y' the locus of σ in its complex plane is a straight line. As y' varies this line envelopes a cardioid. If $y''=0$ (rectilinear second derivative) the locus of σ is a limaçon. For a complete representation it is necessary to parametrize these loci (clocked cardioid, analogous to clocked circle, see Science, December 16, 1927). Higher derivatives are also discussed. This paper has been offered to the Transactions.

17. Professor W. C. Graustein: *On the average number of sides of polygons of a net.*

The leading theorem in this paper, stated in its simplest form, reads as follows: If in a net of regular hexagons covering the plane a finite simply connected set of hexagons is removed and the region thus vacated is covered by polygons in such a way that the number of sides issuing from each vertex in the net remains three, then the average number of sides, per polygon, of the polygons in the net remains six. Similar theorems hold for nets of squares and equilateral triangles.

18. Professor W. C. Graustein: *Note on a certain type of parabola.*

The author discusses certain striking properties of a parabola which is tangent to the line at infinity in a circular point.

19. Dr. J. M. Stetson: *Theorems of permutability.*

It seems to have escaped notice that a part of the classic theory of Levy transformations may be restated thus: the transformations of Laplace and Levy and the inverse of the transformations of Levy are permutable. From this follows the fundamental theorem: Any type of transformation made by combining these three is permutable, and is permutable with any other such type. This general theorem appears to include as special cases all known theorems regarding permutability of transformations of conjugate nets in which no restrictions are made as to the character of the nets, if we note that either radial transformations, or parallel transformations, but not both, may be added to the list of transformations given in the theorem. We obtain a simple proof of the permutability of transformations F, and of the transformations giving derived nets (and their inverses), a fact which seems to be new except for the special case of reciprocal derived nets.

20. Professor R. L. Wilder: *On connected and regular point sets.*

This paper will appear in full in an early issue of this Bulletin.

21. Professor R. L. Wilder: *Concerning irreducibly connected sets and irreducible regular connexes.*

If the point set M be irreducibly connected about a set K but not about any proper subset of K, K is called a basic set about which M is irreducibly connected. Among the results obtained are the following: (1) If K is a basic set about which M is irreducibly connected, then (a) K is identical with the set of non-cut points of M and (b) if K consists of n points, the number of cut points of M of order greater than one is at most $n-2$. (A point P is called a cut point of order n if $M-P$ is the sum of n mutually separated sets.) Furthermore, if P is a point of M such that $M-P$ is the sum of n components $M(i=1, 2, \cdots, n)$, and if $M_i \times K = K_i$, then the set $K_i + P$ is a basic set about which $M_i + P$ is irreducibly connected. (2) If M is closed and bounded, then (a) M contains a basic set about which it is irreducibly connected; (b) M contains at least two non-cut points. Special attention is paid to the case where K consists of a finite number of points as well as to the case where M is regular.

22. Professor R. L. Wilder: *Concerning the relation of sets $F_{\sigma\epsilon}$ to their complements in euclidean space of n dimensions.*

We denote by $F_{\sigma\epsilon}$ a set M which is the sum of a denumerable collection of mutually exclusive, closed and bounded sets $M_1, M_2, \cdots$, whose diameters converge to zero as n tends to infinity. Among the results obtained are the following: (1) If D is a connected domain in euclidean space E_n such that no set M_n contains points of both D and $E_n - D$, and if no set M_n separates E_n, then $D - M \times D$ is a properly non-vacuous, arcwise connected point set. (2) Let A and B be points of a bounded connected domain D in E_n such that M is a subset of D and no set M_n separates A from B in D. Then $D-M$ contains an arc from A to B. (3) If M lies in E_2, no M_n separates E_2, K is a component of a set M_n, and e is any positive number, there exists a simple closed curve J such that every point of J is at a distance from K less then e and such that J and M have no point in common.

23. Mr. L. W. Cohen: *On sets homeomorphic with linear sets.*

Definitions of closed arcs, half open arcs, and open arcs are stated for a separable metric space, and such sets are shown to be homeomorphic respectively with closed, half open, and open intervals on the linear continuum. The following theorem is proved: Necessary and sufficient conditions that a set M in a separable metric space be homeomorphic with a linear set are that (1) the components of M are points of arcs; (2) if p is a point component of M, then $\dim_p M = 0$; (3) if a is an end point of an arc component T in M, then $\dim_a (M - T + a) = 0$; (4) if p is a point of an arc component T distinct from its end points, then p is not a limit point of $M - T$. In this theorem an arc is either closed, half open, or open.

24. Dr. P. A. Smith: *The regular components of a surface transformation.*

Regular components are maximal connected regions of "uniform approach" toward certain invariant limit sets on the surface. These components have been studied by Birkhoff and Smith in a paper to appear in the Journal de Mathématiques, but only on surfaces of genus zero. The results are here extended to the general case.

25. Professor E. W. Chittenden: *On covering theorems in general topological space.*

A family F of sets G of elements of a class P is said to be a *covering* of a subset E of P and to be of *type* T in case each element p of the set E belongs to a set G of F and is also in the relation T to some set G of F. The undefined relation T is assumed to be definite between elements and subsets of P. For example, in a topological space the relation T may be defined to mean that the element p is interior to G. A covering F is called reducible provided it contains a covering of the same type and of lower cardinal number. The covering theorems of Borel, Lebesgue, and Lindelof, together with certain extensions suggested by Alexandroff, Urysohn, and Hildebrandt, are considered in relation to this concept of reducibility. In particular, the paper contains sets of necessary and sufficient conditions for the reducibility of every covering F(of a set E) which is of type T. The results obtained are applied to the theory of topological spaces in general.

26. Dr. P. A. Smith: *Transformations of the boundary of a simply connected invariant region.*

This paper is a report on certain questions which arise in a topological study of the structure of surface transformations.

27. Professor Henry Blumberg: *On various metric properties of general sets and functions.*

In this paper the following theorems are proved: If S is a point set in euclidean n-space, it may be uniquely represented (sets of measure zero regarded as negligible) in the form $S=M+N$, where M is measurable and N is such that at every point of N the exterior metric density of N and also that of the complement of N is equal to 1. If A and B are any two sets whatever, overlapping or not, the exterior metric density of A is 0 or 1 at every point of B, with the possible exception of a set of measure zero. If $f(x)=f(x_1, \cdots, x_n)$ is an arbitrary real function and $u_m(x)$, $l_m(x)$ represent the upper and lower metric limits of f at x, that is, the largest and smallest values of η for which $(x, \eta)=(x_1, x_2, \cdots, x_n, \eta)$ is "approached" but not "infinitesimally approached," in the metrical sense, by the surface $y=f(x)$, then $u_m(x)$, $l_m(x)$ are "metrically upper-semi-continuous," and, moreover, approximately continuous, in the sense of Denjoy, almost everywhere; hence $u_m(x)$ and $l_m(x)$ are measurable. Therefore, given any function $f(x)$, there exist two measurable functions $m_1(x)$ and $m_2(x)$ such that

$m_1(x) \geqq f(x) \geqq m_2(x)$ almost everywhere, and the metrical approach of $y=f(x)$ at $(x, m_1(x))$ as well as at $(x, m_2(x))$ is "full," in the sense of metric density.

28. Professor Henry Blumberg: *Note concerning the inductive principle.*

It seems not to have been remarked that we may formulate the inductive principle for all linearly ordered sets as follows, and thereby comprehend in our formulation the particular applications to the normally ordered sets and to the linear continuum: If the property P holds for all the elements of some segment S of a linearly ordered set A, and if the validity of P for all the elements of a segment S of A implies its validity for all the elements of some proper supersegment S' of A, then P holds for all the elements of A. The principle is extensible to multiply ordered sets.

29. Professor H. M. Gehman: *Concerning certain types of non-cut points, with an application to continuous curves.*

A point P of a continuum M is called a *point of type* 1, if given any positive number ϵ, there exists an M-domain containing P of diameter less than ϵ whose complement is connected. Similarly, P is a *point of type* 2, if there exists an open set in M containing P of diameter less than ϵ whose complement is connected. A point of type 1 is a point of type 2, and a point of type 2 is a cut point, but not conversely, in either case. Certain properties of points of types 1 and 2 are proved in this paper, and an application is found in the proof of the following theorem: If M is a continuous curve (bounded or unbounded) in space of any finite number of dimensions, and if P is a non-cut point of M, then given any positive number ϵ, there exists an M-domain containing P of diameter less than ϵ whose complement is connected; if P is a cut point of M, then given any positive number ϵ, there exists an M-domain containing P of diameter less than ϵ whose complement consists of a *finite* number of maximal connected sets.

30. Professor G. T. Whyburn: *Concerning certain accessible points of plane continua.*

Let K denote the set of all those points of a plane continuum M each of which is accessible from at least two complementary domains of M. Then (1) if H is any uncountable subset of K, there exist points X and Y of H, an uncountable subset E of H, complementary domains R_1 and R_2 of M, and two continua L and N such that (a) $L \cdot N = X + Y$, (b) $L + N = M$, (c) every point of E is a cut point of N and is accessible from both R_1 and R_2; (2) all, save possibly a countable number, of the points of K are points of Menger order two of M; (3) if G is the collection of all the finite subsets of M each of which is an irreducible cutting of M, and T is the sum of all the point sets of the collection G, then all, save possibly a countable number, of the points of T are points of Menger order two of M. An interesting generalization of (1) is given which can be applied in the study of the relation between a subcontinuum of a plane continuous curve and its complement in that curve.

31. Professor G. T. Whyburn: *A generalized notion of accessibility.*

A continuum K will be said to be accessible from a point set R provided that if A is any point of R, a bounded continuum exists which contains both A and K and is a subset of $R+K$. The following results are established. (1) If R_1, R_2, and R_3 are mutually exclusive connected point sets in a plane S, there do not exist, in $S-(R_1+R_2+R_3)$, three mutually exclusive continua each of whch is accessible from both R_1 and R_2 and contains a limit point of R_3. (2) Every component of the M-boundary of every connected open subset R of a continuous curve M (in n dimensions) is *regularly accessible* from R. (3) For open subsets of a continuous curve, accessibility by continua as above defined is equivalent to accessibility by a special kind of a continuum, which is an "ideal arc" all of whose elements are points except possibly one. (4) The following properties are equivalent for continua M (in n dimensions): (α) every subcontinuum of M is a continuous curve, (β) every connected open subset of M has property S, and (γ) every point of the M-boundary of every connected open subset R of M is regularly accessible from R.

32. Mr. P. M. Swingle: *A certain type of continuous curve and related point sets.*

This paper is a study of continuous curves every subcontinuum of which is itself a continuous curve; special sections are devoted to (1) continua irreducible between two points, and (2) sets which are composed of the points in sets of arcs joining two distinct closed sets, with a view to determining the relation between the number of possible distinct arcs and the number of common points. The following conditions are each shown to be necessary and sufficient that every subcontinuum of a continuous curve be itself a continuous curve· (1) If P, A and B are any three points of a subcontinuum N of M such that there exists no point, distinct from P, contained in every arc of N from P to $A+B$, then there is an arc APB in N. (2) If N is a connected subset of M, A and B are distinct points of N, L is the set consisting of the points of arcs AB of N', and $Q=L\times N$, then Q is connected. (3) If L is the set consisting of points of a set of arcs of M joining two distinct closed sets E and F, then every point of L' is contained in an arc of L' joining E and F. It is also shown that in order that every subcontinuum of a continuum M be a continuous curve it is sufficient that if A and B are any two distinct points of M there exist a finite number of points which separate A and B in M.

33. Mr. E. W. Miller: *Note on the relation between component and quasi-component in a closed point set.*

In a metric space, if M is a compact and closed set and P is a point of M, then the quasi-component of M determined by P and the component of M determined by P are identical. If the condition of compactness is removed this is no longer true. However, in a metric space every bounded subset of which is compact, it is shown that if M is a closed set then in order that the quasi-component of M determined by P should be identical with the component of M determined by P it is sufficient that the component be compact.

34. Dr. W. L. Ayres (National Research Fellow): *On continua which are disconnected by the omission of any point.*

There exists a plane continuum M such that (1) every point of M is a cut point of M, (2) every continuum K such that every point of K is a cut point of K is homeomorphic with some subset of M. In order that every point of a plane continuum M be a cut point of M either of the following conditions is necessary and sufficient: (1) every point of M be accessible from two unbounded complementary domains, (2) M be non-dense, the boundary of every complementary domain be an open curve and only a finite number of complementary domains have points within any circle. If P is a non-cut point of a continuous curve M in n dimensions and ϵ is any positive number, there exists a continuous curve N and a positive number δ such that (1) M contains N, (2) the diameter of N is less than ϵ, (3) every point of M whose distance from P is less than δ belongs to N, (4) $M-N$ is connected.

35. Dr. T. W. Moore (National Research Fellow): *Extended results in elimination.*

This paper presents a means of elimination for double binary forms in sets of digredient variables. The fundamental problem is to get the condition that three such forms have a common solution expressed as a determinant involving no extraneous factors and not vanishing identically. The method employed is an extension of that used by Morley and Coble for forms in one set of variables published in their article *New results in elimination*, in the American Journal of Mathematics, vol. 49 (1927), No. 4. A covariant containing the coefficients of all three forms and vanishing when they have a common solution is used to obtain this determinant representation. The method has already been applied to a limited range of cases and appears capable of extension to forms containing other than binary variables.

36. Dr. T. W. Moore: *On the complete system of combinants of two binary quintics.*

This paper contains a system of combinants of two binary quintic forms, derived as the simultaneous system of covariants of their first and third transvectants, an octavic and a quartic respectively. It is a continuation of a paper presented to this Society in February, 1927 (see this Bulletin, vol. 33, p. 260) in which the case of the invariant combinants only was considered. The list of forms found in the system contains 113 members. All these are equivalent to the 69 independent forms in the complete system of the binary octavic apolar to the two given quintics, but the exact connection has not yet been established.

37. Professor J. R. Musselman: *A planar imprimitive group of order* 216.

In a recent paper in the American Journal, the existence of a planar imprimitive group of order 216 connected with four Hesse configurations was shown by the writer. The purpose of the present paper is to discuss the group and its associated geometry in detail. Of special interest is a configuration of twelve triangles. A new canonical form used in discussing four-fold projective triangles leads to a ruler construction for them and a compass construction for a Clebsch six-point. The group in S_n is of order $6^n(n+1)!$.

38. Professor Oystein Ore: *Arithmetical theory of Galois fields*. Second communication.

In the author's first communication on this subject it was shown that the complete system of residues for prime-ideal powers can be derived by successive adjunctions of roots of binomial and trinomial congruences, corresponding to the different subgroups of the "Zerlegungs"-group of Hilbert. In the present paper various consequences of these results are derived, and it is proved that the normal congruences are either all binomial or all trinomial. It then follows that in the binomial cases both the "Trägheits"-group and the "Verzweigungs"-group of Hilbert must be cyclic groups.

39. Dr. Raymond Garver: *A new normal form for quartic equations*.

This paper has appeared in the May-June issue of this Bulletin.

40. Dr. Raymond Garver: *The Tschirnhaus transformation*.

In this paper, which is to appear in the Annals of Mathematics, the author makes what he thinks is the first attempt to collect and consider the various results that have been obtained by the use of the Tschirnhaus transformation. A bibliography of 100 titles accompanies the paper

41. Professor F. D. Murnaghan: *The stress components in an elastic solid subjected to a finite deformation*.

When an elastic solid is subjected to a finite strain it is shown that the stress components are derivable from the elastic energy ϕ, which is assumed to be a function of the six strain components $(e_1, \cdots, e_6)$, by means of the equations $X_x = \partial\phi/\partial e_1 - (2e_1\partial\phi/\partial e_1 + e_5\partial\phi/\partial e_5 + e_6\partial\phi/\partial e_6)$; $Y_z = \frac{1}{2}\partial\phi/\partial e_4 - (2e_4\partial\phi/\partial e_2 + e_3\partial\phi/\partial e_4 + e_5\partial\phi/\partial e_6)$; $Z_y = \frac{1}{2}\partial\phi/\partial e_4 - (2e_4\partial\phi/\partial e_3 + e_2\partial\phi/\partial e_4 + e_6\partial\phi/\partial e_5)$. In these expressions $e_1 = \partial U/\partial x - \frac{1}{2}[(\partial U/\partial x)^2 + (\partial V/\partial x)^2 + (\partial W/\partial x)^2]$; $2e_4 = \partial W/\partial y + \partial V/\partial z - [(\partial U/\partial y)(\partial U/\partial z) + (\partial V/\partial y)(\partial V/\partial z) + (\partial W/\partial y)(\partial W/\partial z)]$, etc., where (U, V, W) is the displacement vector of any particle of the medium. If we assume that the stress tensor is symmetric, we have three equations of the type $Y_z = Z_y$, which form a complete system, and ϕ must be a function of the three strain invariants. Hence for a non-isotropic, that is, crystalline medium, we must give up the assumption that the energy of deformation is a function of the strain components.

42. Professor F. D. Murnaghan and Mr. Carl Kaplan: *On the foundations of electromagnetic theory.* Preliminary communication.

An examination is made of the connection between the two fundamental covariant alternating tensors of rank two, namely the (B, cE) and the $(D, -cH)$ tensors, which appear in Maxwell's equations. It appears that the usual equations $B=\mu H$ and $D=\epsilon E$ are but special cases of a more general theory which would put $B=\alpha D+\beta H$, $E=\gamma D+\delta H$, where α, β, γ, δ are constants.

43. Dr. Hillel Poritsky (National Research Fellow): *n-dimensional potential theory.*

In this paper we indicate the elements of the theory of p-vector potentials in n-dimensional euclidean space, and generalize to n dimensions some of well known propositions of three-dimensional potential theory. As an example, we obtain one chain of theorems that reduces for $n=3$ to three links of which the outer ones are equivalent to Gauss's flux theorem while the middle one is equivalent to the linkage theorem of electromagnetism: the work done in carrying a unit magnetic pole around a closed path C in the field of unit electric current flowing in a circuit γ is equal to $4\pi n$, where n is the number of times C links with γ.

44. Professor W. A. Hurwitz: *On Bell's arithmetic of Boolean algebra.*

This paper suggests a modification of the arithmetic of Boolean algebra given by Bell (Transactions of this Society, vol. 29, pp. 597–611) by proposing a definition of congruence which retains all the properties postulated by Bell and also restores several analogies to rational arithmetic.

ARNOLD DRESDEN,
Associate Secretary.

THE APRIL MEETING OF THE SAN FRANCISCO SECTION

The fifty-fourth regular meeting of the San Francisco Section was held at Stanford University on Saturday, April 7, 1928. In the absence of the regular Chairman, Professor E. R. Hedrick presided during the early part of the meeting and Professor E. W. Brown during the latter part. The total attendance was twenty-seven, including the following twenty members of the Society:

Bernstein, Biggerstaff, Blichfeldt, Bright, E. W. Brown, Buck, Cajori, Crum, E. R. Hedrick, Hoskins, R. L. Jackson, Vern James, D. N. Lehmer, F. R. Morris, Noble, Pehrson, T. M. Putnam, Stager, A. Pell Wheeler, A. R. Williams.

The Secretary announced that the Summer meeting of the Section will be held at Reed College on June 2. It was decided to hold the next Spring meeting at Stanford University on April 6, 1929.

Titles and abstracts of papers read at the meeting follow.

1. Professor E. T. Bell: *Certain class—number relations implicit in the Nachlass of Gauss.*

This paper appears in full in the present issue of this Bulletin.

2. Professor E. T. Bell: *Remark on the number of classes of binary quadratic forms of a given negative determinant.*

The long sought means are indicated for an independent (strictly elementary) proof of the fact that, if p is an odd prime, the number of quadratic residues in the set $1, 2, 3, \cdots, (p-1)/2$ exceeds the number of quadratic non-residues. This paper will be published in the Proceedings of the National Academy.

3. Professor E. T. Bell: *Remarks on the quaternary quadratic identity of Hermite.*

In a recent note in the American Mathematical Monthly, it was observed that Hermite's identity does not admit of generalization by algebraic transformations of the coefficients. The present remarks refer to unpublished work concerning the composition of Hermitian forms in 4 or 8 indeterminates.

4. Professor E. T. Bell: *Note on difference equations defining enumerative arithmetical functions.*

The trivial algebraic origin of an extensive class of definitive difference equations for enumerative numerical functions is indicated. In particular, the important identity of J. V. Ouspensky (Bulletin de l'Académie des Sciences de l'URSS, 1925, p. 647), concerning representations as sums of squares, is traced to its obvious origin in the formula for the derivative of a product. The note will be published in the Bulletin of the Calcutta Mathematical Society.

5. Professor E. T. Bell: *A generalization of circulants.*

This is a self-contained, independent detail in a recasting of the theory of algebraic numbers in terms of finite processes only. It will appear in the Proceedings of the Edinburgh Mathematical Society.

6. Professor E. T. Bell: *Ternary characteristics of primes.*

This paper has appeared in full in the May-June number of this Bulletin.

7. Professor E. T. Bell: *Certain completely solvable systems of simultaneous diophantine equations.*

This paper will appear shortly in the American Mathematical Monthly. The equations considered are of any degree in any number of indeterminates.

8. Professor E. T. Bell: *A property of resultants.*

The factorization property of the resultant of two algebraic equations, one of which is binomial, is extended to any pair of algebraic equations. The paper will appear in the Messenger of Mathematics.

9. Professor Florian Cajori: *Early history of partial differential equations and of partial differentiation and partial integration.*

This paper is a criticism of recent European expositions of the theory of fluxions, according to which partial fluxional equations are impossible. The history of partial processes and of partial differential equations is traced on the Continent to the time of Euler, and in Great Britain to the opening of the nineteenth century.

10. Professor Florian Cajori: *A revaluation of Harriot's "Artis analyticae praxis."*

John Wallis's appraisement of Harriot's algebra was strained by national bias. Conflicting estimates of later date, and unwarranted statements found in the most recent histories, induced the present author to make a re-examination of Harriot's book.

11. Professor E. R. Hedrick: *On derivatives of non-analytic functions.*

Kasner has recently shown (Proceedings of the National Academy, vol. 13 (1928), pp. 75–82) that the increment ratio $\Delta w/\Delta z$ for a non-analytic function $w=f(z)$ approaches points on the circumference of a circle as the increments approach zero. The present paper emphasizes the fact that one diameter of the Kasner circle is determined by $\partial w/\partial x$ and $\partial w/\partial(yi)$, where $z=x+yi$, and that another diameter is determined by the maximum and minimum values of $r=|\partial w/\partial z|^2$, which was discussed by Hedrick, Ingold, and Westfall (Journal de Mathematiques, vol. 2 (1923), pp. 327–342). The Tissot indicatrix modified in a manner that is not essential, discussed in that paper is in fact the polar diagram of the distances from the origin to the points of the Kasner circle. The diameter of the Kasner circle is the *ellipticity* discussed in that paper, when a nonessential modification is made. The fact that the Tissot characteristic lines are orthogonal corresponds to Kasner's theorem that orthogonal directions give points at the ends of a diameter of the Kasner circle. This paper will appear in the Proceedings of the National Academy.

12. Professor E. R. Hedrick: *On the increment-ratio for non-analytic functions.*

The increment-ratio $q=\Delta w/\Delta z=[f(z)-f(a)]/(z-a)$ for a function $w=f(z)$ may be considered as a new function of z. Its properties are partly known through those of its limit, dw/dz, which ordinarily depends upon the slope of the curve on which Δz approaches zero. But $q=r+it$ is defined for all values of z for which $f(z)$ is defined, except perhaps $z=a$. In this paper it is shown by direct calculation that the Jacobian of the transformation from (x, y) to (r, t) vanishes precisely on the Kasner circle. In general, the vanishing of the Jacobian of any non-analytic function defines a curve of which a special case is the branch point of an analytic function; this curve may be called the *edge of regression* on account of its similarity to the edge of regression of a developable surface. The edge of regression of the increment-ratio q is then the Kasner circle, which is the envelope of curves corresponding to any pencil of curves through the point $z=a$ in the z plane. The Riemann surface over the q plane is, in general, two-leaved, joined along the Kasner circle. However, not every function whose edge of regression is a circle can be set up as an increment-ratio. The result of this paper will appear in the Proceedings of the National Academy; a detailed proof will appear in the Commemoration Volume (1928) of the Calcutta Mathematical Society.

13. Professor E. R. Hedrick: *Analytic points of non-analytic functions.*

The derivative of a non-analytic function $w=f(z)$, $w=u+vi$, $z=x+yi$ may exist at a point $z=a$, and it does if $u_x=v_y$, $u_y=-v_x$. Such a point will be called an *analytic point.* If the Jacobian, J, of u and v does not vanish, $f(z)$ near a is given by a finite Taylor expansion in the powers of $(z-a)$, plus a remainder, $(z-a)^n Q_n(z)$, where $Q_n(z)$ is not analytic at $z=a$. The analytic point can be *"removed"* by subtracting from $f(z)$ a polynomial and dividing by $(z-a)^n$. Any finite number of analytic points can be removed. Under obvious conditions, the expansion is infinite and $f(z)$ is analytic in a region. A limit point l of such analytic points is itself an analytic point. The functions $Q_n(z)$ for $z=l$ is non-analytic at $z=l$; hence it is non-analytic except at a finite number of points near l. Then $f(z)$ must have the same property. It follows, under suitable hypotheses, that if $f(z)$ has more than a finite number of analytic points in a closed region, it is analytic in that region. This result generalizes considerably the usual theorems on the independence of the equations $u_x = v_y$, $u_y = -v_x$ when $J \neq 0$.

14. Professor E. R. Hedrick: *On the law of the excluded middle from the axiomatic standpoint.*

It has been assumed by recent writers that the law of the excluded middle would be sustained by those who treat mathematics axiomatically, as well as by believers in a priori validity of the law. In the present paper, it is pointed out that the words "true" and "false" should be defined in an axiomatic discussion, and that the statements "p is false," "p is not true (or untrue)," and "not-p is true" may have different meanings. The following definitions are suggested. Given a set of axioms, a proposition p is called *true* if it can be reached by a finite number of syllogisms. A *contradiction* arises if q is *true* and not-q is *true*. Finally, p is called *false* if the axioms, together with p, lead to a contradiction after a finite number of syllogisms. A set of axioms that lead to a contradiction is called *inconsistent*. It follows as a theorem that if p is *false* and p is *true*, then the axioms are inconsistent: such sets of axioms are possible, though not desirable; to exclude them is to assert the law of contradiction. With such definitions, even if the axioms are consistent, the law of the excluded middle would be open to question, and could not be said to have a priori validity.

15. Dr. A. R. Williams: *Quintic surfaces with two coincident double lines.*

The purpose of this paper is to contrast two kinds of quintic surfaces. Both have a double curve consisting of two coincident lines. But in the first case the lines have approached coincidence remaining in the same plane, and the resulting surface is not, in general, rational. In the second case the lines, in approaching coincidence, have remained skew, and the surface is rational. Five conditions must be added to the general surface of the first type to make it of the second type.

16. Professor E. W. Brown: *Harmonic analysis of the disturbing function, with a remainder formula.*

The author shows how an approximate value for the remainder after some definite term in the expansion of the disturbing function could be obtained, and he gives the general series from which this value can be deduced. With its use, harmonic analysis with two arguments would give the coefficients of the needed periodic terms without the extensive numerical calculations which are usually necessary.

B. A. BERNSTEIN,
Secretary of the Section.

A CORRECTION

BY W. J. TRJITZINSKY

My attention has been drawn to the fact that Theorem I of my paper, *Expansion in series of non-inverted factorials*, (this Bulletin, vol. 34 (1928), pp. 193-196), is not new. This theorem is a special case of a theorem found on page 229 of N. E. Nörlund's *Differenzenrechnung*, and is due to Nörlund; it appeared first in Annales de l'Ecole Normale Supérieure ((3), vol. 39 (1922)). The theorem is also a special case of a theorem less general than that of Nörlund, which was proved by Carlson in Nova Acta Soc. Scient. Upsaliensis ((4), vol. 4 (1915)). Neither of these was known to me at the time of the publication of my paper.

SOME PHILOSOPHICAL ASPECTS OF MATHEMATICS*

BY ARNOLD DRESDEN

> "The older I get," observed Mr. van Koppen, "the more I realize that everything depends upon what a man postulates. The rest is plain sailing." (Norman Douglas, South Wind, p. 441.)

It will be well for me to begin this paper with the remark which would be superfluous later on, that it does not presume to make a contribution to mathematics. It is about mathematics, not of them. That I have nevertheless chosen it for presentation in response to the invitation of the program committee, is a result of my conviction that it is useful to reflect from time to time on the character of the structure which is being developed by an ever increasing group of workers. For, while many parts of this structure have become familiar through long acquaintance, it is steadily reaching out in new directions, opening up vistas which stimulate the imagination to envisaging still further extensions and also such as give hitherto unsuspected views of the familiar parts. Ever greater become the distances which separate those who work at different sections of this structure and it is becoming increasingly difficult for them to keep feeling with the fundamental plan which determines its development. The day is long past when unity could be secured through the coordinating agency of a single mind, or even of a group of closely associated minds; and no one can dream of holding together the lines of communication which connect the different parts of the magnificent structure. Perhaps this state of affairs, well recognized by every one, justifies an occasional speculation on the intrinsic character of our science. It is to indicate their speculative character that the following remarks are collected under the present title.

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* Address presented at the request of the American Mathematical Society at the joint meeting with the Mathematical Association of America and Section A of the A. A. A. S. at Nashville on December 29, 1927.

Two contrasting circumstances, the one old, the other of recent origin, have been the starting point for these considerations. On the one hand, we find a persistent faith, particularly on the part of those who are not very intimately familiar with the subject, in the incontestable validity of the conclusions of mathematics. They find in it an escape from the uncertainties of a doubting world, a firm rock to which to hold on in the midst of a fluid universe. To find something which is "mathematically certain" is still, as it ever was, the desire of every seeker after "truth." It is this persisting faith which gives to the mathematician a rather unique place, which provides his subject with a characterization, apparently sufficient for the uninitiated. At the same time it should lead him to a searching through of his science, to inquire whence this faith comes, whether and why it can be accepted also by those who are initiated and left undisturbed in the minds of others.*

For, on the other hand, there has grown up among mathematicians during the last decade or two, a tendency which seems to cast doubt on a large and important part of mathematics. By some this tendency has been characterized as revolutionary, as "bolshevistic" (oh, misery of words!), as subversive of the wholesomeness which had always characterized mathematics. This tendency, inaugurated by the Dutch mathematician, L. E. J. Brouwer, has several distinct aspects. It has been discussed elsewhere† and has been referred to by Professor Pierpont in the address made at yesterday's meeting. I have but to recall to you Hilbert's animadversions upon this work,‡ to have you realize how serious an attack upon the citadel of our science it is held to constitute. If this is indeed the character of the position of Brouwer, it is necessary for mathematicians either to refute that position decisively, or else to abandon the part of the field that is under attack and to retire to "previously prepared positions." It is the former of these alternatives which was undertaken in a paper by Barzin and Errera, at least with reference to one important aspect of Brouwer's position,* namely, his denial of the unlimited validity for mathematical logic of the Law of the Excluded Middle. And it will also be with this aspect of Brouwer's position that we shall be concerned in this paper.†

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* A more extended discussion of "mathematical certainty" forms the subject of a paper to be published in an early number of Scientia.

† See the author's article, *Brouwer's contributions to the foundations of mathematics*, this Bulletin, vol. 30 (1924), p. 31.

‡ See Pierpont, *Mathematical rigor, past and present*, this Bulletin, vol. 34 (1928), p. 52.

It is well known that the Aristotelian logic which ordinarily is tacitly assumed as the basis for logical reasoning proceeds from three fundamental canons, namely, the Law of Identity, the Law of Contradiction, and the Law of the Excluded

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* See M. Barzin et A. Errera, *Sur la logique de M. Brouwer*, Bulletins de la Classe des Sciences, Académie Royale de Belgique, January, 1927, p. 56.

† Although as a matter of convenience, this point of view is here ascribed to Brouwer, who is indeed its first contemporary representative and, with some of his pupils, its principal exponent, it is not his alone. For arguments on his side of the question, see H. Weyl, *Über die neue Grundlagenkrise der Mathematik*, Mathematische Zeitschrift, vol. 10 (1921), p. 39; Weyl, *Der circulus vitiosus in der heutigen Begrundung der Analysis*, Jahresbericht der Vereinigung, vol. 28 (1919), p. 85; Weyl, *Randbemerkungen zu Hauptproblemen der Mathematik*, Mathematische Zeitschrift, vol. 20 (1924), (particularly p. 146–150); R. Wavre, *Y a-t-il une crise des mathématiques*, Revue de Métaphysique et de Morale, vol. 31 (1924), p. 435; also the list of references given on the last page of the article by Barzin and Errera quoted above. There is also a considerable literature on the opposing view; see J. v. Neumann, *Zur Hilbert'schen Beweistheorie*, Mathematische Zeitschrift, vol. 26 (1927), p. 1, where other important references are given, and D. Hilbert, *Über das Unendliche*, Mathematische Annalen, vol. 95 (1925), p. 161; Hilbert, *Über das Unendliche*, Jahresbericht der Vereinigung, vol. 36 (1927), p. 201. Of interest for their general bearing upon these questions are also the following papers: Cipolla, *Sue fondamenti logici della matematica secondo le recenti vedute di Hilbert*, Annali di Mathematica, (4), vol. 1 (1923), p. 19; B. Levi, *Sui procedimenti transfiniti*, Mathematische Annalen, vol. 90 (1923), p. 164; A. Muller, *Über Zahlen und Zeichen*, Mathematische Annalen, vol. 90 (1923), p. 153; M. Pasch, *Betrachtungen zur Begrundung der Mathematik*, Mathematische Zeitschrift, vol. 20 (1924), p. 231; J. v. Neumann, *Eine Axiomatisierung der Mengenlehre*, Journal für Mathematik, vol. 154 (1925), p. 219.

Middle. In the forms in which they usually appear in textbooks on logic, they state respectively that (1) "A is A"; (2) "A is B" and "A is not B" cannot hold simultaneously; and (3) of the two propositions "A is B" and "A is not B," one must always hold, whatever A and B may be. Brouwer's departure consists in this, that he does not accept the L.E.M. as a valid basis for all mathematical reasoning. I shall not here enter upon the reasons for his lack of confidence in the L.E.M.; but I shall refer to the logical basis which leaves it out of account as non-Aristotelian. This designation, apart from obvious reasons, is selected in order to point to an analogy with non-euclidean geometry, to which, although it is obvious, attention does not seem to have been called.* At the same time I must point out one important difference between the non-euclidean geometries and this non-Aristotelian logic. For, whereas in the former, Euclid's parallel postulate is replaced by another postulate, this logic merely omits the L.E.M. from the fundamental canons of logic. And it is in this respect that Barzin and Errera do not, in my judgment, interpret Brouwer's position correctly. For, in order to prove that this position must inevitably lead to a contradiction, they replace the L.E.M. by a new canon, namely that of the excluded fourth. Now I would be the last one to deny them the right to do this; but in so doing they create a non-Aristotelian logic different from that of Brouwer, so that their conclusion can not, in itself, be taken as a refutation of Brouwer's position. It must indeed be clear that any attempt to show that Brouwer's position which uses only two of the Aristotelian canons, leads to a contradiction, would, if successful, not leave any hope for the consistency of the classical logic. It is not likely therefore that the position can be refuted by this method.

For the purpose of my discussion, I shall make an arbitrary division of the content of the mathematical sciences. We shall distinguish between pure mathematics (the word is to be understood as a plural here) and the application of mathematics, the latter to be taken in a sense rather different from the usual one, in which the line of demarcation is rarely definitely fixed. As a pure mathematic I designate any structure which proceeds from a set of primitive ideas, arbitrarily selected and named, and a set of primitive propositions concerning these ideas (assumptions or postulates), arbitrarily laid down, and which develops combinations of these ideas and propositions without the intervention of extraneous concepts or assumptions; the method of development is thus in an essential sense a constructive process. It is of course clear that the initiative for the making of these combinations, particularly of significant and fruitful combinations, does not lie in the assumptions and primitive ideas, but must arise out of a "capacity of the human mind," without which we could not get a step away from the axiomatic basis. The recognition of the very significant intervention of the human mind is an important element in our point of view. How it operates and why is not a part of our subject, in spite of its title; this must remain a subject for psychological investigation. An application of a mathematic is made whenever a one-to-one correspondence is set up between the primitive ideas of the mathematic and entities which have in some way or other secured objective existence, in such manner that the primitive propositions receive objective verification when these entities are put in place of the primitive ideas entering into the primitive propositions. It is at this point that a peculiarity of my division of the mathematical content into pure mathematics and applications of mathematics makes itself felt. For the entities, through the introduction of which an application of one mathematic is made, may be the primitive ideas of another mathematic, and the objective existence which is attributed to them is of a different order from that which is attached to chairs and tables. If we call such objective existence as is enjoyed by entities which, to speak without making an attempt at great refinement, can be apprehended by the senses, as being of order zero, we may say that primitive ideas which enter into a mathematic of which an application can be made through the introduction of such entities of zero order, have themselves objective existence of order one. Proceeding in this way we introduce objective existence of increasing orders; the higher the order of objective existence of an entity, the farther it is removed from what is ordinarily designated by the term. It is of course clear furthermore, that the designation of an order of objective existence attached to an entity is not a unique process, but depends upon the constructive procedure by means of which it has been reached. It is of secondary importance for our present purpose whether minimum order of objective existence greater than or equal to 1 can be attached in a unique way to every entity with which mathematicians deal, although this appears to be an interesting question for further study. In the same manner, we can assign an order to propositions, calling facts which are directly verifiable in the world of the senses, propositions of order zero; propositions which are verified by means of them, propositions of order one, and so forth. If, for instance, we set up a mathematic consisting of primitive propositions stated in terms of elements a and operations $(+)$ and $(\times)$; and if these primitive propositions are verified by facts that are directly observable when we substitute, say, apples or pebbles for the elements a and some concrete operations for $(+)$ and $(\times)$, then these elements a and the operations $(+)$ and $(\times)$ obtain objective existence of order 1, and the primitive propositions, together with the consequences drawn from them, become propositions of order 1. If now we set up a new mathematic in terms of elements b and operations $((+))$ and $((\times))$, whose primitive propositions reduce to propositions of order 1; when, for example, pairs of elements a and the operations $(+)$ and $(\times)$ are put in place of the elements b and the operations $((+))$ and $((\times))$, respectively, then the elements b and the operations $((+))$ and $((\times))$ obtain objective existence of order 2; and so forth. The situation I am describing is somewhat obscured in many of the examples of postulational theories which are known in the literature; for in those cases, we usually set up postulates for a particular theory, that is to say, we construct a set of postulates that is to be applicable to a mathematic which has secured citizenship not by coming into the country through an immigration office, but by means of settlement before immigration laws were put into effect, or perhaps by having been smuggled in. For instance, when a set of postulates for positive integers in terms of elements a and operators $(+)$ and $(\times)$ is applied, we usually make appeal not to entities and propositions for which an order of objective existence has been definitely established, but to concepts with which we are well acquainted, to neighbors with whom we are on intimate terms, even though we have never seen their naturalization certificate, indeed though we have never inquired whether they had one. But if we were to proceed on a strictly legal basis the testimony of such citizens could not be admitted; we would have to depend on original settlers or on such as possessed definitely traceable citizenship. That is to say, application of a set of postulates would then be made only through verification in terms of entities for which an order of objective existence can be established, and, perhaps preferably, in terms of entities of objective existence of order zero.

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* When we take cognizance of the controversial literature which has grown up in this field, we get a vivid realization of the reasons which may have led Gauss to withhold his discovery of non-euclidean geometry from publication.

This digression on the peculiar character of the division which we have introduced into the content of mathematics seemed desirable in order to avoid misunderstanding; for our main purpose, the only point of importance is that application of a mathematic must ultimately depend upon the intervention as sponsors for our primitive ideas of entities which have, speaking roughly, the objective existence of chairs and tables, that is, what we have called objective existence of order zero, and that the verification of primitive propositions must ultimately depend upon propositions of order zero. We can sum up this conclusion in the statement that the applicability of a mathematic must ultimately depend upon the possibility of setting it into one-to-one correspondence with a part of the content of human experience.

Returning now to the consideration of a pure mathematic, it must be clear that there is no immediate meaning in the question whether it is true.* The only sense in which such a question can be made significant is by interpreting it as an inquiry as to the applicability of the mathematic, in the sense in which this term has just been explained. Each one of us can build up a mathematic, if only we are sufficiently inventive to originate a set of primitive ideas and primitive propositions. Whether these mathematics will be fruitful of results, whether they will command the abiding interest not only of their inventors but also of others, will depend upon the answers to two questions, namely whether they stimulate minds to exercise in a lasting manner that capacity for constructive combination to which we have made reference before, and whether they are applicable to significant parts of the content of human experience. I am very doubtful in my own mind whether a mathematic can have the former of these two characteristics without having the latter; but this question I must also leave aside. Certainly it seems to me so that if a mathematic has neither of the two, it will be completely sterile and will deservedly die an early death. If it have the latter property as well as the former, it will surely arouse the interest of the worker in applied science, of the "practical" man; if the property of applicability is possessed in an easily recognizable way, if the mathematic can, through its applications, penetrate deeply into human experience, it may appeal even to the man in the street. It is, however, on account of the former property primarily that a mathematic will command the attention of mathematicians as such, because it offers them an opportunity for the investigative ability which is peculiarly theirs, namely the initiating of combinations of relatively simple elements which are rich in possibilities for further development. Of especial significance for the mathematician will be the comparison of different mathematics which are related to each other. Such relations may arise through the partial identity of their fundamental content. If two mathematics have their primitive ideas and some of their primitive propositions in common (or if there is a simple isomorphism between their primitive ideas and some of their primitive propositions), the development of either illuminates in a striking way the structure of the other. Indeed it seems to me so, that full insight in a mathematic can be gained only by comparing it with several of its related mathematics. Every one will realize that this statement is little more than a generalization by extension of the marvelous enrichment of geometrical concepts to which the non-euclidean geometries have given rise, and of the enlargement of our ideas concerning geometry and physics which the theory of relativity has brought about.

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* It is perhaps not without interest to quote here the following passage from a definitely non-mathematical source: "What was truth, after all? It was a very sensible question of Pilate's. Perhaps there *was* no truth, nor falsehood either, in any actual set of words arranged in a certain order. Perhaps they were only a neutral surface over which truth or falsehood could be cast by different minds or tongues, as a blue or an amber light is projected by turns on a colourless piece of stage canvas." (C. E. Montague, *Right Off The Map*, p. 183.)

We are now in a position to view in what seems to me to be its true light the second of the circumstances mentioned at the beginning of this paper. Whether or not one agrees with the opinion according to which logic is a part of mathematics, there will be little opposition to the thesis that logic plays a fundamental role in every mathematic. And this is so because the development of a mathematic, the constructive process which we have recognized as the driving force of this development, proceeds according to the laws of logic. Indeed in every mathematic, or at least in nearly every mathematic, that has been developed thus far, it is understood, either explicitly or tacitly, that the development should remain in accord with the canons of Aristotelian logic. But, whether Aristotelian or non-Aristotelian, it must be clear that a logic must be the inherent part of every mathematic, that the primitive ideas and primitive propositions of some logic are a substructure of every mathematic. It follows therefore that every mathematic may be looked upon as involving an application of some logic, because the canons and primitive ideas of a logic must be considered as having obtained in it some sort of objective existence; they have to be appealed to if the mathematic is ever to develop beyond the embryonic stage. This carries with it, of course, that at least in so far as its structure is concerned the logic must, in some way, be looked upon as a mathematic; but for its development we can evidently not rely upon another logic, for this would clearly involve us in an infinite regress. Indeed the logic must be thought of as a part of what I have referred to above* as "the capacity of the human mind"; and the development of the logic has to proceed by the use of the residue of this capacity, left after the separating off of the logic. This residue we shall call the "bare capacity of the human mind."†

It has been recognized that the logic which underlies a mathematic has a very important bearing upon its development; and for this reason, writers, such as Frege, Peano, Whitehead and Russell have set up a pure logic as the preliminary for any mathematic. But this has been done only for the logic of classes, propositions and relations, and has not affected the fundamental canons. The significance of the departure which Brouwer has made lies in this, that he has brought the canons of logic explicitly into the foreground. By suggesting the possibility of modifying these canons as the fundamental basis even of the logic of mathematics, he has called attention to the fact that they also are a part of this basis; that what we have called the "capacity of the human mind" need not include all the Aristotelian canons. Thus he has freed the mind from the compulsory use of the Aristotelian base. The significant question to ask concerning this new logic which Brouwer builds on is not whether it is true, but whether it is applicable; and this question, in the light of our earlier discussion, means whether it is possible to set its primitive ideas and propositions into one-to-one correspondence with entities that have objective existence of order zero in human experience, that is to say in short, whether it is possible for the human mind to operate effectively with it. If it is, then the developments of this logic will aid greatly in giving insight into the capacities and possibilities of the mind, just as the comparison of two closely related postulate systems for one field of mathematics enrich our knowledge of that field. And there are indications that it is indeed possible to do so in the constructive work that has already been done by Brouwer,* Heyting† and others. The developments of a mathematic by means of this non-Aristotelian logic (usually called intuitionist mathematics) should then be looked upon in the same way as we look upon non-euclidean geometry, as an enrichment of our understanding of the way in which the human mind operates. For myself, I am very doubtful whether I could trust myself not to use the L.E.M.; but I am ready to admit that any one who can, may do important work in that way. I would go farther and open the way for using various types of logic. Each one of us can choose to work with one logic to-day and with another to-morrow, just as one's work in non-euclidean geometry does not exclude him forever from working in euclidean geometry, or as dealing with non-commutative algebras one year need not incapacitate one for occupying himself with commutative algebras the next. Whether or not his work will be fruitful is difficult to decide in advance.

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* See page 442.

† The significance of the explicit formulation of the Aristotelian canons lies in the fact that they constituted, as far as I am aware, the first attempt at an analysis of the "capacity of the human mind." Let us not fall into the error of thinking that they are a complete formulation of that capacity

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* See the list of references given by Barzin and Errera in the article quoted on p. 440.

† See *Zur intuitionistischen Axiomatik der projektiven Geometrie,* Mathematische Annalen, vol. 98 (1927), p. 491.

If this were all there is to the position which I am taking, it would probably not be objected to seriously. For it gives greater liberty, leaving open not merely the primitive ideas and primitive propositions of a mathematic to choice, but also the rules which are to govern its development. But there is another aspect to this story, which we must now consider briefly. For the freedom which we gain is obtained at the sacrifice of a certain permanence of existence which mathematicians have usually wanted to attribute to their science, an existence in essence which is independent of any mind, and independent of time, which requires discovery to have it brought down to the realm of human apprehension. Only recently I came across the following passage in a letter from Hermite to Konigsberger, in which this idea finds expression. He wrote: "I add that these notions of analysis have their existence apart from us, that they constitute a whole of which only a part is revealed to us, incontestably although mysteriously associated with that other totality of things which we perceive by way of the senses."* To this belief, which is probably shared by most mathematicians to-day, the position which I have developed leaves little support. For if the entire mathematical structure, not the basis only, but also the guiding principles for its development, is at the choice of the individual, we have to admit that a mathematic exists only in the minds of the individual, and that without the activity of the human mind there would be no mathematics. This carries with it furthermore, that a mathematic exists only in so far as it has been developed, that is, is invented rather than discovered; that not only with regard to primitive ideas and primitive propositions, but also with regard to the logical processes, the question of truth is irrelevant; that mathematical entities may exist to-morrow which do not exist to-day, that their existence depends upon the construction of a process, which can call them into being. It seems to me that Brouwer's position inescapably commits us to this conclusion; it clarifies points in his work, which otherwise present difficulties which it seems impossible to overcome. By disregarding this consequence of his position in their critique of his work,* Barzin and Errera seem to me to misconceive a fundamental aspect of it. In one of his most interesting papers,† Brouwer introduces an integer k which is defined as the order number of that place in the decimal development of π, at which there appears for the first time a zero, immediately followed by the sequence of digits $1, 2, \cdots, 9$. When we use the non-Aristotelian logic which omits the L.E.M., we say that we do not know whether this k exists, nor whether we shall ever be able to answer the question whether it exists, and that we are consequently able to construct a real number of which we do not know whether it is less than, equal to or greater than 0. If, on the other hand, we use the Aristotelian logic, we can say, using the L.E.M. that either this number k exists or else it does not exist, independently of what we know about it, and that the real number determined by the use of this k is either less than, equal to or greater than 0, whether or not we know which of these alternatives holds. That is to say we hold that the mathematical fact has existence, independently of whether we have discovered it.

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* See W. Birkemeier, *Über den Bildungswert der Mathematik*, Teubner, 1923, p. 26.

There is therefore a fundamental difference between the characters of mathematical existence which the two points of view involve. But is this cause for alarm? I think not; in diversity there may lie strength. Indeed the new point of view adds to our insight. For when we learn that, without the use of the L.E.M. the system of real numbers is not ordered, we have increased our knowledge, just as when we learn that in matrix algebra the product of two factors can vanish, while neither factor vanishes. In which of the two ways we proceed is a matter of choice to be determined by each individual, not for all time necessarily, but from time to time, according to his tastes and prejudices.

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* See Barzin and Errera, loc. cit., p. 59.

† See Brouwer, *Über die Bedeutung des principium tertii exclusi in der Mathematik, besonders in der Funktionentheorie*, Journal für Mathematik, vol. 154 (1925), p. 1.

We come finally to the question of the bearing of this point of view on the validity of mathematical conclusions. It would seem as if the freedom which is left to the individual to build his mathematic as he chooses would not leave much basis on which to rest a compelling faith in his conclusions. Indeed, as we have seen before, the question as to the truth of these conclusions would not have any meaning. But the non-mathematician is not interested in a mathematic, but rather in the applications of a mathematic, indeed in the application which leads back from the mathematic to the content of human experience. Now we have seen that if a mathematic is to survive beyond the embryonic stage, it must be capable of development and of application. Thus a natural selection takes place which secures for a surviving mathematic those qualities which link it up with the realm of human experience. The canons of logic by which the development takes place must have their roots in the experiences of the mind; the primitive ideas and the primitive propositions must have their ultimate connections with the objective experience of the race, they must be, as it were, the result of a process of distillation, which continues throughout the history of the race; they must embody the essence of this experience, obtained through successive abstractions from its significant elements. In this way the application of a mathematic brings forth results which are relevant in human experience and which are in harmony with its fundamental characteristics. And it is this harmony of the applications of a mathematic with fundamental aspects of his experience, which constitutes their truth for the man in the street, for the man not directly concerned with mathematics; from it he derives his faith in the validity of mathematical conclusions.

We may sum up our conclusions as follows. A mathematic may be established through the free choice of a logic, and of primitive ideas and primitive propositions; if this choice is guided by wisdom, the mathematic will be capable of development and application. The application will then, sometimes directly, but more frequently through a chain of intermediate stages, have significant bearing upon the content of human experience and furnish results which may be called true. Their truth then gives, retroactively, a sound basis for belief in the validity of the conclusions of mathematics.

SWARTHMORE COLLEGE

RECENT DEVELOPMENTS IN PROJECTIVE DIFFERENTIAL GEOMETRY*

BY E. B. STOUFFER AND E. P. LANE

A. INVARIANTS AND COVARIANTS, CANONICAL FORMS, CANONICAL EXPANSIONS

1. *Introduction.* In a long series of papers appearing first in 1901 and continuing for more than twenty years, Wilczynski made a study of the projective differential properties of many geometrical figures, including plane and space curves, ruled and curved surfaces, and linear congruences. While Wilczynski was not the first to discover projective differential properties of figures, he was the first to proceed in a *systematic* manner in finding these properties. His general scheme in studying a given figure is to set up a system of one or more linear homogeneous differential equations such that the fundamental sets of solutions of the system determine the figure uniquely except for projective transformations. The independent and the dependent variables which appear in the differential equations and also in the parametric equations of the figure can be subjected to certain transformations which do not disturb the figure or change the form of the system of differential equations, but will in general change the coefficients of the differential equations. A function of the new coefficients and their derivatives and of the new dependent variables and their derivatives which is equal, except possibly for a factor, to the same function of the original coefficients and variables is called a *covariant.* A covariant which does not contain the dependent variables or their derivatives is called an *invariant.*

Wilczynski uses a complete and independent system of invariants and covariants as a foundation for each of his geometrical studies. His general method of calculating the invariants and covariants is by means of the Lie theory of continuous groups, a process which is frequently very laborious, although the final results are usually quite elegant. Since the method is purely analytical, there is no assurance that the invariants and covariants will appear in a form which will readily show their geometric significance.

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*Two addresses presented at the request of the program committee at the Western meeting of the Society, April 6, 1928. Part A was given by Professor Stouffer, and Part B by Professor Lane.

Fubini and his school began about 1913 to study projective differential properties by means of differential forms. However, before many geometrical results can be obtained by this method it is necessary in each case to pass from the forms to a system of differential equations of the type used by Wilczynski. Fubini confines his attention almost wholly to systems which have been made canonical, that is, systems in which the coefficients have been specialized by means of the permissible transformations in such a way as to make the fundamental covariants take very simple forms.

One purpose of this paper is to derive several canonical forms for defining systems of differential equations and to show how each properly chosen canonical form leads in a very simple manner to a complete system of invariants and covariants of the Wilczynski type. A second purpose is to show how such canonical forms make possible direct derivation of canonical expansions for the equations of the figures.

In a paper⁽¹⁾ published in 1915 Green showed how the invariants and covariants associated with a curved surface referred to general parametric curves could be obtained from those associated with the surface referred to the asymptotic curves as parametric curves. The principle which Green used in that particular problem is of remarkably wide application. We shall first apply it to the very simple case of a plane curve.

2. *Curves.* The differential equation associated with a plane curve C is of the form⁽²⁾

$$y''' + 3p_1y'' + 3p_2y' + p_3y = 0, \tag{1}$$

⁽¹⁾Numerical references are made to the bibliography at the end.

where differentiation is with respect to the independent variable x and where p_i are functions* of x. The permissible transformations in this case have the form

$$\bar{x} = \phi(x), \tag{2}$$

$$y = \lambda \bar{y}, \tag{3}$$

where ϕ and λ are arbitrary functions of the independent variable.

The transformation (2) changes (1) into a new equation (1a) with coefficients $\bar{p}_i$ given by

$$\tag{4} \begin{cases} \bar{p}_1 = \dfrac{1}{\phi'}\left(p_1 + \dfrac{\phi''}{\phi'}\right), \\ \bar{p}_2 = \dfrac{1}{(\phi')^2}\left(p_2 + p_1\dfrac{\phi''}{\phi'} + \dfrac{\phi'''}{3\phi'}\right), \quad \bar{p}_3 = \dfrac{1}{(\phi')^3}p_3{}^3, \end{cases}$$

and (3) in turn changes (1a) into another equation (1b) in $\bar{y}$ with coefficients $\overline{P}_i$ given by†

$$\tag{5} \begin{cases} \overline{P}_1 = \bar{p}_1 + \dfrac{\bar{\lambda}'}{\bar{\lambda}}, \\ \overline{P}_2 = \bar{p}_2 + 2\bar{p}_1 \cdot \dfrac{\bar{\lambda}'}{\bar{\lambda}} + \dfrac{\bar{\lambda}''}{\bar{\lambda}}, \\ \overline{P}_3 = \bar{p}_3 + 3\bar{p}_2\dfrac{\bar{\lambda}'}{\bar{\lambda}} + 3\bar{p}_1\dfrac{\bar{\lambda}''}{\bar{\lambda}} + \dfrac{\bar{\lambda}'''}{\bar{\lambda}} \cdot \end{cases}$$

It is easy to determine from (4) and (5) an invariant (2), that is, a function θ_3 of p_i which is related to the same function $\bar{\theta}_3$ of $\overline{P}_i$ by the equation $\bar{\theta}_3 = \theta_3/(\phi')^3$.

If we choose ϕ' to satisfy the equation‡ $(\phi')^3 = \theta_3$ so that $\bar{\theta}_3 = 1$ and then choose $\bar{\lambda}$ to satisfy the equation $\bar{\lambda}'/\bar{\lambda} = -\bar{p}_1$

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*Here as everywhere else in this paper we shall assume that all functions which appear are capable of differentiation to as high an order as neccessary.

†In equations (5), $\bar{\lambda}$ is a function of $\bar{x}$ and differentiation is with respect to $\bar{x}$.

‡ It is assumed that C is not a conic, since in that case $\theta_3 = 0$.

so that $\bar{P}_1=0$, we have a canonical form of the desired type for our differential equation. This is exactly the canonical form obtained by Sannia(3) who introduced the differential form a_1dx and then chose $a_1=\theta_3^{1/3}$.

Since the conditions $\bar{\theta}_3=1$, $\bar{P}_1=0$ are maintained by the transformations (2) and (3) only if $\phi''=\lambda'=0$, the coefficients $\bar{P}_i$ of our canonical form and their derivatives are each determined except for possible factors. A simple process of reasoning shows that they are the canonical forms of invariants whose general forms in terms of the coefficients of (1) can be obtained from (5) by direct substitutions from (4) with the now known values of ϕ' and λ' inserted. The same reasoning shows that $\bar{y}$ and its derivatives are the canonical forms of covariants whose general forms follow by mere substitution. The completeness of the system of invariants and covariants thus obtained is evident and the determination of the particular invariants and covariants which are independent is not at all difficult.

The system of invariants and covariants thus obtained is, of course, equivalent to the system derived by Wilczynski and is, in fact, not greatly different from it. The great value of the method outlined above lies in its simplicity and in its complete avoidance of the complicated Lie theory.

A canonical form of the above type for the differential equation of a plane curve leads by a process involving only direct substitutions to an equation for the curve in the form of an expansion of one non-homogeneous coordinate in terms of the other.(4) This method of obtaining a canonical expansion has the advantage that the vertices of the triangle of reference are exactly the points determined by the three covariants $\bar{y}$, $\bar{y}'$, $\bar{y}''$ and are, as a consequence, easily characterized geometrically. Since a canonical expansion is the most natural and simple tool for obtaining results in this geometry, the ability to derive the expansion in a direct analytic manner is of great advantage.

The general scheme applied above to plane curve theory can be applied to space curves.(4) However, it is desirable in this case to use the canonical expansion in order to determine the most satisfactory canonical form for the fundamental differential equations, even though this canonical form is actually used to determine the canonical expansion. We shall illustrate this principle in some detail by means of ruled surface theory. In the study of ruled surfaces it has been long and difficult work to determine the invariants and covariants(2) and a canonical expansion(5) by previous methods.

3. *Ruled Surfaces.* Let us denote by P the point on a ruled surface R in the neighborhood of which we wish to study the surface. If we choose as one directrix curve C_y the asymptotic curve through P and for the other directrix curve C_z any curve on the surface, it is well known(2) that the fundamental system of differential equations has the general form

$$(6) \qquad \begin{aligned} &y'' + 2p_{11}y' + \qquad\qquad q_{11}y + q_{12}z = 0, \\ &z'' + 2p_{21}y' + 2p_{22}z' + q_{21}y + q_{22}z = 0, \end{aligned}$$

where p_{ij} and q_{ij} are functions of the independent variable x. The permissible transformations are

$$(7) \qquad \bar{x} = \phi(x),$$

for the independent variable and

$$(8) \qquad y = \alpha\bar{y}, \quad z = \gamma\bar{y} + \delta\bar{z}, \quad \alpha\delta \neq 0,$$

for the dependent variables, where ϕ, α, γ, δ are arbitrary functions of the independent variable.

Let us now assume that (6) has been transformed by (7) and (8) into a canonical form with the dependent variables and their first derivatives uniquely determined except for possible factors. What the canonical form is we do not know as yet. However, it is easy to determine that the associated canonical expansion for the equation of the surface in the neighborhood of the point P is

$$(9) \qquad \zeta = \xi\eta + \frac{1}{3}\bar{Q}_{12}\xi^3 + \frac{1}{12}\bar{V}_{12}\xi^4 - \frac{1}{3}(\bar{U}_{11} - \bar{U}_{22})\xi^3\eta + \cdots,$$

where $\overline{V}_{12}$ and $\overline{U}_{11}-\overline{U}_{22}$ are simple functions of the coefficients $\overline{P}_{ij}$ and $\overline{Q}_{ij}$ of the canonical form and their derivatives. Equation (9) makes evident the desirable conditions to be imposed in order to produce a canonical form and at the same time simplify the expansion. By means of (7) and (8) we can make*

$$\overline{V}_{12} = \overline{U}_{11} - \overline{U}_{22} = \overline{P}_{11} = \overline{P}_{22} = 0,\ \overline{Q}_{12} = 1.$$

Since these conditions are maintained by (7) and (8) only if $\phi''=\alpha'=\delta'=\gamma=0$, we obtain for (6) a canonical form

$$\left\{\begin{aligned} &\bar{y}'' + \overline{Q}_{11}\bar{y} + \bar{z} = 0, \\ &\bar{z}'' + 2\overline{P}_{21}\bar{y}' + \overline{Q}_{21}\bar{y} + \overline{Q}_{11}\bar{z} = 0 \end{aligned}\right. \tag{10}$$

in which the coefficients and their derivatives are invariants and the dependent variables and their derivatives are covariants, all in their canonical forms. The general form of this complete system of invariants and covariants in terms of the coefficients and variables of (6) can be obtained by direct substitutions just as the case of plane curves.

The expansion (9) is thus reduced to the simple form $\zeta = \xi\eta + \xi^3/3 +$ terms of at least the fifth degree. The vertices of the tetrahedron of reference are given at once by the four simplest covariants. Their geometrical significance is easily determined. In fact, one vertex is the point P, a second is the point $\overline{P}_z$, the harmonic conjugate of P with respect to the flecnode points on the generator through P, a third is the point $\overline{P}_\rho$ on the tangent to C_y at P which is the pole of the generator with respect to the osculating conic of C_y at P, and the fourth is the intersection of the asymptotic tangent at $\overline{P}_z$ and the line of the osculating hyperboloid which passes through $\overline{P}_\rho$.

The geometrical significance of the vanishing of the invariants, which in their canonical forms are the coefficients of (10), can be read directly from the differential equation.

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*These results are not valid if P is a point on a flecnode curve.

In fact, if $\overline{P}_{21}=0$, $\overline{C}_z$ is an asymptotic curve, if $\bar{Q}_{21}=0$ the flecnode curves coincide and if $\bar{Q}_{11}=0$ the tangent at $\overline{P}_\rho$ to $\overline{C}_\rho$ passes through $\overline{P}_z$.

Čech(6) has applied Fubini's methods to ruled surfaces by starting with a specialized case of the differential forms for curved surfaces. He derives from these forms a pair of second order differential equations which he makes canonical by putting a certain invariant θ_4 of Wilczynski equal to unity.

4. *Curved Surfaces.* In applying the above methods to curved surfaces we start with the fundamental equations(7)

$$\begin{aligned} y_{uu} + 2a\, y_u + 2by_v + cy &= 0, \\ y_{vv} + 2a'y_u + 2b'y_v + c'y &= 0, \end{aligned} \tag{11}$$

with the asymptotic curves parametric. The permissible transformations are

$$\bar{u} = \phi(u), \qquad \bar{v} = \psi(v), \tag{12}$$

for the independent variables, and

$$y = \lambda(u,v)\bar{y}, \tag{13}$$

for the dependent variable.

Let us assume that (11) has been transformed by (12) and (13) into a system with coefficients $\overline{A}$, $\overline{B}$, $\overline{C}$, $\overline{A}'$, $\overline{B}'$, $\overline{C}'$, whose dependent variable $\bar{y}$ and its derivatives $\bar{y}_{\bar{u}}$, $\bar{y}_{\bar{v}}$, $\bar{y}_{\bar{u}\bar{v}}$ are all completely determined except for possible factors. It is easily found that an associated canonical expansion for the equation of the surface S in the neighborhood of a point $\overline{P}_y$ given by $\bar{u}=\bar{u}_0$, $\bar{v}=\bar{v}_0$, is

$$\left\{ \begin{aligned} \zeta = {} & \xi\eta + \tfrac{2}{3}\overline{B}\xi^3 + \tfrac{2}{3}\overline{A}'\eta^3 + \tfrac{2}{3}(\overline{B}_{\bar{v}} - 2\overline{B}\overline{B}')\xi^3\eta \\ & + \tfrac{2}{3}(\overline{A}'_{\bar{u}} - 2\overline{A}\overline{A}')\xi\eta^3 + \tfrac{1}{6}(\overline{B}_{\bar{u}} + 4\overline{A}\overline{B})\xi^4 \\ & + \tfrac{1}{6}(\overline{A}'_{\bar{v}} + 4\overline{A}'\overline{B}')\eta^4 + \tfrac{1}{2}(\overline{A}_{\bar{v}} + \overline{B}_{\bar{u}})\xi^2\eta^2 + \cdots . \end{aligned} \right. \tag{14}$$

The fact that $\phi(u)$ and $\psi(v)$ in the transformation (12) are each functions of only a single variable limits the conditions which may be imposed by this transformation. However, we can, for instance, impose by (12) and (13) the conditions

$$(15) \qquad \bar{A}'_{\bar u}(\bar u,\bar v_0) = \bar{B}_{\bar v}(\bar u_0,\bar v) = 0,$$

$$(16) \qquad \bar{A} = \bar{B}' = 0,$$

conditions which are maintained by (12) and (13) only if $\phi_{uu}=\psi_{vv}=\lambda_u=\lambda_v=0$. Since the coefficients in (14) are functions of $\bar u_0$ and $\bar v_0$ we are thus led to a canonical expansion for the equations of the surface. However, we do not obtain in place of (11) a canonical form in the true sense. Nevertheless, by the same kind of simple substitutions as above, the coefficients and variables of the reduced form and their derivatives produce a complete system of invariants and covariants in terms of the coefficients and variables of (11). The expansion (14) becomes under the conditions (15) and (16) essentially that obtained by Wilczynski(8) and also by Green(9), with the vertices of the tetrahedron of reference determined by the simplest covariants and located at the intersection of Wilczynski's directrices of the first and second kind with his canonical quadric(8).

In place of (15) we may impose the more general conditions

$$(17) \quad l\,\frac{\bar{A}'_{\bar u}(\bar u,\bar v_0)}{\bar{A}'(\bar u,\bar v_0)} + m\,\frac{\bar{B}_{\bar u}(\bar u,\bar v_0)}{\bar{B}(\bar u,\bar v_0)} = l\,\frac{\bar{B}_{\bar v}(\bar u_0\ \bar v)}{\bar{B}(\bar u_0,\bar v)} + m\,\frac{\bar{A}'_{\bar v}(\bar u_0,\bar v)}{\bar{A}'(\bar u_0,\bar v)} = 0$$

where l and m are constants. In each case we obtain a canonical expansion with two points $\bar P_z$ and $\bar P_\rho$, one on each asymptotic tangent through P_y, as two vertices of the tetrahedron of reference. The fourth vertex $\bar P_\sigma$ is the intersection of the canonical quadric of Wilczynski with the polar reciprocal of the line $\bar P_z \bar P_\rho$ with respect to the quadric of Lie. For all values of l and m the lines joining corresponding points $\bar P_z$ and $\bar P_\rho$ pass through a point, the canonical point, and the corresponding lines joining P_y and $\bar P_\sigma$ all lie in a plane, the canonical plane.

If $m=0$ we have the case mentioned above. If $l=0$ the vertices are on the canonical edges of Green(9). If $l/m=1$ the line joining P_y and $\bar P_\sigma$ is the pseudo-normal of Green(9), the projective normal of Fubini(10). Other values of the ratio l/m give other covariant lines. In each case the coefficients and variables of the reduced form produce a corresponding system of invariants and covariants which can be expressed in terms of the coefficients and variables of (11) by direct substitutions.

If we apply the transformations (13) alone, we find that it is possible to impose the conditions(11)

$$2\overline{A} + (\log \overline{A}'\overline{B})_u = 2\overline{B}' + (\log \overline{A}'\overline{B})_v = 0. \tag{18}$$

These conditions are not disturbed by the general transformation (12). If we write $\theta = (1/2) \log \overline{A}'\overline{B}$, equations (11) become under the conditions (18)

$$\begin{aligned} \bar{y}_{uu} - 2\theta_u \bar{y}_u + 2\overline{B}\bar{y}_v + \overline{C}\bar{y} &= 0, \\ \bar{y}_{vv} + 2\overline{A}'\bar{y}_u - 2\theta_v \bar{y}_v + \overline{C}'\bar{y} &= 0, \end{aligned} \tag{19}$$

which is the canonical form of Fubini.

If $\overline{A}$ and $\overline{B}'$ are replaced by $-\theta_u$ and $-\theta_v$, respectively, in accordance with (18), the expansion (14) becomes essentially the canonical expansion of Fubini(12). The vertices of the tetrahedron of reference for this expansion are the same as for the case $l/m = 1$ above, except that the fourth vertex is the intersection of the projective normal with Fubini's canonical quadric(20). Fubini(13) has derived the same expansion without the use of a fundamental system of differential equations. He simply assumes an expansion of one non-homogeneous coordinate in terms of the other two and then chooses his tetrahedron of reference properly.

Recently Lane(14) has obtained all the above mentioned canonical expansions for the equation of a curved surface, starting from the equivalent of equations (18). The same author(39) has also derived a significant canonical form for the equations of a curved surface referred to a conjugate net as the parametric net.

The methods outlined above for obtaining complete systems of invariants and covariants and canonical expansions apply equally well in many other cases. However, their consideration must be left to other occasions.

B. GEOMETRY OF SURFACES

1. *Surfaces in Ordinary Space.* The foundations for a theory of the projective differential geometry of an analytic non-ruled surface in ordinary space were laid by Wilczynski in a series of five memoirs which were published in the Transactions of this Society from 1907 to 1909. As the theory grew and expanded, Green made notable contributions to it, especially in a memoir that appeared in the Transactions after his death in 1919. The work of Fubini on this subject dates from about 1913, and that of Bompiani from about 1923.

A considerable portion of the projective differential geometry of a surface as elaborated in the last two decades, together with some of the more or less incidental results found by various geometers before 1907 can be organized about three unifying ideas, namely, quadrics containing the asymptotic tangents, the canonical pencils of lines, and union curves with their generalizations. After establishing an analytic basis for our discussion we shall consider each of these topics in turn.

Let the projective homogeneous coordinates $x^{(1)}, \cdots, x^{(4)}$ of a point P on a surface S be functions of two independent variables u, v. If the asymptotic net is parametric and if the proportionality factor of the coordinates is suitably chosen, then the functions x are solutions of a completely integrable system of differential equations of the form

$$(1) \quad x_{uu} = px + \theta_u x_u + \beta x_v, \quad x_{vv} = qx + \gamma x_u + \theta_v x_v, \qquad (\theta = \log \beta\gamma).$$

The coordinates of a point N on S near P can be represented by Taylor's formula as power series in the increments Δu, Δv corresponding to displacement from P to N. Then by means of (1) and the equations obtained therefrom by differentiation it is possible to express each of these series uniquely in the form $x_1 x + x_2 x_u + x_3 x_v + x_4 x_{uv}$, where $x_1, \cdots, x_4$ are the following series which represent the local coordinates of N referred to the covariant tetrahedron x, x_u, x_v, x_{uv}, with suitably chosen unit point:

$$
(2)\quad \begin{cases}
x_1 = 1 + (p\Delta u^2 + q\Delta v^2)/2 + \cdots, \\
x_2 = \Delta u + (\theta_u \Delta u^2 + \gamma \Delta v^2)/2 + \cdots, \\
x_3 = \Delta v + (\beta \Delta u^2 + \theta_v \Delta v^2)/2 + \cdots, \\
x_4 = \Delta u \Delta v + (\beta \Delta u^3 + 3\theta_u \Delta u^2 \Delta v + 3\theta_v \Delta u \Delta v^2 + \gamma \Delta v^3)/6 + \cdots.
\end{cases}
$$

The equations of the u-tangent are $x_3 = x_4 = 0$, and those of the v-tangent are $x_2 = x_4 = 0$.

2. *Quadrics containing the Asymptotic Tangents.* The equation of any non-singular quadric surface ϕ containing the asymptotic tangents of S at P can be written (15) in the form

$$
(3)\quad x_2 x_3 + x_4(K_1 x_1 + K_2 x_2 + K_3 x_3 + K_4 x_4) = 0 \quad (K_1 \neq 0).
$$

If $K_1 = -1$, then ϕ has contact of the second order with S at P and cuts S in a curve with a triple point at P. If the triple point tangents coincide, they coincide in one of *the three directions of Darboux*(16) for which

$$
(4)\qquad \beta du^3 + \gamma dv^3 = 0.
$$

If $K_1 = -1$ and $K_2 = K_3 = 0$, then ϕ is *a quadric of Darboux*, which has contact of the second order and has the tangents of Darboux for triple point tangents.

Among the quadrics of Darboux there are four that deserve mention. If $K_4 = -(\beta\gamma + \theta_{uv})/2$, then ϕ is *the quadric of Lie*(17), called by Wilczynski *the osculating quadric*, which is the limit of the quadric determined by three asymptotic tangents of one family constructed at points of a fixed curve of the other family as these points approach coincidence along the fixed asymptotic. If $K_4 = -\theta_{uv}/2$, then ϕ is *the canonical quadric of Wilczynski*(8). Bompiani has recently(18) rediscovered this quadric, apparently without recognizing it, by means of the following considerations. If a curve C on S has an inflexion at P then C is tangent to an asymptotic at P. The limit of the quadric determined by three asymptotic tangents of the other family constructed at points of C, as these points approach P along C, is the canonical quadric of Wilczynski. If in this definition the curve C, instead of having an inflexion, has a stationary osculating plane at P, the resulting quadric is the quadric of Fubini(19) for which $K_4 = -(2\beta\gamma + 3\theta_{uv})/6$. Finally, if $K_4 = 0$, then ϕ is *the canonical quadric of Fubini*(20), which is the quadric of Darboux that passes through the covariant point (0, 0, 0, 1).

We shall refer briefly to only two more species of quadrics of the general type (3). At a point P of curve C on S there are two of Bompiani's *asymptotic osculating quadrics*(21); each of these is the limit of the quadric determined by three asymptotic tangents of one family constructed at points of C, as these points independently approach P along C. And *the quadric of Moutard*(22) of S at P in the direction of a tangent t is the locus of the osculating conics at P of the curves of intersection of S and the planes of the pencil with t as axis.

3. *The Canonical Pencils.* The canonical line pencils of S at P may be defined analytically as follows. The line l_1 joining P to the point $(0, K\psi, K\phi, 1)$, where

$$\phi = \frac{\partial}{\partial u} \log \beta\gamma^2, \quad \psi = \frac{\partial}{\partial v} \log \beta^2\gamma, \qquad K = \text{const.}, \tag{5}$$

is a canonical line of the first kind. When K varies, the locus of l_1 is *the first canonical pencil*, with center at P and lying in *the canonical plane* $\phi x_2 - \psi x_3 = 0$. The polar line l_2 of l_1 with respect to the quadric of Lie is a canonical line of the second kind and joins the points $(K\phi, 1, 0, 0)$, $(K\psi, 0, 1, 0)$. When K varies, the locus of l_2 is *the second canonical* pencil with center at *the canonical point* $(0, \psi, -\phi, 0)$ and lying in the tangent plane $x_4 = 0$.

Every canonical line is covariant to the surface, but negative rational values of K give most of the lines that have appeared naturally in geometric investigations. For instance, if $K=-1/2$, then l_1, l_2 are *the directrices*(8) of Wilczynski. These have several characteristic properties, but were discovered by Wilczynski as the directrices of the linear congruence of intersection of the osculating linear complexes of the two asymptotic curves at P. If $K=-1/4$, then l_1, l_2 are *the canonical edges*(9) of Green. Perhaps the most recently discovered characteristic property of these lines is due to B. Segre who shows(23) that all non-composite cubic surfaces having contact of the fourth order with S at P cut the tangent plane in the same cubic curve which has a double point at P and has three inflexions lying on the second canonical edge of Green. If $K=-1/3$, then l_1 is *the axis* of Čech, called(24) by him *the line of Segre*, because the osculating planes of the three curves of Segre $\beta du^3-\gamma dv^3=0$ at P intersect in this line. If $K=0$, then l_1 is *the projective normal*(10) of Fubini, which was discovered independently by Green and called(9) by him *the pseudo-normal.* Other canonical lines of interest are *the principal lines*(25) of Fubini, for which $K=-1/6$, $K=-1/12$; the lines(15) for which $K=-5/12$, $K=-3/4$; and(26) the lines(27) for which $K=-3/8$.

4. *Union Curves.* Let us consider a congruence Γ of lines one of which, l, passes through each point P of S but does not lie in the tangent plane at P. *A union curve of* Γ is defined(28) by Miss Sperry to be a curve on S such that its osculating plane at each of its points contains the line of Γ through the point. The differential equation of the union curves has the form

$$v'' = A + Bv' + Cv'^2 + Dv'^3 \qquad (v' = dv/du), \tag{6}$$

and the osculating planes at P of all the union curves of Γ through P form a pencil with the line l through P as axis. For this reason Bompiani(29) calls the union curves of a congruence *an axial system.* In metric geometry the union curves of the congruence of normals are the geodesics.

The most general equation of the form (6) defines on S a system of curves such that the osculating planes at a point

P of all the curves of the system that pass through P envelope a cone of the third class which touches the tangent plane of S along the asymptotic tangents at P, and which has three cusp-planes that intersect in a line through P, called *the cusp-axis* of P with respect to the given system of curves. We shall mention two examples of cusp-axes. The extremals of the integral

$$\int (\beta\gamma v')^{1/2}du$$

are given by Euler's equation,

$$v'' = \theta_u v'/\theta - \theta_v v'^2/\theta, \qquad (\theta = \log \beta\gamma), \tag{7}$$

and the corresponding cusp-axis is the projective normal. The curves of the Segre-Darboux pencil

$$v'' = \lambda_u v'/\lambda + \lambda_v v'^2/\lambda, \qquad (\lambda = (\beta/\gamma)^{1/3}). \tag{8}$$

have for cusp-axis the axis of Čech.

Bompiani defines(30) *a planar system* of curves as follows. Let us consider a congruence Γ of quadrics one of which is associated with each point P of S and contains the asymptotic tangents through P but does not have contact of order as high as the second with S at P. Then *a planar curve of* Γ is a curve on S such that at each of its points its asymptotic osculating quadric of one family (which contains the asymptotic tangents through the point) intersects the associated quadric of Γ in a residual pair of straight lines, instead of in a residual non-singular conic as would ordinarily be the case. The differential equation of a planar system is of the form (6). A given congruence Γ determines two planar systems of curves because a curve has two families of asymptotic osculating quadrics. But if a planar system is defined by one family of asymptotic osculating quadrics and a congruence Γ, then a second congruence Γ can be determined so that it and the other family of asymptotic osculating quadrics define the same planar system. The reason for the name *planar system* will appear when we discuss Bompiani's extension of this theory to hyperspace.

5. *Method of the Italian School.* Italian geometers of the school of Fubini define a configuration, except for a projective transformation, by means of a system of differential forms, and employ the absolute calculus of Ricci; whereas American geometers of the school of Wilczynski define a configuration projectively by means of a system of differential equations and make use of the Lie theory of continuous groups. Fubini's method(31) has been used by Sannia in studying plane and space curves; by Čech in his theory of ruled surfaces; and by Fubini in his investigations of rectilinear congruences and complexes. But the method finds perhaps its best exemplification in Fubini's theory of surfaces in ordinary space. We shall sketch this theory very briefly.

Let us consider a fundamental binary quadratic differential form

$$(9)\quad G = a_{ij}du_i du_j, \qquad (a_{ij} = a_{ji}\,; A = |a_{ij}| \neq 0\,; i,j = 1,2),$$

which will be completely specified later, and let us also consider a surface S whose parametric equations in projective homogeneous point coordinates are given by the equations $x^{(k)} = x^{(k)}(u, v)$ $(k=1, \cdots, 4)$. Let us define two differential forms F_2, Φ_3 by the equations

$$(10)\, F_2 = (x, x_1, x_2, d^2x)\,|\,A\,|^{-1/2},\ \Phi_3 = (x, x_1, x_2, d^3x)\,|\,A\,|^{-1/2},$$

wherein numerical subscripts of x indicate covariant differentiation with respect to G, and a determinant is indicated by writing a typical row within parentheses. The forms F_2, Φ_3 are of the first and second orders respectively, and are both absolutely invariant under transformation of curvilinear coordinates u, v on S. But under transformation of fundamental form G and of proportionality factor ρ of the homogeneous coordinates, the forms F_2, Φ_3 behave quite differently.

Let us define a differential form F_3 by the equation

$$F_3 = 2\Phi_3 - 3dF_2 + \frac{3}{4}F_2\,d\log\frac{\Delta}{A}, \tag{11}$$

where Δ is the discriminant of F_2. Both forms F_2, F_3 are of the first order of degrees indicated by the subscripts, are absolutely invariant under transformation of curvilinear coordinates, and are cogredient under transformation of G and ρ. In fact both forms are relative invariants under the latter transformation, since we have

$$|A'|^{1/2}F_i' = \rho^4|A|^{1/2}F_i, \qquad (i = 2,3), \tag{12}$$

accents indicating the transformed quantities.

Let us define two forms ϕ_2, ϕ_3 by the formula

$$\phi_i = \rho^4|A|^{1/2}|A'|^{-1/2}F_i, \qquad (i = 2,3), \tag{13}$$

wherein ρ is chosen so that the ratio of the discriminant of ϕ_3 to the cube of the discriminant of ϕ_2 is constant. With this choice of ρ the coordinates are *Fubini's normal coordinates*. Next let us choose the fundamental form G to be ϕ_2. Then the hessian of ϕ_3 is proportional to ϕ_2, so that ϕ_3 is apolar to ϕ_2. If the asymptotic net on S is taken for parametric net, we can write

$$\phi_2 = 2\beta\gamma du dv, \quad \phi_3 = 2\beta\gamma(\beta du^3 + \gamma dv^3). \tag{14}$$

If we attempt to set up the differential equations to obtain a surface which has the forms ϕ_2, ϕ_3 arbitrarily assigned, we find that ϕ_2, ϕ_3 are not sufficient to define a surface to within a projective transformation. But if we adjoin to ϕ_2, ϕ_3 the form ψ_2 defined by

$$\psi_2 = p du^2 + q dv^2, \tag{15}$$

then the forms ϕ_2, ϕ_3, ψ_2 do define a surface, except for a projective transformation, and the differential equations for obtaining such a surface are precisely equations (1). At this point the Italian and American theories merge.

The form ϕ_2 vanishes for the asymptotic curves, and ϕ_3

for the curves of Darboux, while ψ_2 vanishes for the curves corresponding to the developables of the congruence of lines which are polars of the projective normals with respect to the quadrics of Lie.

6. *Extensions to Hyperspace.* Extensive investigations in the projective differential geometry of hyperspace have been made by Segre(32), Bompiani, Fubini, Terracini, Tzitzéica(33) and others. We shall confine our attention to only two of the many topics that might be considered, first discussing Bompiani's *pluri-axial systems*, and then comparing the Italian and American methods as applied in hyperspace.

7. *Pluri-Axial Systems.* In a linear space S_n of n dimensions let us consider a surface S which has on it a conjugate net, so that at every point of S the osculating planes of all the curves on S through the point are contained in an S_4 instead of in the usual S_5. Let a congruence Γ of planes be given, one of which passes through each point P of S and lies in the tangent S_4 without intersecting, except at P, the tangent plane of S at P. Then a curve on S is *a planar curve*(34) *of* Γ in case its osculating plane at each of its points intersects in a straight line the plane of Γ through the point. The differential equation of a planar system of curves has the form (6). It is well known that point geometry on a hyperquadric in S_5 is isomorphic with line geometry in ordinary ruled space R_3, the correspondance being established by interpreting the six Plückerian coordinates of a line in R_3 as the coordinates of a point in S_5. If the surface S under consideration here is the surface in S_5 that corresponds to one congruence of asymptotic tangents of a surface in R_3, then the planar system of curves defined here corresponds to the planar system previously defined.

On a general surface S in S_n a system of curves analogous to a planar system can be defined(35) by considering a congruence of spaces S_3, instead of planes, one of which passes through each point P of S and lies in the tangent S_5 at P. More generally, let us consider at each point P of S

the $S(k)$ which is the ambient space of the osculating S_k at P of every curve on S through P. Then let a congruence Γ of linear spaces S_h be given, one of which passes through each point P of S and lies in the $S(k)$ at P. Then a curve on S is *a pluri-axial curve of* Γ in case its osculating S_k at each of its points intersects the S_h of Γ through the point in a linear space of dimensions greater than would ordinarily be the case.

Still further generalizations can be obtained(36) by replacing the surface S by a variety V_k of k dimensions and associating with each point of V_k a linear space S_h. Or instead of linear spaces S_h, cones may be used. And instead of a pluri-axial system of curves, it would seem possible to define a pluri-axial system of varieties V_j on V_k $(1<j<k)$.

8. *Comparison of Italian and American Methods.* The analytic and synthetic methods used by Italian geometers in studying the projective differential geometry of hyperspace possess great power and elegance. Nevertheless there does not yet exist a general theory of a V_k in S_n comparable with the Italian and American theories of a surface in ordinary space. The Italian method of differential forms has failed for a V_k in S_n $(1<k<n-1,\ n>4)$, either because of the lack of a covariant quadratic differential form, or because of the lack of an absolute calculus for an n-ary p-adic differential form. The American method of differential equations is theoretically available, but the amount of labor involved in making the necessary calculations seems at present to be practically prohibitive in so far as establishing a general theory is concerned. In special cases, however, the American method is applicable. For instance, Miss Beenken has recently(37) applied this method in a study of surfaces in S_5. And there is hope that it may be possible to modify the American method so as to avoid much of the labor of the calculations ordinarily involved by reducing all of the differential equations of a defining system to equations of the first order(38).

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FACTORIAL SERIES IN TWO VARIABLES*

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Because of its connection with the study of partial difference equations, the author has had occasion to examine the factorial series

$$(A) \qquad \sum_{n=0}^{\infty} \frac{(n!)^2 a_n}{x(x+1)\cdots(x+n)y(y+1)\cdots(y+n)}.$$

It may be of some interest to observe that the methods employed by Landau in his fundamental paper on factorial series† can immediately be extended to deduce the essential convergence properties of the series (A). We give here eight theorems corresponding to Theorems I–VIII of Landau. They may all be established by proofs parallel to his in every detail; some of them may also be inferred otherwise as indicated below.

THEOREM 1. *If the series* (A) *converges for the place*‡ (x_0, y_0), *it converges for any place* (x_1, y_1) *satisfying the conditions*§ $R(x_1) \geqq R(x_0)$, $R(y_1) \geqq R(y_0)$ *(both equality signs not to hold simultaneously).*

Only the following three possibilities may therefore occur: (a) the series (A) converges everywhere; (b) it converges nowhere; (c) there exist two associated real numbers λ_1, λ_2, such that the series converges for $R(x) \geqq \lambda_1$, $R(y) \geqq \lambda_2$ and diverges for $R(x) \leqq \lambda_1$, $R(y) \leqq \lambda_2$ (both equality signs not to hold simultaneously in either case). One would scarcely expect the pair of numbers λ_1, λ_2 to be uniquely determined; the relation between them will presently be evident.

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† *Über die Grundlagen der Theorie der Fakultatenreihen*, Sitzungsberichte der Munchener Akademie (math.-phys.), vol. 36 (1906), pp. 151-218.

‡We think of the complex variables x and y as represented by points in two distinct planes. Following a common terminology a pair of values x, y will be spoken of as the *place* (x, y).

§$R(x)$ is used to denote the *real part* of x.

THEOREM 2. *The series* (A) *converges uniformly in the neighborhood of every place* (x, y) *satisfying the conditions* $R(x) > \lambda_1$; $R(y) > \lambda_2$; *and* $x, y \neq 0, -1, -2, \cdots$.

Then from a well known theorem in the theory of functions* follows

THEOREM 3. *The series* (A) *represents an analytic function within its related half-planes of convergence* $(x, y \neq 0, -1, -2, \cdots)$ *and in this region may be differentiated partially with respect to either variable as many times as may be desired.*

THEOREM 4. *The region of absolute convergence of the series* (A), *unless it converges everywhere or nowhere, is a pair of related half-planes bounded on the left by lines* $R(x) = \mu_1$, $R(y) = \mu_2$; *the lines themselves may be included or not.*

THEOREM 5. *If the series* (A) *converges for the place* (x_0, y_0) *and if we have* $R(x_1) > R(x_0) + a$, $R(y_1) > R(y_0) + b$, *and* $a + b = 1$, *the series* (A) *is absolutely convergent for the place* (x_1, y_1).

THEOREM 6. *The regions of convergence of the series* (A) *and of the Dirichlet series*

$$\text{(B)} \qquad \sum_{n=1}^{\infty} \frac{a_n}{n^{x+y}}$$

are the same; that is, for any place (x, y) $(x, y \neq 0, -1, -2, \cdots)$ *both series converge or both diverge.*

From this theorem, which is proved independently of the preceding ones, Theorem 1 could be inferred. In conjunction with Landau's Theorem VI it also forms the basis for the rather surprising result that *the regions of convergence of* (A) *and of the series*

$$\sum_{n=0}^{\infty} \frac{n!a_n}{(x+y)(x+y+1)\cdots(x+y+n)}$$

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*See Osgood, *Lehrbuch der Funktionentheorie*, vol. II, 1924, p. 13.

are identical. Hence we conclude that *the related abscissas of convergence of the series* (A) *have a constant sum.*

THEOREM 7. *The places of absolute convergence of the series* (A) *and* (B) *are the same.*

From Theorems 6 and 7 and Landau's Theorems VI–VIII follows at once

THEOREM 8. *If the sum of the abscissas of* [*absolute*] *convergence* $\lambda_1+\lambda_2[\mu_1+\mu_2]$ *is* $\geqq 0$, *it is given by*

$$\limsup_{t\to\infty} \frac{\log\left|\sum_{n=1}^{t} a_n\right|}{\log t} \quad \left[\limsup_{t\to\infty} \frac{\log \sum_{n=1}^{t} |a_n|}{\log t}\right].$$

It may be remarked that Landau's work can similarly be extended to prove the convergence properties of the series

$$\sum_{n=0}^{\infty}(n!)^m a_n/[x_1(x_1+1)\cdots(x_1+n)x_2(x_2+1)\cdots(x_2+n)\cdots x_m(x_m+1)\cdots(x_m+n)],$$

where m is any positive integer.

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THE NODES OF THE RATIONAL PLANE QUARTIC*

BY L. T. MOORE

The projective properties of the rational quartic curve may be studied from two points of view. If the curve is given parametrically, its invariants are expressible in terms of the coefficients of the fundamental involution. If the curve is given by the plane $(KX)=0$, and the Steiner quartic surface $(X^{1/2})=0$, the invariants are expressible in symmetric functions of the coefficients of $(KX)=0$.† An invariant condition obtained for one form of the equations of the curve may readily be expressed in the invariants of the other form. The relations connecting the two systems are

$$(1)\qquad \begin{cases} 4I_2 = MS_1S_3, \\ I_2' = M(S_1S_3 - 16S_4), \\ I_4 = (16M)^2(S_2S_3^2 - S_1S_3S_4 + S_4^2), \\ I_6 = M^3S_3^2(S_1S_2S_3 - S_1^2S_4 - S_3^2), \end{cases}$$

where M is a positive constant. These equations may also be solved for any function of S_1, S_2, S_3, S_4, whose weight is a multiple of four.‡

To determine the nature of the nodes from the invariants, it is desirable to refer the quartic to a triangle whose vertices are at the nodes. Such a quartic is given by the equations§

$$X_0^2X_1^2 + X_1^2X_2^2 + X_0^2X_2^2 - 2X_0X_1X_2X_3 = 0,$$

$$\alpha_0X_0 + \alpha_1X_1 + \alpha_2X_2 + \alpha_3X_3 = 0.$$

Eliminating X_3 we have

$$(2)\quad \alpha_3(X_0^2X_1^2 + X_1^2X_2^2 + X_0^2X_2^2) + 2X_0X_1X_2(\alpha_0X_0 + \alpha_1X_1 + \alpha_2X_2) = 0,$$

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* Presented to the Society, May 7, 1927.

† J. E. Rowe, Transactions of this Society, vol. 12 (1911), pp. 295–310.

‡ L. T. Moore, American Journal, vol. 48 (1926), p. 251.

§ The surface $(\sqrt{X})=0$ referred to a tetrahedron having the three double lines as edges. Salmon-Rogers, *Geometry of Three Dimensions*, vol. 2, p. 213.

which may be derived from the conic

$$(3)\quad \alpha_3(X_0^2 + X_1^2 + X_2^2) + 2\alpha_2 X_0 X_1 + 2\alpha_1 X_0 X_2 + 2\alpha_0 X_1 X_2 = 0,$$

by the transformation $X_0 = 1/X_0'$; $X_1 = 1/X_1'$; $X_2 = 1/X_2'$.

The points of intersection of (3) with the sides of the triangle of reference correspond to the nodes of (2). Therefore, if (3) cuts a side in two real, coincident, or imaginary points, the corresponding node is, respectively, a crunode, a cusp, or an acnode.*

It is evident that the expressions $(\alpha_0^2 - \alpha_3^2)$, $(\alpha_1^2 - \alpha_3^2)$, $(\alpha_2^2 - \alpha_3^2)$ determine the nature of the nodes, and from a consideration of the functions

$$\begin{aligned} F_3(\alpha) &= (\alpha_0^2 - \alpha_3^2)(\alpha_1^2 - \alpha_3^2)(\alpha_2^2 - \alpha_3^2), \\ F_2(\alpha) &= (\alpha_0^2 - \alpha_3^2)(\alpha_1^2 - \alpha_3^2) + (\alpha_0^2 - \alpha_3^2)(\alpha_2^2 - \alpha_3^2) \\ &\qquad + (\alpha_1^2 - \alpha_3^2)(\alpha_2^2 - \alpha_3^2), \\ F_1(\alpha) &= (\alpha_0^2 - \alpha_3^2) + (\alpha_1^2 - \alpha_3^2) + (\alpha_2^2 - \alpha_3^2), \end{aligned}$$

the following classification can be obtained:

Three crunodes: $F_3(\alpha) > 0$, $F_2(\alpha) > 0$, $F_1(\alpha) > 0$;

Two crunodes, one acnode: $F_3(\alpha) < 0$, $\{F_1(\alpha) > 0$ or $F_1(\alpha) < 0$, $F_2(\alpha) < 0\}$;

Three acnodes: $F_3(\alpha) < 0$, $F_2(\alpha) > 0$, $F_1(\alpha) < 0$;

One crunode, two acnodes: $F_3(\alpha) > 0$, $F_2(\alpha)$ or $F_1(\alpha) < 0$.

The following relations connect the coefficients of $(\alpha X) = 0$ and $(KX) = 0$:†

$$\begin{aligned} S_1 &= K_0 + K_1 + K_2 + K_3 = 4\alpha_3, \\ S_2 &= K_0K_1 + K_0K_2 + K_0K_3 + K_1K_2 + K_1K_3 + K_2K_3 \\ &= 2(3\alpha_3^2 - \alpha_0^2 - \alpha_1^2 - \alpha_2^2), \\ S_3 &= 4(\alpha_3^3 - \alpha_3(\alpha_0^2 + \alpha_1^2 + \alpha_2^2) + 2\alpha_0\alpha_1\alpha_2), \\ S_4 &= \alpha_0^4 + \alpha_1^4 + \alpha_2^4 + \alpha_3^4 - 2\alpha_1^2\alpha_0^2 - 2\alpha_0^2\alpha_2^2 - 2\alpha_0^2\alpha_3^2 \\ &\qquad - 2\alpha_1^2\alpha_2^2 - 2\alpha_1^2\alpha_3^2 - 2\alpha_2^2\alpha_3^2 + 8\alpha_0\alpha_1\alpha_2\alpha_3. \end{aligned}$$

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* Salmon, *Higher Plane Curves*, 1879, p. 254.

† L. T. Moore, loc. cit., p. 248.

From these relations we have

$$64F_3(\alpha) = (S_1^2S_4 - S_1S_2S_3 + S_3^2),\ 16F_2(\alpha) = (S_2^2 + S_1S_3 - 4S_4),$$
$$2F_1(\alpha) = -S_2.$$

Again, after multiplying $F_1(\alpha)$ by S_1^2 and $F_3(\alpha)$ by S_3^2, we have from equations (1),

$$M^3S_3^2(S_1^2S_4 - S_1S_2S_3 + S_3^2) = -I_6,$$

$$1024M(S_2^2 + S_1S_3 - 4S_4)$$
$$= \frac{\{I_4 + 64I_2(4I_2 - I_2') - (4I_2 - I_2')^2\}^2 + 256I_6'(I_2' + 12I_2)}{I_6'},$$

$$MS_1^2S_2 = \frac{4I_2^2\{I_4 + 64I_2(4I_2 - I_2') - (4I_2 - I_2')^2}{I_6'},$$

where $I_6' = I_2I_4 - I_2(I_2' - 4I_2)^2 - 64I_6$.

Thus we have established sufficient criteria, in each of the three systems of invariants, to determine the nature of the nodes of the curve.

We can determine the nature of the nodes of rational tacnodal and oscnodal quartics by the same method used for trinodal quartics.* The results obtained may be stated as follows:

Tacnode and crunode: $I_6 < 0$, $I_2' - 4I_2 < 0$;
Tacnode and acnode: $I_6 > 0$, $I_2' - 4I_2 < 0$;
Isolated tacnode and crunode: $I_6 < 0$, $I_2' - 4I_2 > 0$;
Isolated tacnode and acnode: $I_6 > 0$, $I_2' - 4I_2 > 0$;
Isolated oscnode: $I_6 < 0$.

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* L. T. Moore and J. H. Neelley, *Rational tacnodal and oscnodal quartic curves considered as plane sections of quartic surfaces*, to appear soon in the American Journal. In this paper, equations corresponding to equations (1), (2), (3) above are developed for tacnodal and oscnodal quartics.

A THEOREM ON RULED SURFACES

BY A. F. CARPENTER

On page 229 of volume 1 of Fubini-Čech, *Geometria Proiettiva Differenziale*, appears the following theorem.

If the two flecnode curves of a ruled surface are plane, then the complex curves and the harmonic curves are plane, and all of their planes belong to a pencil.

The authors furnish a proof of this, and they call it "an interesting theorem by Sullivan."

In a letter to the author of this note, Professor C. T. Sullivan says, with reference to the above, "The theorem you refer to is consequently not to be found in any of my publications." It appears probable that the Italian geometers have confused this theorem with one of a somewhat similar nature published by the author in 1915,* or perhaps with a somewhat more general theorem contained in a later paper by the author.†

In this paper it is shown that the planes osculating the two branches of the flecnode curve and the planes osculating the two branches of the complex curve at the four points in which these curves cut a line element of their supporting ruled surface, will form a pencil, if and only if,

$$D \equiv p_{21}^2 \Delta_1 - p_{12}^2 \Delta_2 = 0, \tag{1}$$

where

$$\Delta_1 = p_{12}^2 q_{22} - p_{12} q_{12}' + 3q_{12}^2, \quad \Delta_2 \equiv p_{21}^2 q_{11} - p_{21} q_{21}' + 3q_{21}^2,$$

p_{ik}, q_{ik} being coefficients of the flecnode canonical form of the system of differential equations defining the surface. The theorem of 1915 is a special case of that of 1923 since

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*Transactions of this Society, vol. 16 (1915), p. 520. This volume also contains a paper by Professor Sullivan.

†Tôhoku Mathematical Journal, vol. 23 (1923), p. 114.

$\Delta_1=0$, $\Delta_2=0$ are necessary and sufficient conditions that the flecnode curves be plane.

It is the further purpose of this note to extend the theorem of the Tôhoku paper to include the harmonic curves.*

In the notation there employed the harmonic points are given by the factors of the covariant $p_{21}y^2+p_{12}z^2$. If $\eta=(p_{21})^{1/2}y+i(p_{12})^{1/2}z$, $\zeta=(p_{21})^{1/2}y-i(p_{12})^{1/2}z$, then the equations of the two planes osculating the curves C_η, C_ζ, at points P_η, P_ζ, are respectively

$$(2)\quad \begin{aligned} p_{21}(p_{12})^{1/2}(p_{12}p_{21}(p_{12}p_{21})^{1/2} + iP)(i(p_{12})^{1/2}x_1 - (p_{21})^{1/2}x_2) \\ + f_1(i)x_3 + f_2(i)x_4 = 0, \end{aligned}$$

$$(3)\quad \begin{aligned} p_{21}(p_{12})^{1/2}(p_{12}p_{21}(p_{12}p_{21})^{1/2} - iP)(i(p_{12})^{1/2}x_1 + (p_{21})^{1/2}x_2) \\ - f_1(-i)x_3 - f_2(-i)x_4 = 0, \end{aligned}$$

where

$$\begin{aligned} f_1(i) &\equiv D + 2p_{12}q_{21}P - p_{12}^3p_{21}^3 \\ &\qquad - ip_{12}p_{21}(p_{12}p_{21})^{1/2}(2p_{12}q_{21} + P), \\ (p_{12})^{1/2}f_2(i) &\equiv i(p_{21})^{1/2}(D + 2p_{21}q_{12}P - p_{12}^3p_{21}^3) \\ &\qquad + p_{12}p_{21}^2(p_{12})^{1/2}(2p_{21}q_{12} - P), \\ P &= p_{12}q_{21} - p_{21}q_{12}. \end{aligned}$$

Two of the pencil of planes determined by (2), (3) are seen to be

$$(4)\quad \begin{aligned} (p_{12}^3p_{21}^3 + P^2)(p_{12}x_2 + p_{12}^2x_3 - 2q_{12}x_4) \\ - p_{12}D(p_{12}^2p_{21}x_3 - Px_4) = 0, \end{aligned}$$

$$(5)\quad \begin{aligned} (p_{12}^3p_{21}^3 + P^2)(p_{21}x_1 - 2q_{21}x_3 + p_{21}^2x_4) \\ - p_{21}D(Px_3 - p_{12}p_{21}^2x_4) = 0. \end{aligned}$$

If, and only if, $D=0$, will (4), (5) reduce to

$$(6)\quad p_{12}x_2 + p_{12}^2x_3 - 2q_{12}x_4 = 0,$$

$$(7)\quad p_{21}x_1 - 2q_{21}x_3 + p_{21}^2x_4 = 0.$$

But (6) and (7) are the equations of the planes osculating

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*These curves were called by Wilczynski the *involute curves*. See his *Projective Differential Geometry*, p. 208.

the flecnode curves. Combining this result with the theorem of the Tôhoku paper we have the following theorem.

THEOREM. *The planes osculating the flecnode curve, the complex curve and the harmonic curve at the six points in which these curves cut a line element of their supporting ruled surface, will belong to a pencil, if, and only if,*

$$D \equiv \overset{2}{p}_{21}\Delta_1 - \overset{2}{p}_{12}\Delta_2 = 0.$$

Since, when $\Delta_1 = \Delta_2 = 0$, the flecnode curves are plane, the theorem of Fubini-Čech appears as a special case of the preceding theorem.

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NETS OF CONICS IN THE GALOIS FIELDS OF ORDER 2^n*

BY A. D. CAMPBELL

We consider a net of conics in the Galois fields of order 2^n

$$\lambda C_1 + \mu C_2 + \nu C_3 = 0, \tag{1}$$

where

$$C_i \equiv a_i x^2 + b_i y^2 + c_i z^2 + f_i yz + g_i zx + h_i xy, \quad (i = 1,2,3),$$

and where C_i has the discriminant†

$$\Delta_i \equiv f_i g_i h_i + a_i f_i^2 + b_i g_i^2 + c_i h_i^2 .$$

Such a field is denoted for brevity by the symbol $GF(2^n)$. Along with (1) we consider the cubic curve in λ, μ, ν that is obtained by equating to zero the discriminant of the general conic of (1)‡

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* Presented to the Society, December 31, 1926.

† See A. D. Campbell, *Plane cubic curves in the Galois fields of order* 2^n, Annals of Mathematics, vol. 27 (1926), p. 395.

‡Compare C. Jordan, *Réduction d'un réseau de formes quadratiques*, Journal des Mathématiques, (6), vol. 2 (1906), p. 412.

$$(2)\qquad K \equiv a\lambda^3 + b\mu^3 + c\nu^3 + d\lambda^2\mu + e\lambda\mu^2 + f\lambda^2\nu + g\lambda\nu^2 + h\mu^2\nu + j\mu\nu^2 + k\lambda\mu\nu = 0,$$

where $a \equiv \Delta_1,\ b \equiv \Delta_2,\ c \equiv \Delta_3,$ and

$$d \equiv f_1g_1h_2 + f_1g_2h_1 + f_2g_1h_1 + a_2f_1^2 + b_2g_1^2 + c_2h_1^2,$$

with similar expressions for e, f, g, h, j; and where

$$k \equiv f_1g_2h_3 + f_1g_3h_2 + f_2g_1h_3 + f_2g_3h_1 + f_3g_1h_2 + f_3g_2h_1.$$

In this paper we shall classify these nets of conics, deriving a typical net for each class. First we shall study the nets that have no degenerate pencil of conics* and hence have a non-composite cubic $K=0$; then we shall consider the nets with composite cubics. For the first type of nets we reduce the cubic to a standard form and then put the net in a canonical form.

If the net (1) has a double line this must correspond to a cusp on the cubic (2), a fact easy to prove. We put (2) in the form†

$$\lambda^2\mu + \lambda^2\nu + \mu\nu^2 = 0$$

with a cusp at (0, 1, 0) and an inflection at (1, 0, 0). To the tangent at the inflection on the cubic there must correspond in the net a pencil with just one degenerate conic C which corresponds to the inflection. This conic C must be a real line pair, because there is no such pencil with C a pair of conjugate imaginary lines.‡ We can now put (1) in the form

$$\lambda yz + \mu x^2 + \nu C_3 = 0,$$

$$K \equiv \lambda^2\mu + a_3\lambda^2\nu + g_3h_3\lambda\nu^2 + f_3^2\mu\nu^2 + \Delta_3\nu^3 = 0.$$

We must have $a_3=1$, $g_3h_3=0$, $f_3^2=1$, $\Delta_3=0$; hence $h_3=0$ or $g_3=0$. We can assume that $h_3=0$, because the transformation

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* That is, a pencil with $\Delta\equiv 0$.

† Campbell, loc. cit., p. 398. See this paper also for the other typical forms of cubics.

‡ See A. D. Campbell, *Pencils of conics in the Galois fields of order 2^n*, American Journal of Mathematics, vol. 49 (1927), pp. 401–406 for such facts and for typical pencils of conics.

$x=x'$, $y=z'$, $z=y'$ reduces the case $g_3=0$ to the case $h_3=0$. Therefore we have $h_3=0$, $c_3g_3^2+1=0$. If $c_3\neq 0$, we put*

$$\lambda = \lambda' + \nu', \qquad \mu = \mu' + \nu', \qquad \nu = \nu' ;$$

$$x = \frac{1}{g_3}x', \qquad y = g_3 y', \qquad z = \frac{1}{c_3^{1/2}}z' ;$$

then

$$\lambda' = \frac{c_3^{1/2}}{g_3}\lambda'', \ \mu' = g_3^2\mu'', \ \nu' = \nu'' ;$$

and after dropping the primes and double primes from variables and parameters we obtain

$$\lambda yz + \mu x^2 + \nu(y^2 + z^2 + zx) = 0, \ K \equiv \lambda^2\mu + \nu^3 = 0, \tag{3}$$

whose cubic is reducible to the above typical cubic. If $c_3=0$, we get

$$\lambda yz + \mu x^2 + \nu(y^2 + zx) = 0, \ K \equiv \lambda^2\mu + \nu^3 = 0. \tag{4}$$

The cubics of these two nets are the same, but any transformation that is to send (3) into (4) must send the pencil $\nu=0$ of (3) into the same pencil of (4)† and so must have the form

$$T : \rho x = \alpha_1 x', \quad \rho y = \beta_2 y', \quad \rho z = \gamma_3 z' ;$$

or

$$\rho x = \alpha_1 x', \quad \rho y = \gamma_2 z', \quad \rho z = \beta_3 y'.$$

However, T sends (3) into a net containing the term in z^2, hence (3) and (4) are non-equivalent.

Next we suppose the cubic (2) to have a crunode and no real inflection, and we reduce the cubic to the form

$$\alpha\lambda^3 + \mu^3 + \lambda\mu\nu = 0,$$

with crunode at (0, 0, 1), where $\alpha\neq$cube and $2^n=3m+1$. As we discussed above the sort of conic in the net that must correspond to an inflection on the cubic so here we can readily show that to the crunode there must correspond a real line pair in the net. We put our net in the form

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* We call this change of parameters "using C_1 to rid C_3 of yz, and using C_2 to rid C_3 of x^2."

†Because the double line has to go into a double line, also $\nu=0$ in the (λ, μ, ν) plane joins the cusp to the inflection on the cubic and so must go into the same line on the other cubic.

$$\lambda C_1 + \mu(y^2 + zx) + \nu xy = 0, K^* \equiv \Delta_1\lambda^3 + (fh + g^2)\lambda^2\mu + b\lambda\mu^2 + fg\lambda^2\nu + c\lambda\nu^2 + f\lambda\mu\nu + \mu^3 = 0.$$

We must have $b=c=g=0$, $f=1$, $h=0$, $a=\alpha\neq$cube; hence we get

$$\lambda(\alpha x^2 + yz) + \mu(y^2 + zx) + \nu xy = 0. \tag{5}$$

Now we consider the cubic (2) to have a crunode and just one real inflection, and put the cubic in the form

$$\lambda^3 + \mu^2\nu + \lambda\mu\nu = 0,$$

with crunode at (0, 0, 1) and inflection at (0, 1, 0), which cubic exists only in the $GF(2)$. We reduce the net to the form

$$\lambda(y^2 + zx) + \mu C_2 + \nu xy = 0, \quad K \equiv \lambda^3 + b\lambda^2\mu + (fh + g^2)\lambda\mu^2 + f\lambda\mu\nu + fg\mu^2\nu + c\mu\nu^2 + \Delta_2\mu^3 = 0.$$

We have $b=0$, $f=g=h=a=1$, $c=0$. We use C_3 to rid C_2 of xy and we get

$$\lambda(y^2 + zx) + \mu(x^2 + yz + zx) + \nu xy = 0. \tag{6}$$

If for a net (1) the cubic (2) has a crunode and at least two real inflections, we put this cubic in the form

$$\lambda^2\mu + \lambda\mu^2 + \lambda^2\nu + \mu^2\nu + \lambda\mu\nu = 0,$$

existent only in $GF(2^n)$ for $n>1$, with crunode at (0, 0, 1) and inflections at (1, 0, 0) and (0, 1, 0). The net can be written

$$\lambda C_1 + \mu xy + \nu(z^2 + zx) = 0, \quad K \equiv \Delta_1\lambda^3 + fg\lambda^2\mu + c\lambda\mu^2 + (fh + h^2)\lambda^2\nu + b\lambda\nu^2 + \mu^2\nu + f\lambda\mu\nu = 0.$$

We find that $fg=c=fh+h^2=1$, $b=0$, $f=1$, $\Delta_1\equiv h+a+h^2=0$;

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* We shall from now on omit the subscripts from such coefficients as f_1, h_1, etc., and a_2, b_2, etc., and a_3, b_3, etc.

hence $g=h^2+h=a=1$. We use C_2 to rid C_1 of xy and C_3 to rid C_1 of z^2+zx, and we obtain

$$\lambda(x^2 + yz) + \mu xy + \nu(z^2 + zx) = 0. \tag{7}$$

We consider the net (1) to have a cubic (2) with an acnode and no real inflections, and we reduce the cubic to the form

$$\lambda^2\mu + \lambda^2\nu + \alpha\mu^2\nu + \lambda\mu\nu = 0,$$

with acnode at (0, 0, 1) and $\lambda^2+\alpha\mu^2+\lambda\mu=0$ as tangents. If we suppose that to an acnode there corresponds a real line pair in the net we are led to a contradiction, hence to the acnode there must correspond a conjugate imaginary line pair. Therefore we can put (1) in the form

$$\lambda C_1 + \mu xy + \nu(x^2 + z^2 + \alpha zx) = 0, \tag{n}$$

$$K \equiv \Delta_1\lambda^3 + fg\lambda^2\mu + c\lambda\mu^2 + (h^2 + f^2 + \alpha fh)\lambda^2\nu + \alpha^2 b\lambda\nu^2 + \mu^2\nu + \alpha f\lambda\mu\nu = 0.$$

We must have $c=b=\Delta_1=0$, $fg=\alpha f=h^2+f^2+\alpha fh$; hence $g=\alpha$, $fgh+af^2\equiv\alpha fh+af^2=0$. We use C_2 to rid C_1 of xy, and put

$$\lambda = \frac{f}{\alpha h}\lambda', \quad \mu = \frac{f^2}{\alpha h}\mu', \; \nu = \nu' \; ; \; x = x', \; y = \frac{\alpha h}{f^2}y', \; z = z'$$

and we get

$$\lambda(x^2 + yz + fzx/h) + \mu xy + \nu(x^2 + z^2 + \alpha zx) = 0, \tag{8}$$

where $h^2+f^2+\alpha fh+\alpha f=0$.

We deal with a net whose cubic has an acnode and just one real inflection, and we put this cubic in the form

$$\lambda^3 + \lambda^2\nu + \alpha\mu^2\nu + \lambda\mu\nu = 0,$$

which is nonexistent in $GF(2)$, with acnode at (0, 0, 1) and inflection at (0, 1, 0). We can put the net in the above form (n) with its cubic $K=0$. Hence we see that $fg=b=c=0$, $\alpha f=h^2+f^2+\alpha fh=\Delta_1\neq 0$; therefore $g=0, f\neq 0, af^2=\alpha f, a=\alpha/f$. We use C_2 to rid C_1 of xy, then put

$$\lambda = \frac{f}{\alpha}\lambda', \quad \mu = \frac{f^2}{\alpha}\mu_1', \quad \nu = \nu' \ ; \ x = x', \quad y = \frac{\alpha}{f^2}y', \quad z = z'$$

and we obtain

$$(9) \qquad \lambda(x^2 + yz) + \mu xy + \nu(x^2 + z^2 + \alpha zx) = 0.$$

Next we study the net (1) when it has a cubic (2) with an acnode and at least two real inflections, and we reduce this cubic to the form

$$\lambda^2\mu + \lambda\mu^2 + \lambda^2\nu + \mu^2\nu + \lambda\mu\nu = 0,$$

existent only in $GF(2)$, with acnode at (0, 0, 1) and one real inflection at (0, 1, 0). We can put our net in the above form (n) with $\alpha=1$. We must have $fg=c=h^2+f^2+fh=f=1$, $b=0$, $\Delta_1=0$; hence $f=g=c=1$, $h=1$ and $a=0$, or $h=0$ and $a=0$. We use C_2 to rid C_1 of xy and we get

$$(10) \qquad \lambda(z^2 + yz + zx) + \mu xy + \nu(x^2 + z^2 + zx) = 0.$$

Now we assume that the net (1) has a cubic (2) with no node or cusp and no real inflection, and we put this cubic in the form

$$\lambda^2\mu + \lambda\mu^2 + \alpha\lambda\nu^2 + \beta\mu\nu^2 + \lambda\mu\nu = 0.$$

The point (0, 1, 0) on this cubic was any point on the cubic, so we take a point that corresponds to a real line pair in the net, and put our net in the form

$$\lambda C_1 + \mu(z^2 + zx) + \nu xy = 0, \qquad K \equiv \Delta_1\lambda^3 + (fh + h^2)\lambda^2\mu + b\lambda\mu^2 + fg\lambda^2\nu + c\lambda\nu^2 + \mu\nu^2 + f\lambda\mu\nu = 0.$$

We see we must have $\Delta_1=0$, $f\neq 0$, $fg=0$, $b=fh+h^2=f\neq 0$; hence $g=0$, $\Delta_1\equiv af^2+ch^2=0$. We can therefore reduce our net to the form

$$(11) \quad \lambda\left[\frac{c(h^2 + 1)(h + 1)}{h^4}x^2 + y^2 + \frac{c(h + 1)}{h^2}z^2 + yz\right] + \mu(z^2 + zx) + \nu xy = 0,$$

with a cubic that is transformable into the above typical cubic with $\alpha=c(h+1)/h^2$, $\beta=(h+1)/h^2$.

We consider the net (1) when its cubic (2) has no node or cusp and just one real inflection and can be put in the form

$$\lambda^2\mu + \lambda\mu^2 + \alpha\lambda\nu^2 + \mu^2\nu + \lambda\mu\nu = 0,$$

with inflection at (0, 1, 0). Since there are two real tangents to the cubic from the point (0, 0, 1), this point must correspond to a real line pair in the net.* We can reduce the net to the above form (n). We have then $fg=c=1$, $fh+h^2=0$, $b=\alpha$, $f=1$, $\Delta_1=0$; hence $h=1$ or $h=0$, $g=1$, $a=\alpha$. We use C_2 to rid C_1 of xy and get

$$(12)\quad \lambda(\alpha x^2 + \alpha y^2 + z^2 + yz + zx) + \mu xy + \nu(z^2 + zx) = 0.$$

We suppose the cubic (2) has no node or cusp but just one real inflection and can be put in the form

$$\beta\lambda^3 + \lambda^2\nu + \lambda\nu^2 + \alpha\mu^2\nu + \lambda\mu\nu = 0.$$

We reduce the net (1) to the form

$$\lambda(y^2 + zx) + \mu xy + \nu C_3 = 0,\quad K \equiv \lambda^3 + b\lambda^2\nu + (fh + g^2)\lambda\nu^2 + c\mu^2\nu + fg\mu\nu^2 + f\lambda\mu\nu + \Delta_3\nu^3 = 0.$$

This gives us $b=fh+g^2=f\neq 0$, $fg=0$, $\Delta_3=0$; hence $g=0$, $h=1$, $a=c/f^2$. By an obvious transformation on λ, μ, ν alone, we get

$$(13)\quad \lambda(y^2 + zx) + \mu xy + \nu\left(\frac{c}{f^3}x^2 + y^2 + \frac{c}{f}z^2 + yz\right) = 0,$$

with cubic reducible to the above typical form with $\alpha=c/f$, $\beta=1/f$.

Next we consider the net (1) when its cubic (2) has no node or cusp but at least two real inflections, and reduce this cubic to the form

$$\alpha\nu^3 + \lambda^2\mu + \lambda\mu^2 + \lambda\mu\nu = 0,$$

with inflections at (0, 1, 0) and (1, 0, 0). We put our net in the form

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* This fact can be observed by noting the pencils of conics in the $GF(2^n)$ that have discriminants of the form $\lambda^2\mu$.

$$\lambda C_1 + \mu xy + \nu(y^2 + zx) = 0, \quad K \equiv \Delta_1\lambda^3 + fg\lambda^2\mu + c\lambda\mu^2$$
$$+ (fh + g^2)\lambda^2\nu + b\lambda\nu^2 + \nu^3 + f\lambda\mu\nu = 0.$$

We must have $fg = c = f \neq 0$, $fh + g^2 = b = \Delta_1 = 0$; hence $g = 1$, $c = f$, $h = 1/f$, $a = (f+1)/f^3$. Using C_2 to rid C_1 of xy we get

$$(14)\quad \lambda\left(\frac{f+1}{f^3}x^2 + fz^2 + fyz + zx\right) + \mu xy + \nu(y^2 + zx) = 0,$$

with cubic reducible to the above type form with $\alpha = 1/f$.

Finally we suppose our net (1) has no degenerate conic and therefore its cubic (2) is imaginary. We put our cubic in either one of the two forms

$$\lambda^3 + \alpha'\mu^3 + \beta'\nu^3 + \gamma'\lambda\nu^2 + \mu\nu^2 + \lambda\mu\nu = 0$$

or

$$\lambda^3 + \alpha'\mu^3 + \beta'\nu^3 + \lambda\mu\nu = 0,$$

where α' and β' are non-cubes and $2^n = 3m + 1$. Then the net (1) can be put in the form

$$\lambda(z^2 + xy) + \mu(\alpha x^2 + \beta y^2 + yz) + \nu C_3 = 0,$$
$$K \equiv \lambda^3 + \alpha\mu^3 + c\lambda^2\nu + (fg + h^2)\lambda\nu^2 + \alpha\mu^2\nu$$
$$+ (gh + \alpha f^2 + \beta g^2)\mu\nu^2 + g\lambda\mu\nu + \Delta_3\nu^3 = 0.$$

We must have $c = fg + h^2 = 0$, $g \neq 0$. We put

$$x = \frac{1}{g}x' + \frac{f}{g}y', \quad y = y', \quad z = \frac{h}{g}y' + z',$$

and we get

$$\lambda(z^2 + b_1y^2 + h_1xy) + \mu(a_2x^2 + b_2y^2 + f_2xz)$$
$$+ \nu(a_3x^2 + b_3y^2 + zx) = 0,$$
$$K \equiv h_1^2\lambda^3 + b_1\lambda\nu^2 + a_2f_2^2\mu^3 + a_3f_2^2\mu^2\nu + b_2\mu\nu^2$$
$$+ b_3\nu^3 + f_2h_1\lambda\mu\nu = 0.$$

This cubic must have one of the above forms, hence we see that $h_1^2 = f_2h_1 \neq 0$, $a_3 = 0$, so $f_2 = h_1 \neq 0$, and we get

$$(15)\quad \lambda(z^2 + \gamma y^2 + xy) + \mu(\alpha x^2 + \delta y^2 + yz) + \nu(\beta y^2 + zx) = 0,$$

where $\alpha\beta \neq 0$, α and β non-cubes, γ may be zero, $\delta = 1$ or 0.

We shall not derive the nets with degenerate cubics but merely list them below, because their derivation is so simple by the method employed by C. Jordan, loc. cit.

(16) $\lambda x^2 + \mu y^2 + \nu(z^2 + yz) = 0.$

(17) $\lambda x^2 + \mu y^2 + \nu yz = 0.$

(18) $\lambda x^2 + \mu y^2 + \nu(z^2 + xy) = 0.$

(19) $\lambda x^2 + \mu y^2 + \nu xy = 0.$

(20) $\lambda x^2 + \mu y^2 + \nu z^2 = 0.$

(21) $\lambda x^2 + \mu xy + \nu(z^2 + yz) = 0.$

(22) $\lambda x^2 + \mu xy + \nu yz = 0.$

(23) $\lambda x^2 + \mu xy + \nu(y^2 + z^2 + \alpha yz) = 0,$
$y^2 + z^2 + \alpha yz = 0$ irreducible.

(24) $\lambda x^2 + \mu xy + \nu(z^2 + zx) = 0.$

(25) $\lambda x^2 + \mu xy + \nu(y^2 + zx) = 0.$

(26) $\lambda x^2 + \mu xy + \nu xz = 0.$

(27) $\lambda x^2 + \mu(y^2 + xy) + \nu(z^2 + yz) = 0.$

(28) $\lambda x^2 + \mu(y^2 + xy) + \nu yz = 0.$

(29) $\lambda x^2 + \mu(y^2 + xy) + \nu(\alpha y^2 + z^2 + zx) = 0,$
$\mu^2 + \alpha\nu^2 + \mu\nu = 0$ irreducible.

(30) $\lambda x^2 + \mu(y^2 + xy) + \nu(z^2 + zx) = 0.$

(31) $\lambda xy + \mu xz + \nu(x^2 + y^2 + z^2 + \alpha yz) = 0.$

(32) $\lambda xy + \mu xz + \nu(y^2 + z^2 + \alpha yz) = 0.$

(33) $\lambda xy + \mu xz + \nu(x^2 + yz) = 0.$

(34) $\lambda xy + \mu xz + \nu yz = 0.$

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CERTAIN CLASS-NUMBER RELATIONS IMPLIED IN THE NACHLASS OF GAUSS*

BY E. T. BELL

1. *Introduction.* In a previous note† it was pointed out that the fundamental expansions of the elliptic theta constant products of the third degree, due to Hermite and Kronecker, follow by a mere change of sign in an algebraic identity from sections 292-3 of the *Disquisitiones Arithmeticae* of Gauss. The posthumous notes of Gauss, on what would now be called the transformation of the elliptic theta functions, contain implicitly a great many further class-number relations of types much more abstruse than the famous eight of Kronecker. To justify this assertion we need translate only two of the simplest of Gauss' identities into their equivalents in terms of class numbers. These will be sufficient; several of the remaining identities give class-number relations of equal simplicity and elegance. The present relations are allied to some that were stated without any clue to their origin by Liouville,‡ also to others deduced by Gierster§ from the theory of modular correspondences. It is surprising that these simple and extremely elegant relations should have been overlooked, lying as they do on the very surface of Gauss' work. All are written down by the usual device of comparing coefficients in different expansions of the same functions. It will not be necessary to preserve the algebraic details beyond the statement of the relevant identities (those given by Gauss for the transformation of the third order), as all the technique is standard. The final formulas (1)–(18) have been verified numerically.

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*Presented to the Society, San Francisco Section, April 7, 1928.

†This Bulletin, vol. 30 (1924), pp. 236-8.

‡For references, see Dickson's *History*, vol. 3, Chap. 6; also H. J. S. Smith, *Report on the Theory of Numbers*.

§Mathematische Annalen, vol. 21 (1883), p. 49.

2. *The Functions A, B, C, H, K.* If μ is an odd integer, we shall write $(-1\,|\,\mu)\equiv(-1)^{(\mu-1)/2}$, so that $(-1\,|\,-\mu)=-(-1\,|\,\mu)$. The functions $A, \cdots, K$ are defined as follows, where m is an odd integer >0, and in each case the $\sum$ refers to all μ such that, for m constant, the indicated (indeterminate) equation is satisfied; ν denotes an arbitrary (odd or even) integer $\gtreqless 0$, and μ, μ_1 are odd integers $\gtrless 0$.

$$\begin{aligned}
m &= 16\nu + 3\mu^2 &:&\quad A(m) = \textstyle\sum(-1\,|\,\mu)\mu.\\
m &= 4\mu_1^2 + 3\mu^2 &:&\quad B(m) = \textstyle\sum(-1\,|\,\mu)\mu.\\
4m &= \mu_1^2 + 3\mu^2 &:&\quad C(4m) = \textstyle\sum(-1\,|\,\mu)\mu.\\
m &= 12\nu^2 + \mu^2 &:&\quad H(m) = \textstyle\sum(-1\,|\,\mu)\mu.\\
4m &= 3\mu_1^2 + \mu^2 &:&\quad K(4m) = \textstyle\sum(-1\,|\,\mu)\mu.
\end{aligned}$$

For example, if $m=19$, the values of (ν, μ) from which $A(m)$ is computed are $(\nu, \mu)=(\pm 1, \pm 1)$, and hence $A(19)=4$; for $m=27$ we have (ν, μ) $(0, \pm 3)$, and hence $A(27)=-6$.

For these we find the following theorems.

(1) $m \equiv 3 \bmod 8$:

$$6A(m) = 3C(4m) = 2H(3m) = K(12m).$$

(2) $m \equiv 11 \bmod 24$:

$$A(m) = C(4m) = H(3m) = K(12m) = 0.$$

(3) $m \equiv 23 \bmod 24$:

$$B(m) = K(12m) = H(3m) = 0.$$

(4) $m \equiv 5 \bmod 6 : C(4m) = 0.$

(5) $m \equiv 7 \bmod 8$:

$$2H(3m) = K(12m) = -6B(m).$$

(6) $m \equiv 3,\ 19 \bmod 24 : 6A(m) = -K(4m).$

Certain of these can be seen by inspection, on noting what numbers are not represented in the given forms; all, however, follow at once by the means indicated in §4.

3. *Class-Number Relations.* For uniformity we shall express all the relations in terms of E alone. The notation

is that of H. J. S. Smith's *Report on the Theory of Numbers*; it agrees with Dickson's *History* (vol. 3, chap. 6); $F(n)$, $F_1(n)$ are respectively the numbers of odd, even classes for the negative determinant $-n$; $E(n)\equiv F(n)-F_1(n)$, so that 12 $E(n)$ is the total number of representations of n as a sum of three squares. By convention $F(0)=0$, $E(0)=1/12$, and a class equivalent to $a(x^2+y^2)$ ($a\equiv$an integer) contributes 1/2 to F or to F_1, while a class equivalent to $a(2x^2+2xy+2y^2)$ contributes 1/3 to F_1. For expressing E in terms of F and F_1 there are the following well known relations, $(n\geqq 0)$,

$$3E(8n+3)=2F(8n+3),\ E(8n+7)=0,\ E(4n)=E(n),$$
$$E(4n+1)=F(4n+1),\ E(4n+2)=F(4n+2),\ F(4n)=2F(n).$$

A convenient table for numerical verifications was given in a former paper.*

We shall call two relations *independent* if neither can be inferred from the other either by the theorems (1)–(6) of §2, or by the relations between E, F, F_1 stated above. The sense in which class-number relations are n-tuply infinite, $n>0$, was defined and amply illustrated elsewhere†; briefly, a class-number relation involving arbitrary uniform functions, or arbitrary odd or even uniform functions of precisely n independent variables, was called n-fold infinite; if no arbitrary function occurs in the relation, (we say that) the relation is *constant.* All the class-number relations reported in Dickson's *History* (loc. cit.) are constant in this sense; they all are degenerate cases of singly infinite relations. The relations next stated are equivalent to all those constant relations deducible from the transformation of the third order of the elliptic theta functions, combined with the transformation of the first order, and hence the corresponding Gauss formulas can give nothing further. It is to be particularly noted that we have included in this statement the transformations of the first order, as otherwise certain of the formulas may not be evident. The relations are not all independent in the sense above defined; we have given the dependent ones on account of their relative simplicity. The set is as follows. The $\sum$ refers to all integers $\nu \lesseqgtr 0$, or to all odd integers $\mu \gtrless 0$, such that the arguments of the summands are $\geqq 0$.

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* Tôhoku Mathematical Journal, vol. 19 (1921), p. 116.

† Quarterly Journal of Mathematics, vol. 49 (1923), pp. 322–337.

(7) $m \equiv 11 \bmod 24$:

$$\sum [E(m - 48\nu^2) - E(m - 3\nu^2)] = 0.$$

(8) $m \equiv 5 \bmod 6$:

$$\sum (-1)^\nu E(m - 3\nu^2) = 0.$$

(9) $m \equiv 23 \bmod 24$:

$$\sum [E(m - 12\mu^2) - E(m - 3\mu^2)] = 0.$$

(10) $m \equiv 3,\ 19 \bmod 24$:

$$4 \sum [E(m - 48\nu^2) - E(m - 3\mu^2)] = A(m).$$

(11) $m \equiv 1,\ 3 \bmod 6$:

$$8 \sum (-1)^\nu E(m - 3\nu^2) = C(4m).$$

(12) $m \equiv 7,\ 15 \bmod 24$:

$$4 \sum [E(m - 12\mu^2) - E(m - 3\mu^2)] = -B(m).$$

(13) $m \equiv 1 \bmod 24$:

$$12 \sum \left[E\left(\frac{m - \mu^2}{12}\right) - E\left(\frac{m - 4\nu^2}{3}\right)\right] = H(m),$$

$$\mu \equiv 1, 5 \bmod 6, \qquad \nu \equiv 2, 4 \bmod 6;$$

namely, the $\sum$ refers only to such odd $\mu \gtrless 0$, and to such arbitrary $\nu \lesseqgtr 0$, as indicated, and similarly in (14), (17).

(14) $m \equiv 13 \bmod 24$:

$$12 \sum \left[E\left(\frac{m - \mu^2}{12}\right) - E\left(\frac{m - 4\nu^2}{3}\right)\right] = -H(m),$$

$$\mu \equiv 1,\ 5 \bmod 6, \qquad \nu \equiv 1,\ 5 \bmod 6.$$

(15) $m \equiv 3 \bmod 8$:

$$12 \sum \left[E\left(\frac{m - 3\mu^2}{4}\right) - E(m - 48\nu^2)\right] = H(3m).$$

(16) $m \equiv 7 \bmod 8$:

$$12\sum\left[E\left(\frac{m-3\mu^2}{4}\right) - E(m-12\mu^2)\right] = -H(3m).$$

(17) $m \equiv 1 \bmod 6$:

$$24\sum\left[E\left(\frac{m-\mu^2}{3}\right) - E\left(\frac{m-4\nu^2}{3}\right)\right] = K(4m),$$

$$\mu \equiv 1,\ 5 \bmod 6, \qquad \nu \equiv 1,\ 2 \bmod 3.$$

(18) $m \equiv 7 \bmod 8$:

$$24\sum[E(m-3\mu^2) - E(m-12\mu^2)] = K(12m).$$

4. *Proofs*. As remarked in §1, it is unnecessary to go into details, although some of the formulas in §§2–3 follow only after several reductions. It is essential however to state from what identities of Gauss all of the preceding follow as indicated. The requisite relations are given (in a slightly different notation) by Gauss (*Werke*, vol. 3, p. 471), and are

$$3Pp = \frac{r^3}{R} - \frac{q^3}{Q}, \qquad Pp = \frac{Q^3}{q} - \frac{R^3}{r}$$

where, for $|x| < 1$,

$$p \equiv \sum x^{\nu^2}, \quad q \equiv \sum(-1)^\nu x^{\nu^2}, \quad r \equiv \sum x^{\mu /4},$$

the $\sum$ extending to all integers $\nu \gtreqless 0$, or to all odd integers $\mu \gtrless 0$, and P, Q, R are obtained from p, q, r respectively on replacing x by x^3. Independent proofs of Gauss' identities from the theta formula of Schröter have been given by Goring.*

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*Mathematische Annalen, vol. 7 (1874), pp. 325 et seq.

HOMOGRAPHIC CIRCLES OR CLOCKS*

BY LULU HOFMANN AND EDWARD KASNER

1. *Introduction.* This paper contains some theorems on the homographic transformation of one circle into another. The distribution of points on the transformed circle is studied under the assumption that the distribution on the original circle is uniform. The varying *density* on the transformed circle is characterised *im Grossen* by the *centroid,* a point defined analytically by a mean-value process. At an individual point it is measured by the absolute value of the ratio of corresponding arc elements on the two circles, the ratio taken from the original to the transformed element.

The distribution of points on the transformed circle proves to be uniform when and only when the centroid coincides with the center. When this coincidence does not occur, the density varies as follows.

It reaches its maximum and its minimum, two reciprocal values, in the end points of the diameter through the centroid, the *main diameter.*

It assumes every intermediate value twice and to every value occurring the reciprocal value likewise occurs.

Points of equal density lie symmetrically with regard to the main diameter, and points of reciprocal densities are collinear with the centroid.

If we draw a chord through the centroid and number the end points and the corresponding segments determined by the centroid arbitrarily as the first and second, then the density in the first point will be equal to the ratio of the length of the second segment to that of the first segment.

2. *Homographic Clocks. The Centroid.* We shall treat the problem analytically. As the most convenient tool we choose the linear fractional transformation with complex coefficients of one complex variable into another

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*Presented to the Society, February 6, 1928.

$$z = \frac{A + Bt}{C + Dt} \equiv \Omega(t).$$

This transforms the unit circle T,

$$t = e^{i\theta}, \qquad (\theta \text{ real}),$$

into the circle Z

$$z = \frac{A + Be^{i\theta}}{C + De^{i\theta}} \equiv w(\theta),$$

and these are the two circles we shall study.* They are homographically related on account of the linearity of the transformation formula. Geometrically the transformation is determined by three pairs of corresponding points on the two circles, because, since we are dealing with a homography, the anharmonic ratio of four points on T is equal to that of the four corresponding points on Z. A circle whose points are in definite homographic correspondence with the points of the unit circle, distinguished points we term a *homographic clock*; and more precisely a *positive* or *negative homographic clock* according as when t moves on the circle T in a certain sense, z moves on the circle Z in the same or in the opposite sense (the sense is uniquely determined on account of the one-to-one character of the transformation).

The concept of *homographic clock* belongs to both *inversive* and *projective* geometry.

With regard to the unit circle T, we note that $t = -C/D$ and $t = -\bar{D}/\bar{C}$ are corresponding points in inversion. The same relation must therefore hold with regard to the circle Z for the transformed points

$$z = \Omega\left(-\frac{C}{D}\right) = \infty, \quad z = \Omega\left(-\frac{\bar{D}}{\bar{C}}\right) = \frac{A\bar{C} - B\bar{D}}{C\bar{C} - D\bar{D}}.$$

And since the point at infinity is the conjugate of the center of the circle of inversion, it follows that Z has the center

$$N = \frac{A\bar{C} - B\bar{D}}{C\bar{C} - D\bar{D}}.$$

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*When in the following z occurs and nothing is expressly remarked to the contrary, then $z = w(\theta)$ is always meant.

Furthermore, forming $|N-w(0)|$ (we could also have chosen any other definite value for θ) we find that Z has the radius,

$$R = \left|\frac{AD - BC}{C\bar{C} - D\bar{D}}\right|.$$

The transformed circle Z will therefore degenerate into a line when and only when $|C| = |D|$, which we henceforth exclude. To obtain a general idea of the distribution of points on the new circle, we form the mean value M of z with regard to θ

$$M = \frac{1}{2\pi}\int_0^{2\pi}\frac{A + Be^{i\theta}}{C + De^{i\theta}}\,d\theta.$$

The point M indicates the *mean point* into which the points of T are transformed. We call it the *centroid* of the clock. For $D=0$ and $C=0$ the formulas simplify; we have

$$D = 0:\quad z = \frac{A}{C} + \frac{B}{C}e^{i\theta},\quad N = \frac{A}{C},\quad M = \frac{A}{C};$$

$$C = 0:\quad z = \frac{A}{D}e^{-i\theta} + \frac{B}{D},\quad N = \frac{B}{D},\quad M = \frac{B}{D}.$$

When neither $D=0$ nor $C=0$, the integrand becomes

$$\frac{B}{D} + \frac{AD - BC}{D}\cdot\frac{1}{C + De^{i\theta}},$$

and

$$M = \frac{1}{2\pi}\left[\frac{B}{D}\theta + \frac{1}{i}\left(\frac{A}{C} - \frac{B}{D}\right)\left(i\theta - \log\{C + De^{i\theta}\}\right)\right]_0^{2\pi}$$

$$= \frac{A}{C} - \frac{1}{2\pi i}\left(\frac{A}{C} - \frac{B}{D}\right)\epsilon,$$

where $\epsilon = 0$ when $|C| > |D|$ and $\epsilon = 2\pi i$ when $|C| < |D|$. Including the cases $D=0$ and $C=0$, we therefore have the following result.

The centroid M of the clock Z lies at the point

$$M = \frac{A}{C}, \quad |C| > |D|,$$

$$M = \frac{B}{D}, \quad |C| < |D|.$$

As the formulas show, the center and the centroid coincide when $D=0$ or $C=0$. Vice versa, these are the only cases for which this coincidence occurs. For according to the two values of M we have

$$N - M = \frac{A\bar{C} - B\bar{D}}{C\bar{C} - D\bar{D}} - \frac{A}{C} = \frac{\bar{D}(AD - BC)}{C(C\bar{C} - D\bar{D})},$$

$$N - M = \frac{A\bar{C} - B\bar{D}}{C\bar{C} - D\bar{D}} - \frac{B}{D} = \frac{\bar{C}(AD - BC)}{D(C\bar{C} - D\bar{D})},$$

and these expressions vanish when and only when $D=0$ or $C=0$, because $AD-BC=0$ would make z a constant independent of θ.

Among all homographic clocks, those of the types $z=A+Be^{i\theta}$ and $z=Ae^{-i\theta}+B$ corresponding to $D=0$ and $C=0$ are thus distinguished by the coincidence of N and M. It follows immediately from the formulas that when the point t moves with uniform velocity on the circle T, the point z moves with uniform velocity on the circle Z, in the same sense as t in the first case and in the opposite sense in the second case. We therefore term these clocks *uniform positive* and *negative clocks.*

The uniform positive and negative clocks

$$z = A + Be^{i\theta}, \quad z = Ae^{-i\theta} + B$$

are the only homographic clocks for which the center and the centroid coincide.

From the preceding discussion, by reason of continuity, we may deduce the following theorem.

Any homographic clock

$$z = \frac{A + Be^{i\theta}}{C + De^{i\theta}}$$

is a positive or negative clock according as $|C| > |D|$ *or* $|C| < |D|$.

3. *The Density.* After having thus studied the distribution of points on Z *im Grossen* by constructing the mean point or centroid M, we now proceed to study it in the infinitely small, that is, at the individual points of Z. It will be represented by a quantity which for obvious reasons we call the *density* and abbreviate by δ. The density will be characterised by $|dz/d\theta|$, the ratio of the corresponding arc lengths on the circles Z and T. For example, let us consider the circle T as uniformly covered with mass in such a manner that the mass on $d\theta$ is measured by $|d\theta|$. Then, since under the transformation which transforms $d\theta$ into dz, the mass may be considered constant, the density δ with which it is distributed on dz must be such that $|dz| \cdot \delta = |d\theta|$, that is, the density will be the reciprocal of $|dz/d\theta|$.

The formulas simplify, if we do not study $|dz/d\theta|$ itself but the related expression $|dz'/d\theta|$, where

$$z' = (z - N)\frac{D\bar{D} - C\bar{C}}{AD - BC} = \frac{\bar{D} + \bar{C}e^{i\theta}}{C + De^{i\theta}} \cdot$$

The circle Z' is obtained from Z by three simple operations in the z-plane: the translation represented by the vector $-N$ which carries the center of Z into the origin; the rotation through the argument of $(D\bar{D} - C\bar{C})/(AD - BC)$, and the magnification from the origin with the factor $1/R$. Hence it is obvious that the distribution of points on Z' is essentially the same as on Z (namely equivalent by similitude). We therefore omit the primes and continue with the circle

$$z = \frac{\bar{D} + \bar{C}e^{i\theta}}{C + De^{i\theta}} \equiv w(\theta)$$

which we now call Z. It is a circle of unit radius with its center at the origin. To fix on a definite case, let us assume $|C|>|D|$, so that

$$M = \frac{\bar{D}}{C}.$$

It will be seen immediately from the results obtained that they hold equally for the other case.

By differentiation we obtain

$$\left|\frac{dz}{d\theta}\right| = \left|\frac{ie^{i\theta}(C\bar{C} - D\bar{D})}{(C + De^{i\theta})^2}\right| = \frac{C\bar{C} - D\bar{D}}{C\bar{C} + D\bar{D} + \bar{C}De^{i\theta} + C\bar{D}e^{-i\theta}}.$$

Putting

$$|C| = c, \quad |D| = d, \quad \bar{C}D = cde^{ik},$$

we obtain

$$\left|\frac{dz}{d\theta}\right| = \frac{c^2 - d^2}{c^2 + d^2 + 2cd\cos(\theta + k)} \equiv \frac{1}{\delta(\theta)}.$$

The density on the transformed circle Z is independent of θ *when and only when* $D=0$ *or* $C=0$, *that is when the center and the centroid coincide.* This means that the clock is uniform.

By substituting $\theta'=\theta+k$ we obtain

$$z = \frac{\bar{C}d}{Dc}\cdot\frac{d + ce^{i\theta'}}{c + de^{i\theta'}} = w'(\theta'), \quad \left|\frac{dz}{d\theta}\right| = \frac{c^2-d^2}{c^2+d^2+2cd\cos\theta'} \equiv \frac{1}{\delta'(\theta')}.$$

It is obvious that the points $z=w'(\theta')$ and $z=w'(-\theta')$ lie symmetrically with regard to the diameter through the point $z=\bar{C}d/(Dc)$; and since $\delta'(\theta')=\delta'(-\theta')$, the density at these points is the same. (Vice versa any two points of equal density have the above described mutual position.) At the point $z=\bar{C}d/(Dc)$ itself, $\theta'=0$, and δ reaches its maximum

$$\delta_{max} = \frac{c + d}{c - d}.$$

At the diametrically opposed point $z=-\bar{C}d/(Dc)$, $\theta'=\pi$ and δ assumes its minimum

$$\delta_{min} = \frac{c-d}{c+d} = \frac{1}{\delta_{max}}.$$

Referring to the maximum and minimum value of the density, we denote these points by z_{max} and z_{min}:

$$z_{max} = \frac{\overline{C}d}{Dc}, \quad z_{min} = -\frac{\overline{C}d}{Dc}.$$

The diameter through z_{max} and z_{min} we term *the main diameter.*

The distribution of points on the circle Z is symmetric with regard to a definite diameter, the main diameter. The density is equal in points lying symmetrically with regard to this diameter and reaches its maximum and minimum, which are reciprocal, in its end points.

We have

$$\frac{z_{max}}{M} = \frac{c}{d}, \quad \frac{z_{min}}{M} = -\frac{c}{d},$$

$$\left|\frac{z_{max} - M}{z_{min} - M}\right| = \frac{c-d}{c+d} = \delta_{min} = \frac{1}{\delta_{max}}.$$

This means:

The centroid lies on the main diameter and divides it in such a manner that the ratio of its distance from the first end point to its distance from the second end point is equal to the density at the second point.

It is obvious, from the fact that the maximum and minimum density are reciprocal, that to every value of the density occurring the reciprocal value must also occur. We shall now show that for points with reciprocal densities a theorem holds which is the immediate extension of the one just stated.

Denote by $z = w(\theta)$ and $z^* = w(\theta^*)$ any two points collinear with the centroid and by l and l^* their respective distances from this point. We have

$$l = \left| \frac{\bar{D} + \bar{C}e^{i\theta}}{C + De^{i\theta}} - \frac{\bar{D}}{C} \right| = \left| \frac{(c^2 - d^2)e^{i\theta}}{C(C + De^{i\theta})} \right| .$$

Without calculating θ^* itself, one easily finds

$$l^* = \left| \frac{c^2 + d^2 + 2cd \cos(\theta + k)}{C(C + De^{i\theta})} \right| ,$$

so that

$$\frac{l}{l^*} = \frac{c^2 - d^2}{c^2 + d^2 + 2cd \cos(\theta + k)} = \frac{1}{\delta(\theta)} .$$

But similarly of course

$$\frac{l^*}{l} = \frac{1}{\delta(\theta^*)} .$$

Therefore

$$\delta(\theta) = \frac{1}{\delta(\theta^*)} .$$

On the transformed circle Z the density is reciprocal in two points z and z^* which are collinear with the centroid. The density at the point $z(z^*)$ is equal to the ratio of the distances of $z^*(z)$ and $z(z^*)$ from the centroid.

Appendix on Polygenic Functions

BY EDWARD KASNER

The geometric concept of a *clock* first presented itself in the theory of *polygenic functions*[†]

$$w = \phi(x,y) + i\psi(x,y),$$

as follows. The derivative

$$\frac{dw}{dz} = L + le^{-2i\theta} ,$$

where

$$L = \mathfrak{D}w = \tfrac{1}{2}(D_x - iD_y)w ,$$

$$l = \mathfrak{P}w = \tfrac{1}{2}(D_x + iD_y)w .$$

[†] See E. Kasner, Science, vol. 66 (1927), pp. 581–582, and Proceedings of the National Academy of Sciences, vol. 14 (1928), pp. 75–82.

This gives for each point $x+iy$ a uniform negative clock with center vector L and phase vector l.

It is to be noticed that the angle θ of the preceding joint paper is here 2θ because we are dealing with a correspondence between the points of the derivative circle and a pencil of directions, rather than with a unit circle.*

If w_1 and w_2 are any two polygenic functions, then the derivative is

$$\frac{dw_1}{dw_2}=\frac{L_1+l_1e^{-2i\theta}}{L_2+l_2e^{-2i\theta}},$$

and so defines a general homographic clock. In particular, the clock degenerates into a point when and only when the two functions are of the same class.† The clock becomes a straight line (the case omitted in the joint paper) when and only when the jacobian of ϕ_2 and ψ_2 vanishes. Finally, the clock becomes uniform (negative or positive) when and only when the denominator function w_2 is an analytic function of $x+iy$ or $x-iy$. Hence the mean value (centroid)

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{dw_1}{dw_2}d\theta$$

coincides with the center of the derivative clock when and only when w_2 is an analytic function of $x+iy$ or $x-iy$. The theory of congruences of clocks will be studied elsewhere.

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* See the abstracts in this Bulletin, vol. 34 (1928), 152, pp. 263–264.

† According to the terminology introduced by Hedrick, Ingold, and Westfall, *Theory of non-analytic functions of a complex variable*, Journal de Mathématiques, (9), vol. 2 (1923), 327–342.

CONCERNING ACCESSIBILITY IN THE PLANE AND REGULAR ACCESSIBILITY IN n DIMENSIONS*

BY G. T. WHYBURN

1. *Accessibility in the Plane.* The point set K is said to be everywhere accessible from a point set R provided that if A and B are any two points belonging to K and R respectively, then there exists a simple continuous arc AB from A to B such that $AB-A$ is a subset of R. In my paper *Concerning plane closed point sets which are accessible from certain subsets of their complements*,† among other results, I proved the following theorem.

THEOREM A. *If, in a plane S, K is a point set such that there exist three mutually exclusive connected subsets R_1, R_2, and R_3 of $S-K$ such that K is everywhere accessible from R_1 and R_2 and every point of K is a limit point of R_3, then K contains not more than two points.*

In this section it is purposed to establish some results which are related to Theorem A and in the proof of which use will be made of Theorem A. It is obvious from Theorem A that there does not exist any plane continuum, which is everywhere accessible from each one of a particular group of three of its complementary domains. It is not so apparent, however, that there does not exist a plane continuum each point of which is accessible from each one of some group of three of the complementary domains of this continuum. This result will be demonstrated in this present paper as an immediate consequence of a somewhat more general theorem. It will also be shown that if each point of a bounded plane continuum M is accessible from each of two complementary domains of M, then M is a Menger regular curve* which has at most a countable number of cut points.

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* Presented to the Society, December 28, 1927.

† Offered to the Proceedings of the National Academy of Sciences. See Theorem 1, and remark at the end of proof of Theorem 1.

THEOREM 1. *If G is any countable collection of mutually exclusive connected point sets in the plane S, T denotes the sum of all the point sets of the collection G, and K denotes the set of all points X in $S-T$ such that X is accessible from at least three of the sets of G, then K is countable.*

PROOF.† Suppose, on the contrary, that K is uncountable. Then since G is countable and each point of K is accessible from at least one set of the collection G, it follows that there exists a set R_1 of the collection G such that each point of an uncountable subset K_1 of K is accessible from R_1. Since K_1 is uncountable and the collection G_1 obtained by omitting R_1 from the collection G is countable, and since each point of K is accessible from at least one set of G_1, it follows that there exists a set R_2 of the collection G_1 such that each point of an uncountable subset K_2 of K_1 is accessible from R_2. Let G_2 be the collection obtained by omitting the set R_2 from the collection G_1. Then since K_2 is uncountable and G_2 is countable, and each point of K_2 is accessible from at least one set of the collection G_2, there exists a set R_3 of the collection G_2 such that each point of an uncountable subset K_3 of K_2 is accessible from R_3. But then every point of K_3 is accessible from each of the three mutually exclusive connected subsets R_1, R_2, and R_3 of $S-K_3$, and hence, by Theorem A, K_3 can contain at most two points. But K_3 is uncountable. Thus the supposition that K is uncountable leads to a contradiction; and the truth of Theorem 1 is therefore established.

Since the complementary domains of every plane continuum are countable, we have the following immediate corollaries.

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* See the definition of this term below.

† Professor W. A. Wilson has kindly called my attention to the following very concise proof of Theorem 1: Since G is countable, the number of groups of three sets of G is countable. It follows from Theorem A that at most two points of K are accessible from each set of any one of these groups. Hence the set of points of K must be countable.

COROLLARY 1. *If M is any plane continuum, then not more than a countable number of points of M are accessible from each of three complementary domains of M.*

COROLLARY 2. *There does not exist, in the plane, a continuum M every point of which is accessible from each of three distinct complementary domains of M.*

DEFINITION. A continuum M is said to be a Menger regular curve* provided that if P is any point of M and ϵ is any positive number, then there exists a connected open subset R of M which contains P, is of diameter $<\epsilon$, and whose M-boundary is finite. The M-boundary of an open subset R of a point set M is the set of all those points of $M-R$ which are limit points of R.

THEOREM 2. *If every point of a plane continuum M is accessible from at least two of the complementary domains of M then M is a Menger regular curve; and if M is bounded, then it has not more than a countable number of cut points.*

PROOF. I shall first show that M is a Menger regular curve. Suppose, on the contrary, that M is not a Menger regular curve. Then by a theorem of Menger's† there exists a subcontinuum H of M no point of which is a regular point of M. Now let $D_1, D_2, D_3, \cdots$ be the complementary domains of M. For each i, let B_i denote the boundary of D_i, and let A_i denote the set of points $H\cdot B_i$. Then, for each i, A_i is a closed point set. Since, by hypothesis, every point of H is accessible from one of the domains $D_1, D_2, \cdots$, it follows that $H=A_1+A_2+A_3+\cdots$. It is well known that no continuum is the sum of a countable number of closed point sets each of which is totally disconnected. Therefore, for some integer p, A_p contains a continuum K. Now for each integer $i\neq p$, let C_i denote the set of points $K\cdot B_i$. Then since, by hypothesis, every point of K is accessible from at least one of the domains $D_1, D_2, \cdots, D_{p-1}, D_{p+1}, \cdots$, it follows that $K = C_1 + C_2 + \cdots + C_{p-1} + C_{p+1} + \cdots$. And since, for each i, C_i is closed, it follows just as above that there exists an integer $q \neq p$ such that C_q contains a bounded continuum T. The continuum T belongs to the boundary of each of the domains D_p and D_q.

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* See K. Menger, *Grundzuge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925), pp. 277–306.

† See K. Menger, loc. cit., p. 288, Theorem 8.

Now all, save possibly a countable number, of the points of T must be accessible from D_p. For if on the contrary, an uncountable subset Q of T exists, no point of which is accessible from D_p, then since, by hypothesis, every point of T is accessible from at least two complementary domains of M, it follows by an argument similar to that given in the proof of Theorem 1 that there exists an uncountable subset W of Q and two integers j and k each distinct from p, and such that each point of W is accessible from each of the domains D_j and D_k. But since every point of W is a limit point of D_p, this contradicts Theorem A. Hence all, save possibly a countable number, of the points of T are accessible from D_p. Likewise all, save possibly a countable number of the points of T, are accessible from D_q. Therefore there exists a countable, or vacuous, subset U of T such that every point of $T - U$ is accessible from each of the domains D_p and D_q.

Let X and Y be two points of $T - U$. Then since X and Y are accessible from each of the domains D_p and D_q, it readily follows that there exist points P and Q in D_p and D_q respectively and arcs PXQ and PYQ from P to Q which have in common only the points P and Q, and such that the point set $PX + PY - (X + Y)$ is a subset of D_p and the set $QX + QY - (X + Y)$ is a subset of D_q. Let J denote the simple closed curve $PXQYP$. Then since T is bounded and has only the points X and Y in common with J, it follows by a theorem of Rosenthal's* that either the interior or the exterior of J

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* A. Rosenthal, *Teilung der Ebene durch irreduzible Kontinua*, Sitzungsberichte der Mathematisch-Physikalischen Klasse der Bayerischen Akademie der Wissenschaften zu München, 1919, p. 104.

contains a connected subset g of T such that each of the points X and Y is a limit point of g. The two cases are practically alike, so let us suppose that g is a subset of the interior of J. Let G denote the subset of M which lies within J, and let N denote the set of points $G+X+Y$. Then since $g+X+Y$ is connected it follows that N is a continuum.

Let E denote the set of all those points of $T-U$ which belong to g. Then since U is countable, and g is a connected subset of T, it follows that E is uncountable. Let Z be any point of E. Then since Z is accessible from each of the domains D_p and D_q, it readily follows that there exists an arc AZB, where A is a point on the arc XPY of J and B a point on the arc XQY of J, and such that $AZ-Z$ is a subset of D_p and $BZ-Z$ is a subset D_q. The arc AZB divides the interior of J into two regions R_x and R_y whose boundaries contain X and Y respectively. And if S_x denotes the set of points $N\cdot R_x+X$ and S_y the set of points $N\cdot R_y+Y$, then it is clear that S_x and S_y are mutually separated sets, and that $S_x+S_y=N-Z$. Therefore each point Z of E is a cut point of N. But since E is uncountable, then by a theorem of the author* it follows that E contains at least one point V which is a regular point (in fact a point of Menger order two) of N; and since $M-N$ contains no point within J, it follows that V is a regular point of M. But V belongs to H, and by supposition no point of H is a regular point of M. Thus the supposition that M is not a Menger regular curve leads to a contradiction.

Now suppose M is bounded, and that, contrary to the second part of this theorem, it has uncountably many cut points. Then by the above quoted theorem of the author, at least one of these cut points of M must be a point of Menger order two of M. But by another theorem of the author† a cut point of Menger order two of a bounded continuum M can belong to the boundary of at most one complementary domain of M; and by our hypothesis every point of M must belong to the boundaries of at least two complementary domains of M. Thus the supposition that M has uncountably many cut points leads to a contradiction, and the theorem is proved.

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* G. T. Whyburn, *Concerning the cut points of continua*, Transactions of this Society, vol. 30 (1928), No. 3. The theorem used here states that *all, save possibly a countable number, of the cut points of a plane continuum M are points of Menger order two of that continuum.*

† Loc. cit., Theorem 10.

It is of interest to note that every bounded continuum satisfying the hypothesis of Theorem 2 is a type of continuous curve which has elsewhere[*] been called a *two-way continuous curve*, that is, every two of its points can be joined by two different arcs which lie wholly in the curve.

2. *Regular Accessibility in n Dimensions.* A limit point P of a point set R is said to be regularly accessible[†] from R provided that if ϵ is any positive number, then there exists a $\delta_{\epsilon p} > 0$ such that every point X of R whose distance from P is $< \delta_{\epsilon p}$ can be joined to P by an arc XP of diameter $< \epsilon$ such that $XP - P$ is a subset of R.

THEOREM 3. *In order that the boundary point P of a connected open subset*[‡] *R of a continuous curve M in a euclidean space of n dimensions should be regularly accessible from R, it is necessary and sufficient that the point set $R+P$ should be connected im kleinen at P.*

PROOF. That the condition is necessary is an immediate consequence of the definition of regular accessibility. I shall show that the condition is sufficient. The case where P belongs to R is evident, so let us suppose that P belongs to $M - R$. Let ϵ be any number > 0. By hypothesis there exists a $\delta_{\epsilon p} > 0$ such that every point X of R at a distance $< \delta_{\epsilon p}$ from P can be joined to P by a connected subset of $R+P$ of diameter $< \epsilon/2$. Let G denote the set of all those points of R whose distance from P is $<\delta_{\epsilon p}$, let X_1 be any point of G, and let R_1 denote the maximal connected subset of the set of all those points of R whose distance from P is $<\epsilon/2$ which contains X_1. Then R_1 is a connected open subset of M which is of diameter $<\epsilon$; furthermore, P is a limit point of R_1. For suppose it is not. By hypothesis a connected subset N of $R+P$ exists which contains both P and X_1 and is of diameter $<\epsilon/2$. Since N is connected and contains the point X_1 of R_1 and the point P which is neither a point nor a limit point of R_1, it therefore contains a point Q, distinct from P, which belongs to $R-R_1$ and is a limit point of R_1. But Q is at a distance $<\epsilon/2$ from P, and by definition R_1 is a maximal connected subset of the set of all points of R a distance $<\epsilon/2$ from P. Thus the supposition that P is not a limit point of R_1 leads to a contradiction. Hence P is a limit point of R_1. In a similar way it follows that there exists a connected open subset R_2 of M which is a subset of R_1, is of diameter $<\epsilon/2$, and has P for a limit point; and so on. In general, for each positive integer n, a connected open subset R_n of M exists which is a subset of R_{n-1}, is of diameter $<\epsilon/n$, and has P for a limit point. For each n, the set R_n contains a point X_n and an arc* X_nX_{n+1}. It is easy to see that the point set $P+X_1X_2+X_2X_3+\cdots+X_nX_{n+1}+\cdots$ is closed and that it contains an arc X_1P of diameter $<\epsilon$ such that X_1P-P is a subset of R. And since X_1 is any point of G, it follows that P is regularly accessible from R.

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[*] See G. T. Whyburn, *Two-way continuous curves*, this Bulletin, vol. 32 (1926), pp. 659–663.

[†] See G. T. Whyburn, *Concerning the open subsets of a plane continuous curve*, Proceedings of the National Academy, vol. 13 (1927), pp. 650–657.

[‡] The subset R of a closed point set M is an open subset of M provided that $M - R$ is either vacuous or closed.

COROLLARY. *In order that a boundary point P of a domain D, in n-dimensional space, should be regularly accessible from D it is necessary and sufficient that the set of points $D+P$ should be connected im kleinen.*

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* See R. L. Moore, Mathematische Zeitschrift, vol. 15 (1922), Theorem 1.

BELL ON ALGEBRAIC ARITHMETIC

Algebraic Arithmetic. By E. T. Bell. Colloquium Publications of the American Mathematical Society, volume 7, 1927. 180 pp.

This book of marked originality is of vital interest to advanced students in various branches of mathematics, including the theory of numbers, abstract algebra, elliptic and theta functions, Bernoullian numbers and functions, and the foundations of mathematics.

A central feature is the new presentation of the author's principle of arithmetical paraphrases, which won him the Bôcher prize in 1924, jointly with Professor Lefschetz. This general principle serves to unify and extend many isolated results in the theory of numbers. In a modified form, this principle may be stated very simply for a special, but typical, case. Suppose that by two developments of an elliptic or theta function we have found a linear identity between the cosines of the angles a_ix+b_iy for $i=1, 2, \cdots$; then if $f(a,b)$ is any function which is unaltered by the change of a to $-a$ or of b to $-b$, we have the same linear identity between the $f(a_i,b_i)$. Since we are free to choose the even function f, we obtain an infinitude of arithmetic facts.

The general principle of paraphrase (pp. 66–68) starts with a linear identity between terms each a product of r cosines and s sines and leads to the same identity between values of any function which is even in r variables and odd in s. It holds also in the extended sense that the variables are any one-rowed matrices. Several illustrations are given to show in detail how the initial identities are obtained from theta functions of one or more arguments. The applications include many classic problems in the theory of numbers.

A leading feature of the book seems to the reviewer to be its success in a systematic attempt to find a unified theory for each of various classes of related important problems in the theory of numbers, including its interrelations with algebra and analysis. An older case was the development of a theory of abstract groups which includes as special cases the really vital results and processes of the various theories of concrete groups, such as groups of permutations of letters, groups of linear or other substitutions, and groups of motions. The gain is not merely in great economy, but in the clarification which results from the exclusion of the irrelevant and the concentration on the relevant. Here it was clearly necessary to proceed abstractly. To the immature mind, abstract means obscure. To anyone really initiated in modern mathematics, an abstract theory is the really clear one, since irrelevant facts have been discarded, and the vital facts and processes alone are retained. As between an abstract theory and a concrete one, the former is far more likely to be clear and sound, since its very nature requires that the assumptions be explicitly stated (as postulates) and that the deductions be made by logical processes and without lapse into intuition, consciously or not.

It is therefore wholly to its credit that Bell's book is so largely abstract. In this way he has secured the utmost generality and hence insured permanency to his work. There is also the following important immediate gain. After constructing abstract theories for various classes of problems, he was in a position to decide if those theories are distinct or are (abstractly) identical. In the latter case we have not merely the often astonishing conclusion that two quite different theories are really equivalent, but the great advantage of being able to apply in a particular case whichever of the two theories is best adapted to it.

The book is however not abstract for the sake of being abstract, but because it gives the essence of a vast array of concrete results traced to their true sources and easily deduced from these few sources. It is not to be overlooked that the author has a minute first-hand acquaintance with the vast array of concrete facts in the theory of numbers. It would require many volumes to expound them and the result would bewilder the reader. How much better to have a brief book which epitomizes all these facts under a few abstract theories. This original and scholarly book is an honor to American mathematics.

L. E. DICKSON

FORSYTH ON CALCULUS OF VARIATIONS

Calculus of Variations. By A. R. Forsyth. Cambridge University Press, 1927. xxii+656 pp.

Weierstrass made three very important contributions to the theory of the calculus of variations. Earlier writers had deduced necessary conditions on a minimizing arc $y=y(x)$ $(x_1\leqq x\leqq x_2)$ by comparing the value which such an arc gives to the integral to be minimized with the value given by neighboring arcs of the form $y=y(x)+\delta y(x)$, where $\delta y=a\eta(x)$. By the use of variations δy of another type, Weierstrass deduced a new necessary condition. He also introduced the parametric representation of curves, $x=x(t)$, $y=y(t)$, into the theory. This was important from the standpoint of the older writers because it enables one to vary impartially either of the variables x or y. The effort to do this had caused considerable difficulty and misunderstanding in the earlier theory of the non-parametric case. The parametric theory is even more valuable, however, because it removes the geometric restrictions on the form of curves which are imposed by the non-parametric representation $y=y(x)$, a removal which is necessary for the complete investigation of many geometric problems. Finally Weierstrass formulated clearly the problems which he studied, distinguished clearly between conditions which are necessary for a minimum and those which are sufficient, and devised an ingenious sufficiency proof which under certain circumstances established the minimizing property of an arc $y=y(x)$ as compared with all other neighboring arcs $y=y(x)+\delta y(x)$, irrespective of the form of the variation δy.

In the book here reviewed Professor Forsyth shows that he has been influenced by the first two of the contributions which have just been mentioned as being due to Weierstrass, the so-called Weierstrassian necessary condition and the introduction of the parametric representation of curves. The method which he uses is that of expansion in series of powers of the variations of the functions defining the curves, the various necessary conditions being deduced by the study and transformation of the first and second variations of the integral to be minimized. The cases which he considers are problems in the plane for integrals whose integrands contain first, or first and second, derivatives of the functions defining the arc, in both non-parametric and parametric forms, the analogous problems in three-space, isoperimetric problems, double integrals containing first or second partial derivatives and isoperimetric problems involving double integrals, and triple integrals involving first partial derivatives.

In the first paragraph of his introduction Professor Forsyth says: "The range of Mathematical Analysis, usually known as the Calculus of Variations, deals with one of the earliest problems of ordinary experience. The requirement was, and is, to obtain the most profitable result from imperfectly postulated data; and the data may possibly be subjected to conditions, which likewise are imperfectly postulated. When data and conditions are expressed in analytical form, the necessary mathematical calculations can not be effected directly, because of some essential deficiency in the information." It seems to me that this rather vague type of language, which is also used in other places in the book, gives a very vague impression of the theory of the problems of the calculus of variations. The data of these problems can be postulated with accuracy, and the properties deduced for their solutions will in general characterize those solutions completely.

There is a strong tendency among modern writers to avoid as far as ossible the elaborate transformations of the second variation which characterized earlier theories. Legendre's condition, which was originally deduced from the second variation, is in many cases an immediate consequence of the condition of Weierstrass which can be proved without using the second variation at all. Jacobi's condition is a consequence of the beautiful envelope theorem of Darboux, Zermelo, and Kneser, which is a generalization of the string property of the evolute of a curve, or it may be deduced from the second variation by a method which requires almost no manipulation.* For these reasons I always regret to see the elaborate transformation theory of the second variation given such a prominent place as Professor Forsyth has given it in his book.

In a paper read at the Toronto Congress, and which has recently been printed for publication in the Proceedings of the Congress, I showed that the result of the transformation which Clebsch devised for the second variation, in the very general case of the Lagrange problem, is obtained as an immediate consequence of a theorem of Weierstrass expressing the difference between the values of an integral on two neighboring curves as an integral of its E-function. So much effort has been spent on these transformations that it seemed to me worth while to try to coordinate the transformation theory with other well known results in the calculus of variations. If the second variation must be transformed it is desirable to do it in some such way as this.

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* See the references in the footnote, this Bulletin, vol. 32 (1926), p. 392; for the parametric case, see the Transactions of this Society, vol. 17 (1916), p. 195.

It has frequently surprised me to find how much effort the early writers on the calculus of variations spent on the theory of problems for which the integrand functions contain higher derivatives than the first. There are very few special problems of this sort in the literature. Furthermore the theory of such problems is a special case of the theory of the Lagrange problem which has a much wider application. Professor Forsyth discusses only relatively simple forms of the problem of Lagrange but he studies at length problems in the plane and in three-space for which second derivatives occur in the integrand. The theory of the second variation is of course very complicated for these problems. The parametric theory for such cases could doubtless be simplified by the introduction of invariants under parametric transformation in place of higher derivatives in the integrand, as has been done by Radon.*

The last two hundred pages of the book are devoted to double and triple integrals, with applications to minimal surfaces, and the usual necessary conditions are deduced from the first and second variations. The method of transforming the second variation of parametric problems is a modification due to Professor Forsyth of that used by Kobb. The two final chapters are devoted to double integrals whose integrands contain second derivatives, and to triple integrals, and the theory of the second variation for these cases is more completely treated than I have seen it done elsewhere.

G. A. Bliss

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* *Über das Minimum des Integrals* $\int_{s_0}^{s_1} f(x, y, \theta, \kappa)ds$, Wiener Sitzungsberichte, vol. 119 (1910), p. 1257.

ELDERTON ON CURVES AND CORRELATION

Frequency Curves and Correlation. By W. Palin Elderton. Second edition. London, C. & E. Layton, 1927. viii+239 pp. Price 15 shillings net+6d postage.

In such a rapidly developing subject as statistics for an early book to celebrate its twenty-first birthday with an enlarged second edition is phenomenal. The substantial value of the first edition of this book has been enhanced by addition of new material, rearrangement of the old, and a more attractive appearance. The formidable-looking "Key to the Actuarial Terms and Symbols Used" which for the non-actuarial reader stood as a *cave canem* at the beginning of the book has been transferred to the end, and other such rearrangements have been made, rendering the book more readable by avoiding the diversion of attention from the context. There are 67 more pages than in the original edition. This increase is due to the growth in number of Pearson's types of curves from seven to twelve, to the addition of a chapter on partial correlation of the simplest type, and to a somewhat fuller treatment of many topics.

The chief distinctive feature of the book is still the full discussion of the Pearson frequency curves. Now it is true that this system of curves is strictly empirical and does not offer the possibilities, suggested by seminvariants, of incorporating into the curve-fitting process ideas of a general sort regarding properties of compound distributions. Nevertheless they are convenient. Moreover, the modern development of statistics has been so strongly influenced by Karl Pearson that anyone wishing to make his way through statistical literature needs a knowledge of these frequency curves for the same reason that one needs a knowledge of the classics. Such a knowledge may be gained by the abridged reading of the book suggested in an appendix. A considerable part of the book, given to the details of talculation, may be omitted by the casual reader but is indispensible to che curve-fitter.

In a chapter discussing the curves of Edgeworth and the Scandinavians the author concludes on the basis of goodness of fit in examples, and on account of the difficulties connected with series converging, if at all, at uncertain speed, that these systems lack the practical advantages of Pearson's.

The chapter on correlation is radically changed by addition of new material, some of the old material being removed to an appendix. The increased space on correlation is in accord with the increased importance of the subject. The Hardy summation method, which ought to be better known, is applied to the calculation of correlations as well as of univariate moments.

The treatment of correlation of characters not quantitatively measurable presents clearly the chief work of Pearson on the subject. However there ought to be a warning against overworking these methods. In several cases which have come to the reviewer's attention biserial and tetrachoric correlations have been computed with vast labor when all that was wanted was to decide whether there was any correlation at all. These functions, with their dubious assumption of normal correlation and their large and hazy probable errors, are far less efficient for the purpose than the simpler method of contingency, which yields an immediate and correct interpretation in terms of probability. To take the classic example used in the text, the calculation of an exact value for the correlation between vaccination and recovery from smallpox might well be postponed until, by a much easier and more dependable process, it has been shown that this correlation is significantly greater than zero. So much use of correlation is heuristic that the point is important.

Inverse probability lurks in the book without ever coming into full view. The revolutionary work of R. A. Fisher has found no place in the new edition, so that Chapter X, on probable errors, and Chapter XII, on the correlation ratio and contingency, are partially obsolete. For example, page 187 is directly contradicted by Fisher's conclusion in Metron (vol. 5 (1925), No. 3, p. 96). Still more serious from the standpoint of this book are the modifications in the Pearson curve-fitting theory which Fisher has shown to be necessary.

The argument used on page 181 in deriving the formula for the correlation between sampling deviations in class frequencies is unconvincing; for why, in case the number found in a class falls short of expectation, must the corresponding excess in other classes be distributed in exact proportion to their sizes? This can certainly not be expected to happen in all or even in many cases; only by some process of averaging not fully defined can the idea be used. The correlation formula in question may however be reached in a purely algebraic manner by generalizing to n dimensions the ordinary derivation of the standard deviation of a Bernoulli distribution.

On page 48 there is an allusion to some formulas not readily found, and on page 43 it is not clear why, when the criterion is infinite, a root of the quadratic must be infinite rather than zero. It is difficult to see the use or derivation of the functional equation on page 153, and it may possibly mislead the incautious. However the book is on the whole excellent. It supplies a considerable body of valuable information in compact form, and will probably always be the standard book of reference on the Pearson curves.

HAROLD HOTELLING

CHARBONNIER ON EXTERIOR BALLISTICS

Traité de Balistique Extérieure. By P. Charbonnier. Volume II. *Problème Balistique Principal, Troisième Partie; Les Théories Balistiques.* Paris, Doin, Gauthier-Villars, 1927. 797 pp.

This volume follows Volume I, (of 637 pages, 1921) devoted to *Les Théorèmes Généraux de la Balistique,* and is to be succeeded by a third volume in rational ballistics concerning secondary ballistic problems. Volume I was divided into four books and this volume also is divided into four books numbered from five to eight as follows: Book V, Monomial Resistance; Book VI, Flat Fire; Book VII, Ballistic Series; Book VIII, The Calculation of Trajectories by Successive Arcs. From the Introduction to Volume I, one learns that three further volumes on ballistic science are planned, Volume IV, Experimental Exterior Ballistics, Volume V, History of Exterior Ballistics (with bibliography), Volume VI, Tables. The subject of interior ballistics, which differs considerably in its problems and methods, is not to be touched upon in this series.

The present undertaking is essentially a careful (save for numerous misprints) revision with amplification of the author's two volume work (*Balistique Extérieure Rationnelle*) of 1907. Not only is the subject treated throughout in greater detail, but additional carefully written chapters on newer methods have been introduced. In 1907, Charbonnier's work placed him as one of the conspicuous ballisticians of the world. Since that time he has been made a general in the French Army, and is here progressing with customary French lucidity upon the most ambitious text on ballistics ever undertaken. This should ensure respectful consideration from the few technical scientists seriously interested in ballistics.

Works on ballistics from the time of Bashforth, tend to have a curiously provincial air. They can hope to be of significance chiefly to artillery officers, who, however, seldom have the preparation, interest, or opportunity to study them. Most experimental work is kept in confidential army files. There is also a strong patriotic tendency toward nationalistic bias that army training and army contacts naturally accentuate. Most books on ballistics give the reader the impression that progress in the subject is practically confined to the compatriots or even to the friends of the author. This is noticeable even since the war. This book is no exception.

Ballistics is inherently a branch of engineering. Not only are there a distractingly large number of physical factors present in the problem, but the number of independent variables that are actually taken into account in firing is so large, and these variable factors are to a great extent so uncontrollable, that simple accurate formulas are universally regarded as out of the question. Rules based on the normal probability curve are used in actual field conditions, and the numerical tables upon which original computations are constructed are themselves obtained by statistical methods involving more or less arbitrary assumptions. The only methods used in the past have been to relegate the less important variations to a secondary place (not to be ignored, but to be treated as linear differential perturbations) and to incorporate only the most important features into the fundamental differential equation for a trajectory. What this equation is, depends in part on the particular problem, but is in no case as simple today as it was before the World War. In the sense of "rational mechanics", there is no rational ballistics.

For ballistics, as for every other developing branch of engineering, there is a group of present-day problems involved on the one side in seeking to test present-day assumptions, and if possible to establish these on a rational basis, and on the other side in devising means for application of these accepted principles with an accuracy consistent with the data and yet with the greatest possible simplification concordant therewith.

An excellent survey of the history of ballistics, replete with illustrative comparisons, explicit data, and critical comments on physical assumptions, is given by the long classical work of C. Cranz. The only criticism is that despite supplementary notes in later editions, the work closes essentially with the period before the World War, which changed the perspective even on old problems.

This work of Charbonnier bids fair to be considerably more extensive than that of Cranz. Unfortunately it is subject to the same limitation that the author's point of view is that of 1914 and this is not relieved by the mere insertion of additional chapters on new methods. Unlike the work of Cranz, this book attempts to deal with what appears to be a non-existent subject. In open and deserved admiration of the secure position of rational mechanics (in which French writers from Laplace to Appell appear to surpass themselves) this army officer effectually divorces ballistics at this stage from its only logical bases, historical and physical. We meet, consequently, for at least the first seven hundred pages of this volume, a collection of mutually contradictory theories. These are given no historical justification and one cannot make comparisons among them since the author himself fails to compare their relative accuracy and withholds as yet all physical data from the reader in the effort to be strictly rational. It may be of interest to American readers that a condensed but just account (covering five pages) of American methods with reference to Major Moulton is found in the chapter, "Exposé de quelques méthodes de calcul."

Perhaps the most severe criticism of this book by Charbonnier (in view of its date of publication) is contained in his own words, à propos of the artificial solutions of Drach. "Les solutions, . . . , quelque intéressantes qu'elles soient, ne remplissent pas les deux conditions simultanées qu'exigent les balisticiens: 1° s'appliquer à une forme de la fonction $F(v)$, acceptable, c'est-à-dire suffisamment voisine de la fonction expérimentale de la résistance de l'air; 2° conduire, pour les autres éléments du mouvement à une solution formelle, susceptible d'être utilisée pour les applications numériques."

Possibly reference to future volumes of this projected series will serve to restore in part the needed perspective here so effectively eradicated.

A. A. BENNETT

DOWSETT ON CONSTRUCTIVE GEOMETRY

Advanced Constructive Geometry. By J. F. Dowsett. Oxford University Press (American Branch, New York), 1927. viii+340 pp. Price $9.00.

The author of this work, who is lecturer and chief instructor in geometry and graphics, London County Council School of Building, Brixton, states in his preface that the book is intended for advanced students and for teachers of building geometry.

It is encouraging to know that such a thorough course in descriptive geometry, with applications to carpentry and masonry, is available for students of building geometry. Much too often, craftsmen have no adequate knowledge of the principles of descriptive geometry, and also no facilities for carrying out its constructions when they have need for them. Thus the author remarks (Chapter XVII, p. 278), "The absence of facilities for setting out elaborate geometrical constructions on the building site is the chief factor in perpetuating crude rule-of-thumb methods, and is responsible for much of the poor craftsmanship to be found in quite important structures." For this reason the author devotes a chapter to methods of placing lines to which to cut bevels, and for other purposes, directly on the materials of construction.

This is but one illustration of the practical nature of the work. However, aside from applications, the author gives the reader a very good working knowledge of Mongean Descriptive Geometry. In addition to this, the student will find much of interest and of value in the numerous applications to wood and stone cutting.

The book is well planned in that necessary material is developed and ready for use when needed. Thus, for example, the first six chapters, which constitute Section I, are devoted to constructions in plane geometry. Chapter I deals with some constructions (necessarily approximate) such as finding tangents and radii of curvature of curves which are given graphically (i.e., of curves for which the law of generation is not known). It also gives approximate constructions for rectifying and dividing arcs of circles. In the following chapters of this section much useful information is given concerning the circle and the other conic sections. Such applications are given as the construction of scroll curves and volutes, the drawing of moulds for easings to connect two straight hand-rails, etc.

The second section of the work, comprising Chapters VII to XX inclusive, is devoted to descriptive geometry, that is, to the solution of space problems by means of constructions which can be executed in the plane.

With the exception of Chapter XX, which deals with perspective, the method of treatment is principally that of double-orthographic projection (i.e., the method of Monge). Aside from the applications, the geometric problems treated are those usually considered in a thorough course on descriptive geometry.

No classification of problems, such as a division of them into problems of pure geometry of position, perpendicularity problems, and metrical problems, is given. Much, almost excessive, use is made of the device of "changing the ground line."*

Following the problems involving the point, line and plane, there is a consideration of the so-called "three round bodies": sphere, cylinder, cone, to each of which a chapter is devoted.

Among the applications to masonry in these chapters are methods of obtaining moulds (or templates) used in the construction of voussoirs for hemispherical domes, pendentives, cylindrical vaulting, groined arches, skew arches, etc. Equally important problems in carpentry are also considered in much detail.

A further chapter is devoted to what the author calls the geometry of double circular work. Here such problems as the design of arches over openings in cylindrical walls are considered for both stone and wood construction. A whole chapter is devoted to the geometry and construction of wreathed hand rails. Then follows the chapter, already mentioned, on the placing of lines directly on the materials of contruction, and another on mouldings and the design of moulding cutters. A chapter on shadows and one on perspective close the main body of the text.

Following the index is a well selected group of miscellaneous problems derived from the examination papers of various institutions.

The book is rather free from error. Most of the typographical errors are easily detected. In the statement of Problem 78, p. 145, concerning the determination of a point of a given plane for which the lines connecting this point with two given points of space shall make equal angles with the plane, it should be further stipulated that the plane of these two connecting lines be perpendicular to the given plane. For, as solved, the problem is nothing more than that of finding the brilliant point of a plane with respect to two given points of space.

In Fig. 126, the vertical trace of the plane containing the point *P* is not in proper position. It should also be stated that this plane is parallel to the ground line.

The work in a number of places might have been made easier of understanding if a statement of the space construction (i.e., the operations in space of which the constructions in the drawing plane are counterparts) had preceded the description of the construction in the drawing plane.

The drawings are well executed and the pictures showing space relations (which the author calls pictorial views) are excellent. However, in several cases the scale is a little too small for the easy reading of the letters.

W. H. ROEVER

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* This consists in assuming a new vertical (or horizontal) plane of projection to which the space object under consideration is more simply related than it is to the old one, and then revolving (or "rebatting") this new plane into the horizontal (or vertical) plane around its horizontal (or vertical) trace, that is, around the new ground line.

KLEIN ON NINETEENTH CENTURY MATHEMATICS

Vorlesungen uber die Entwicklung der Mathematik im 19. *Jahrhundert.* By Felix Klein. Teil I. Berlin, Springer, 1926. xiii+385 pp.

Twenty years ago, during a series of walks in the forest about Hahnenklee, in the Hartz Mountains, the conversation between Klein and a companion covered, as would naturally be the case, a wide range. Three statements, however, impressed his listener* very strongly. One was political: "There was a time when we looked up to England socially, politically, and as a naval power,—but that is a thing of the past." The second was political and of military significance: "America has no standing army today; twenty-five years hence she will have a large one." It is not strange that his auditor wondered at the real significance of these two statements by a man of Klein's vision and prominence. The third remark was in response to a statement to the effect that he of all men was the one to write a history of mathematics in the 19th century. "I am too old," was the reply, "It needs a young man who could devote years to its preparation." When it was urged that he had seen the development and had taken part in it as few if any others living had done, he remarked, "No, all that I could do would be to give a few lectures on the great events, but I am too much occupied to prepare even these." Ten years later, when the war was on, and his family had been sorely stricken, he gave these very lectures in his home in Göttingen, before a small group of listeners anxious to receive from a master that which only a master could give.

The lectures have been edited by Professors Courant and Neugebauer and are published as Band XXIV of Die Grundlehren der mathematischen Wissenschaften, a recent series, already well known to all mathematical students. Rightly did they say in their Vorwort: "Diese Vorlesungen sind die reife Frucht eines reichen Lebens inmitten der wissenschaftlichen Ereignisse, der Ausdruck überlegener Weisheit und tiefen historischen Sinnes, einer hohen menschlichen Kultur und einer meisterhaften Gestaltungskraft; sie werden sicherlich auf alle Mathematiker und Physiker und weit über diesen Kreis hinaus eine grosse Wirkung ausuben."

The work is divided into eight chapters. The first naturally begins with the founder of the modern German school of pure and applied mathematics, —Gauss. It considers his work in the applied field with respect to astronomy, geodesy, and physics, the last in connection with A. von Humboldt and Wilhelm Weber. In pure mathematics the attention is given chiefly to the number theory, forms, and the function theory, with a succinct statement as to the claim that Gauss is entitled to the award of priority in the discovery of non-euclidean geometry, although not to priority of publication (pp. 57–60).

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* The reader may suspect the identity of the "listener," who modestly conceals it. The editors may state at least that the conversation was actual and that this report is a first-hand report.

(The Editors)

Chapter II relates to France and the École Polytechnique in the first decade of the century. In this Klein pays high tribute to the work of men like Fourier, Cauchy, Poncelet, Monge, and Galois. Of Cauchy he speaks as one "der sich mit seinen glänzenden Leistungen auf allen Gebieten der Mathematik fast neben Gauss stellen kann."

Chapter III concerns the founding of Crelle's Journal and the rise of pure mathematics in Germany. In this period the names mentioned as most prominent are those of Abel, Jacobi, Moebius, Plücker, and Steiner.

Chapter IV considers the development of algebraic geometry by Moebius, Plücker, and Steiner, with reference to Lagrange, Chasles, Cayley, Sylvester, Salmon, Beltrami, and Clifford, as well as to his own countrymen,—Riemann, Hesse, and Grassmann.

Chapter V deals with mechanics and mathematical physics in Germany and England before 1880,—the period of Hamilton, Thomson (Kelvin), and Maxwell in England, of Gibbs in America, and of Franz Neumann in Germany.

Chapter VI is devoted to the development of the theory of functions of a complex variable, chiefly at the hands of Riemann and Weierstrass, but with mention of the influence of Dirichlet, H. A. Schwarz, Fuchs, C. Neumann, Kovalevski, and others.

Chapter VII is taken up with the study of algebraic forms, and Chapter VIII with that of the theory of groups and automorphic functions, with special reference to the work of Galois, C. Jordan, Hermite, Riemann, and Poincaré.

It is proposed, in Band II, to treat chiefly of the theory of invariants and of relativity.

All who knew Professor Klein with any degree of intimacy are aware of his broadmindedness and catholicity of spirit. His love for science did not permit him to allow political prejudices to warp his judgment as to the tribute due to scholarship beyond the boundaries of his own country. It speaks well for his judicial spirit to observe that, of the scientists mentioned, nearly half were not of German nationality. Nearly a fourth were French, about an eighth were British, and a fair number were from Italy, Scandinavia, Russia, and our own country. It was natural that the work of the Germans should have been best known to him, and it was entirely fair that he should have ranked them higher than others of the last half of the century, not alone numerically but on the score of achievement.

David Eugene Smith

EVANS ON LOGARITHMIC POTENTIAL

The Logarithmic Potential, Discontinuous Dirichlet and Neumann Problems. By Griffith Conrad Evans. Colloquium Publications, American Mathematical Society, volume 6, 1927. viii+150 pp.

The appearance of this volume marks a broadening of the scope of the Colloquium Publications, in that these will no longer be confined to lectures given from time to time, by invitation of the American Mathematical Society, at its summer meetings. The new departure, however, could hardly have been initiated more smoothly or appropriately. For in the first place, Professor Evans has in the past been one of the Society's Colloquium lecturers, and secondly, his book is very closely related in character to the traditional Colloquium publications in that it is pre-eminently concerned with recent progress.

The book is characterized by the role played by Stieltjes and Lebesgue integrals and by functions of limited variation, and by the degree to which these concepts have enabled the author to express the conditions for his theorems in necessary and sufficient form. That these tools are particularly adapted to physical problems is a fact which the author early recognized. The material centers about a study of Poisson's integral in two dimensions and of the corresponding Stieltjes integral. The results are then extended to the integrals in terms of Green's functions for general regions.

In the introductory chapter, the fundamental properties of functions of limited variation and of Stieltjes integrals are succinctly developed, and some properties of Lebesgue integrals and theorems on the termwise integration of sequences are stated, with references or proofs. The last eight pages are given to Fourier series.

The second chapter develops properties of the Poisson-Stieltjes integral

$$\text{(A)} \qquad u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1+r^2-2r\cos(\phi-\theta)}dF(\phi),$$

where $F(\phi)$ is of limited variation, and of the Poisson integral

$$\text{(B)} \qquad u(r,\theta)=\frac{1}{2\pi}\int_0^{2\theta}\frac{1-r^2}{1+r^2-2r\cos(\phi-\theta)}f(\phi)d\phi,$$

where $f(\phi)$ is summable.

The function $u(r,\theta)$ defined by (A) is harmonic ($r<1$); the function

$$F(r,\theta)=\int_0^{\theta}u(r,\theta)d\theta$$

approaches, as $r\rightarrow 1$, θ being fixed, the value

$$\tfrac{1}{2}\{F(\theta+0)+F(\theta-0)\}-\tfrac{1}{2}\{F(0+)+F(0-)\};$$

$u(r,\theta)$ is the difference of two non-negative functions for $r<1$; $u(r,\theta)$ is a function of θ of limited variation, uniformly as to r; at every point $P(1,\phi)$ at which $F(\phi)$ has a derivative $F'(\phi)=f$ (i.e., almost everywhere), $u(r,\theta)\rightarrow f$

as $r \to 1$, under the sole restriction that (r,θ) remain between two chords of the unit circle terminating in P. These are among the properties derived. They are paralleled by those of the function $u(r,\theta)$ defined by (B).

The third chapter centers in the fundamental theorem, stated as follows in terms of the above function $F(r,\theta)$, and an infinite sequence

$$r_1, r_2, r_3, \cdots, r_n < 1, \lim r_n = 1.$$

(1) If $u(r,\theta)$ is harmonic for $r<1$, and if $F(r_n,\theta)$ is of limited variation as a function of θ, uniformly as to n, then $u(r,\theta)$ admits the representation (A).

(2) If $u(r,\theta)$ is harmonic for $r<1$, and if $F(r_n,\theta)$ is absolutely continuous as a function of θ, uniformly as to n, then $u(r,\theta)$ admits the representation (B).

This theorem, in combination with Chapter 2, gives necessary and sufficient conditions that a harmonic function admit the representation (A) or (B), as well as a considerable number of properties of such harmonic functions. A section is devoted to interesting special cases of functions given by (B). Of course the results of Chapter 2 include existence proofs of a very general character for the Dirichlet problem for the circle, and Chapter 3 then furnishes the desirable complements in the way of proofs of uniqueness. The formulation of these results constitutes the closing section of Chapter 3.

Chapter 4 is concerned with the potential of a simple distribution on the circumference of a circle of radius a:

$$\text{(C)} \qquad v(r, \theta) = -\frac{a}{2\pi}\int_0^{2\pi} \log\,[a^2 + r^2 - 2ar\cos(\phi - \theta)]dF(\phi) + A,$$

where $F(\phi)$ is periodic and of limited variation. A necessary and sufficient condition that $v(r,\theta)$, harmonic for $r<a$, admit a representation (C) is that

$$\int_0^{2\pi}\left|\frac{\partial}{\partial r}v(r,\theta)\right| d\theta$$

be bounded. Other properties of $v(r,\theta)$ follow, among them that

$$\lim_{r\to a}\int_0^{\theta}\frac{\partial v}{\partial r}\,d\theta = F(\theta) - F(0).$$

But this is in effect the statement that the flux of the vector field whose potential is v, across any given portion of the boundary, is a given quantity, and so we are led to a solution of the problem of Neumann having validity even when the normal derivative of v does not exist on the boundary. The chapter is concluded by a study of the possibility of representing a function of the complex variable, analytic within a circle, by the Cauchy integral formula, the integral being extended over the circumference.

Chapter 5 contains the extension of results for the circle to the general simply connected region of the plane with more than one boundary point, and Chapter 6 to regions of finite connectivity. The theorems are stated in a form invariant under conformal transformation in terms of Green's function and its conjugate, but careful reasoning is necessary to establish them. They are substantial generalizations, not merely obvious ones. Incidentally Chapter 6 contains some interesting results on isolated singularities of harmonic functions. The theorem ascribed to Lebesgue, by the way, on page 86, is due to Bôcher (see the reference on page 111).

The last chapter is a development of Professor Evans' symposium at the meeting of the Southwestern Section of the American Mathematical Society in November, 1926. At the beginning it reacts to simply connected regions, and establishes, among others, the result that the analytic transformation which carries the interior of the unit circle into the interior of a region with rectifiable boundary on a Riemann surface is conformal almost everywhere on the boundary. For regions of infinite connectivity a theorem is established which is concerned with the unique determination by its boundary values of a function harmonic and bounded in a region of infinite connectivity. The chapter closes with a discussion of convergence in the mean of various orders, relating a theorem of Noaillon with those of the author, and with a section on the characterization of harmonic functions by integral properties.

There are some thirty exercises at intervals throughout the text. These are designed in part to familiarize the reader with the concepts introduced, and in part to complement the theory, and they are to be taken seriously. This does not mean, however, that the book is to be regarded as a text, or a systematic introduction to potential theory. For satisfactory results, the reader should know something of potential theory, functions of a complex variable, and Lebesgue integrals. The author's hope, expressed in the preface, that the book will be found suitable for graduate students of a moderate degree of advancement will be fulfilled in the case of those with initiative: potential investigators. Others will have difficulty in supplying the details of reasoning suppressed by the author's succinct style. It is for investigators that the book will find its greatest usefulness, whether at the beginning of their careers or later.

In general, and in all essentials, the book is written with the care and thoroughness we should expect of its author. There are a few minor points where oversights have occurred. Thus Corollary 3 on page 7 is wrong, and the word uniformly has been omitted from the phrase "and (4) is convergent" in the last line of the second paragraph on page 126. These are the only cases noticed worth mentioning, and nothing is affected in the sequel.

The theory presented is Professor Evans' creation, most of the theorems developed being the results of his own researches. These here find a systematic and consecutive presentation. The arrangement is excellent. In spite of a well rounded character of the investigations set forth, the book leaves the reader with a sense of vistas opened. The methods and results cannot fail to influence further work on potential theory, Taylor's series, and elliptic differential equations. It will be indispensable to those occupied in these fields, and will hold an honored place in the colloquium publications.

O. D. KELLOGG

SHORTER NOTICES

Die Theorie der Gruppen von endlicher Ordnung mit Anwendungen auf algebräische Zahlen und Gleichungen sowie auf die Kristallographie. By Andreas Speiser. Second edition, Berlin, Julius Springer, 1927. ix+254pp. Price 6.50 marks.

The first edition of this work appeared in 1923 and was reviewed in this Bulletin (vol. 29 (1923) p. 372). Hence it is only necessary to note here the improvements which the second edition presents. The fact that such an edition appeared so soon after the publication of the first tends to show that the work filled a real need. It is to be hoped that the new material which appears in the present edition will make the work still more useful and will tend to increase its popularity as an introduction to some of the most interesting and most modern developments of the large subjects to which it explicity relates.

The size of the volume has been increased by more than one-fourth of its original extent, while the number of different authors to whom reference is given has been more than doubled, being now 77. No additional name of an American author appears in this list, but the number of page references to the five which appeared in the original list has been more than doubled, while the total number of such references was increased relatively much less. One name which appeared in the original list does not appear in the present one, viz., that of E. Netto, who was credited in the original edition with a material simplification of the proof of Frobenius's Theorem. No reference to such credit is made in the present edition.

The most striking feature of the new edition is the emphasis on plane symmetries and on the theory of ornamental figures. These subjects are here recommended as suitable first steps towards an understanding of abstract group theory. The present volume aims to provide three distinct entrances to this subject. The first of these starts from the postulates, the second begins with ornamental figures, while the third employs the classic theory of substitutions, which is here called the theory of permutations. The wide differences between these methods of entering into the same general field tend to interest the student and to exhibit at the outset the extensive range of applications of the general theory.

The present volume aims also to arouse a deep historical interest by referring to the fundamental question of the source and the nature of the earliest higher mathematics in ancient times. The view is expressed here that this source is found in architecture and in ornamental figures, and hence the earliest study of higher mathematics was inspired by a sense of beauty rather than by an interest in elementary geometry, which is necessarily a somewhat tedious subject. It thus appears that the history of higher mathematics may begin about 1000 years earlier than has been commonly assumed by mathematical historians (p. 3). We refer here to these historical remarks, some of which will probably not be accepted without protest, to exhibit the attractive style in which the subject matter of the present volume has been presented. The fact that the author sometimes expresses general views which may possibly be challenged may tend to awaken interest and discussions on the part of the readers.

The number of chapters has been increased from 15 to 16 while the number of sections has been extended from 63 to 74. An "Einleitung" has been added, in which prehistoric group theory and the derivation of the group concept from permutations is considered. The heading of the last section has been changed from "Anwendung der Substitutionsgruppen" to "Die Kleinsche Gleichungstheorie," but besides the correction of slight errors and the addition of a foot-note no changes were made in this section. In the list of authors the name of *Owen Jones* appears under the letter *O* instead of under the letter *J*. The developments relating to abelian groups have been greatly extended but the ϕ-subgroups did not receive any attention. As in the first edition so in the present one the applications are expecially stressed in accord with the expressed purpose of the valuable series of which the present volume is a worthy part.

G. A. MILLER

Das Rechnen der Naturvolker. By E. Fettweis. Leipzig, B. G. Teubner, 1927. iv+96 pp. Price 5 marks.

This is a compilation of the number words used by many savage races. Gestures for numbers and some of the extremely simple calculations which these races make, are also described. An extensive bibliography of nearly 300 titles is given. It does not include L. L. Conant's *Number Concept*, New York, 1896, though the author makes a second-hand allusion to Conant, (p. 54); and it has few titles in common with the bibliography prepared from the mathematical standpoint by David Eugene Smith, (*History of Mathematics*, vol. 1, p. 14). The work seems likely to be of more interest to the ethnologist, the linguist and, perhaps, the psychologist, than to the mathematician.

J. W. A. YOUNG

Ptolémée, Composition Mathématique, traduite pour la première fois du grec en français par M. Halma (avec le texte grec): et suivie des notes de M. Delambre. Paris, J. Hermann, Volumes 1, 2. Réimpression facsimilé. Price 210 fr.

The edition of Ptolemy's *Almagest*, in Greek and French, printed in parallel columns, which was prepared in 1813–16 by the noted French scholar l'Abbé Halma and supplied with notes by the astronomer Delambre, is generally regarded as the best in existence. Before that time a translation from Greek into Latin had appeared at Basel which was based on a less careful scrutiny of different manuscript texts. Halma made a searching comparison of manuscripts in the libraries in Paris, Venice, Florence and the Vatican. Now, more than a century after the first appearance of Halma's edition, a facsimile reimpression is placed within the reach of readers of our time. It is well that Ptolemy should be available to modern students in a modern language, for Ptolemy's Almagest is the text from which astronomers have drawn knowledge and inspiration for over a thousand years, a longer period of authority than that of any other astronomical author.

The first volume of Halma's edition begins with an elaborate Preface in French which is an historical and critical account of Ptolemy's great book, emphasizing its value to the modern astronomer. This part, though rich in detail, might have been profitably rewritten so as to embody later studies due to Paul Tannery and others. An historical result so very recent that it could not have been included even in a revision, unless written since 1926, is the demonstration that the precession of the equinoxes was known before the time of Hipparchus to the Babylonian astronomer Kidinnu.

Of special historical interest to mathematicians is Ptolemy's spherical trigonometry which he develops more fully than plane trigonometry. In his computation of a table of "chords" he establishes the theorem on the inscribed quadrilateral, that the product of its diagonals is equal to the sum of the products of the opposite sides. A simple and elegant geometric construction (*Ptolemy* I, p. 27), which yields simultaneously the sides of a regular inscribed pentagon and decagon is known to modern engineers, but is not given in elementary texts on geometry because of the comparatively greater difficulty of the proof.

FLORIAN CAJORI

Sieben- und mehrstellige Tafeln der Kreis- und Hyperbelfunktionen und deren Produkte sowie der Gammafunktion, nebst einem Anhang: Interpolations- und sonstige Formeln. By Keiichi Hayashi. Berlin, Julius Springer, 1926. vi+283 pp.

A review of these tables by T. H. Gronwall has already appeared in this Bulletin (vol. 32 (1926), p. 718). Dr. Gronwall's review concerned the contents and scope of the tables; the present review relates to their accuracy.

Wishing to construct a table of two functions auxiliary to the hyperbolic functions, and having read Dr. Gronwall's review, I decided that these tables would facilitate the compilation. Accordingly I procured a volume and used it. A computation formula was used which checked not only the computed values but also the consistency of the tabular values from which they were obtained, and the results have been rather disconcerting.

The first error encountered was in sinh x, $x=.0783$ to $x=.0799$, in which a column of 7's should be a column of 8's. This however is quite probably a proof-reader's error and such errors may easily be corrected by the publisher. But in sinh .548 and cosh .548 the tabulated values are each too large by 0.000005 and the chances of this being an error in proof-reading are small. The chances are diminished by the fact that the same kind of thing occurs in eight other places, and for $x=.782$ and .923 the corresponding error is carried into e^x; cosh .872 and sinh .872 are each in error by 0.0000091618! Such errors could hardly arise from bad proof-reading and one wonders what method of systematic checking would fail to reveal them.

All told, from the calculations made which would involve roughly 5% of the digits of the table, 25 groups of errors, as shown below, have been discovered which involve 55 functional values and 81 digits. Further argument seems unnecessary. One cannot fail to have the deepest sympathy with Mr. Hayashi in the enormous loss and disappointment which he sustained when his first calculations were as he states destroyed by fire, and to have the greatest admiration for his perseverance in immediately duplicating the entire work. But after all the value of the table is a matter of its accuracy and until the whole is thoroughly checked the table must be regarded as unreliable and used with caution. When made dependable these tables will be a great asset to any library.

ERRORS IN HAYASHI'S TABLES

Function		*Tabular Value*	*Correct Value*	*Error (T–C)*
sinh	.0783	0.0783700326	0.0783800326	−0.00001
"	.0784	4703397	4803397	"
"	.0785	5706476	5806476	"
"	.0786	6709563	6809563	"
"	.0787	7712657	7812657	"
"	.0788	8715760	8815760	"
"	.0789	9718870	9818870	"
"	.0790	0.0790721988	0.0790821988	"
"	.0791	1725114	1825114	"
"	.0792	2728248	2828248	"
"	.0793	3731390	3831390	"
"	.0794	4734540	4834540	"
"	.0795	5737698	5837698	"
"	.0796	6740864	6840864	"
"	.0797	7744037	7844037	"
"	.0798	8747219	8847219	"
"	.0799	9750409	9850409	"
tanh	.199	0.19461409	0.19641409	−0.0018
"	.332	0.22034672	0.22031672	−0.00003
sinh	.548	0.5758475557	0.5758425557	+0.000005
cosh	"	1.1539524204	1.1539474204	"
sinh	.554	0.5827816469	0.5827766469	+0.000005
cosh	"	1.1574282675	1.1574232675	"
sinh	.672	0.7237367615	0.7237317615	+0.000005
cosh	"	1.2344229449	1.2344179449	"
sinh	.719	0.7825752763	0.7825702763	+0.000005
cosh	"	1.2698145280	1.2698095280	"

Function		*Tabular Value*	*Correct Value*	*Error* ($T-C$)
sinh	.745	0.8158*38*5675	0.8158*53*5675	−0.000015
cosh	"	1.2905*72*8675	1.2905*87*8675	"
$e^{.782}$		2.1858*38*5759	2.1858*39*5759	−0.000001
sinh	.782	0.86417*42*302	0.86417*47*302	−0.0000005
cosh	"	1.32166*43*457	1.32166*48*457	"
cosh	.838	1.37215*86*405	1.37215*68*405	+0.0000018
sinh	.872	0.9867*783226*	0.9867*874844*	−0.0000091618
cosh	"	1.4049*111297*	1.4049*019679*	+ "
$e^{.923}$		2.51682*65*642	2.51682*95*642	−0.000003
sinh	.923	1.05975*06*444	1.05975*21*444	−0.0000015
cosh	"	1.45707*59*199	1.45707*74*199	"
sinh	.928	1.067*1*508087	1.067*0*508087	+0.0001
cosh	1.137	1.71*89*910180	1.71*90*910180	−0.0001
sinh	1.361	1.8218*8*35997	1.8218*4*35997	+0.00004
cosh	1.363	2.081*9906*903	2.081*8956*903	+0.000005
sinh	1.615	2.414*5*986265	2.414*4*986265	+0.0001
cosh	1.933	3.5*3*74616062	3.5*2*74616062	+0.01
sinh	1.942	3.41*64*327851	3.41*46*327851	+0.0018
cosh	1.952	3.5*8*23744152	3.5*9*23744152	−0.01
"	1.953	*8*58265966	*9*58265966	"
"	1.954	*8*92823738	*9*92823738	"
cosh	2.422	5.67*2*5587463	5.67*8*5587463	−0.006
cosh	2 554	6.468*0*023773	6.468*1*023773	−0.0001
$e^{2.717}$		15.134849*1*214	15.134849*5*214	−0.0000004
sinh	2.717	7 5343*932*230	7.5343*884*230	+0.0000048
cosh	"	7.60046*58*984	7.60046*10*984	+0.0000048
sinh	2.789	8.101*5*321270	8.101*6*321270	−0.0001
sinh	2.803	8.216*4*134403	8.216*7*134403	−0.0003

J. W. CAMPBELL

Lehrbuch der Algebra. By Robert Fricke. Volume II. *Ausführungen über Gleichungen niederen Grades.* Vieweg und Sohn, Braunschweig, F., 1926. vii+418 pp.

Fricke's Algebra is a worthy successor to Weber's Algebra, which it henceforth displaces. The present volume is a very attractive exposition of the modern theory of equations of degrees 5, 6, 7. The approach is quite different from that by Weber and more attractive since extensive use is made of the geometric and function-theoretic methods developed by Klein in his *Ikosaeder* and books on elliptic modular functions. Fricke's long experience with the latter subject made it easy for him to give a simple authoritative exposition of those portions of it which suffice for the transcendental solutions of equations of low degrees.

Part I treats finite groups of linear substitutions on two variables and quintic equations (181 pages). Part II deals with finite linear groups on three variables and equations of degrees 6 and 7 (158 pages). Part III gives applications to the inflexion points of a plane cubic curve and to the bitangents to a quartic curve (85 pages); it is similar to Weber's account, but with important improvements.

Any one interested either in the theory of algebraic equations or in elliptic modular functions will find this book indispensable.

L. E. Dickson

The Theory of Functions of a Real Variable and the Theory of Fourier's Series. By E. W. Hobson. Volume I. Third edition. Cambridge University Press, 1927. xv+736 pp.

For reviews of volumes I and II of the second edition see this Bulletin, vol. 28 (1922), pp. 266–270, and vol. 33 (1927), pp. 115–118, respectively. In preparing the third edition of volume I the author has revised his work throughout and has enlarged the volume by more than sixty pages. (He has also included three pages of additions and corrections to volume II.) There has been no change in plan, and section numbers have been left unaltered so that the references in volume II to the material in volume I are still applicable. Various sections have been rewritten and several new sections have been inserted. In the theory of Riemann-Stieltjes integration particularly changes and additions have been made. The references are ample though they do not provide a means of compiling a complete bibliography of the subject; they are rendered more useful in this new edition by the insertion of dates. In view of the size of the volume and the extraordinary richness of its material, the short general index of but little more than two pages seems quite inadequate; even after having read the volume with some care, one frequently finds it unduly difficult, owing to the inadequacy of the index, to find the treatment of a specific topic which he wishes to reexamine. The improvements and the addition of new matter in this edition will increase the value of this work, already far and away the most useful existent treatise on the theory of functions of a real variable.

R. D. Carmichael

Étude Élémentaire de l'Hyperbole Équilatère et de quelques Courbes Dérivées. By J. Lemaire. Paris, Vuibert, 1927. 172 pp

Book one contains a detailed exposition of the elementary properties of the equilateral hyperbola; much attention is given to the osculating circles and the bitangent circles. The second book is concerned with the strophoid and lemniscate as inverses of the equilateral hyperbola, and with the three-cusped hypocycloid. The many properties of the osculating and of the bitangent circles of these inverse curves follow from the detailed treatment in book one. The discussion of these curves is more extensive than that in any standard treatise. The development is synthetic, well ordered, and the book is easy reading. The text is supplemented by nearly one hundred exercises.

On page 163, example 58, a 3-bar link-work is defined which gives an easy mechanical construction for the lemniscate. The author, apparently following Teixeira, attributes this to Carbonnelle (Lemaire says "Carbonelle"). Carbonnelle proposed this in Nouvelle Correspondance (vol. 5 (1879), pp. 220 and 249). But the theorem had already been given by Phillips in the American Journal of Mathematics vol. 1 (1878), p. 386).

B. H. Brown

Coup d'Oeil sur la Théorie des Déterminants Supérieurs dans son Etat Actuel. By Maurice Lecat. Bruxelles, Lamertin. 1927, viii+97+100 pp.

This synopsis of the present state of hyperdeterminants is a continuation of the researches of M. Lecat on the properties of determinants of N dimensions. It is to be followed by a treatise in three volumes, soon to appear, which will include the applications. The synopsis contains 19 chapters and a bibliography. It is intended to be of a critical character, the author remarking that almost all the investigations before 1910 in this field were incorrect, or at least spotted with serious errors. The exceptions are Cayley and Sylvester. Gegenbauer's work is cited as containing many errors, his results on the adjoint and upon skew symmetric determinants being completely wrong.

This field of course is intimately connected with matrices of cubic, and higher arrays, and the first thing taken up is the "topology" of such arrays. In the next chapter "activity" is considered. This property belongs to a matrix when the elements (written with subscripts) are symmetric as to at least two subscripts. The succeeding two chapters introduce considerable terminology. The next four chapters deal with developments of the determinant. In the second part of the synopsis, which has a separate paging, though the chapters run consecutively, is considered multiplication of determinants, and some special forms, such as continuants, adjoints, circulants, etc.

The synopsis will be useful to the student who desires to follow up the investigations of this field, particularly the very important ones of M. Lecat himself. It really demands the "*Traité*" to make it thoroughly useful.

J. B. Shaw

NOTES

The April, 1928, number of the Transactions of this Society (volume 30, No. 2) contains the following papers: *The foundations of a theory in the calculus of variations in the large*, by M. Morse; *Topological invariants of knots and links*, by J. W. Alexander; *On the expansion of analytic functions in series of polynomials and in series of other analytic functions*, by J. L. Walsh; *Primitive groups which contain substitutions of prime order p and of degree 6p or 7p*, by M. J. Weiss; *Generalized Lagrange problems in the calculus of variations*, by C. F. Roos; *On Jacobi's arithmetical theorems concerning the simultaneous representation of numbers by two different quadratic forms*, by J. V. Uspensky; *On relative content and Green's lemma*, by H. L. Smith; *On Bell's arithmetic of Boolean algebra*, by W. A. Hurwitz; *A theorem on orthogonal functions with an application to integral inequalities*, by L. L. Dines; *A theorem on orthogonal sequences*, by L. L. Dines.

The April, 1928, number of the American Journal of Mathematics (volume 50, No. 2) contains: *A theorem on correspondences on algebraic curves*, by S. Lefschetz; *Concerning the structure of a continuous curve*, by G. T. Whyburn; *Compound statements on four classes*, by F. E. Baker; *A memoir on the invariants of biternary quantics*, by O. E. Glenn; *Transitive substitution groups containing regular subgroups of lower degree*, by F. E. Johnston; *Generalized Green's functions for compatible differential systems*, by W. W. Elliott; *An expansion involving p inseparable parameters associated with a partial differential equation*, by C. C. Camp; *On the asymptotic evaluation of functions defined by contour integrals*, by D. M. Wrinch; *Generalized quaternion algebras and the theory of numbers*, by L. W. Griffiths.

It is announced by the Chairman, Professor J. C. Fields, that the Proceedings of the Mathematical Congress held at Toronto in 1928 will be published while this issue of this Bulletin is in press. There will be two volumes aggregating nearly 2000 quarto pages, bound in cloth. Among other things, there will be found in the first volume a group photograph of the Congress, and the *procès verbal* of the meeting of the International Mathematical Union held during the Congress. In addition to the eight lectures printed in the first volume, the Proceedings will include about one hundred fifty communications printed in full, and over fifty abstracts. There are about one hundred eighty authors, representing twenty-six countries. The price of the entire Proceedings will be five dollars plus postage to registered members of the Congress, and ten dollars plus postage to others; orders may be sent to the University of Toronto Press. Separate copies of the group photograph may be had at fifty cents each.

The papers read at the bicentenary Newton celebration held in New York under the auspices of the History of Science Society (see this Bulletin, vol. 34, p. 247) have been published in book form with the title *Sir Isaac*

Newton, 1727–1927, A Bicentenary Evaluation of his Work. The book was prepared for publication by F. E. Brasch, and has an introduction by D. E. Smith; it is published by Williams and Wilkins, of Baltimore.

In response to a request by G. Albini, Senator, Rector of the University of Bologna, and President of the Organization Committee of the International Congress of Mathematicians, the American Mathematical Society has named the following six persons as its representatives: Professors Archibald, Birkhoff, Blichfeldt, Kasner, Snyder, Veblen. If a chairman be necessary, Professor Birkhoff will act.

The French Association for the Advancement of Science will meet at La Rochelle, July 23–28, 1928. Professor L. G. Du Pasquier, of the University of Neufchatel, is president of the section of mathematics.

The Louisiana Academy of Sciences held its first annual meeting at Louisiana College, Pineville, May 5, 1928. Professor I. Maizlish, of Centenary College, is president, and Professor H. L. Smith, of Louisiana State University, vice-president. Professor H. A. Wilson, of Rice Institute, delivered the principal address, entitled *Recent theories of atomic structure.* Mathematical papers were read by Professors J. A. Hardin, C. D. Smith, and H. L. Smith, and Mrs. Yetta V. Maizlish. The next annual meeting will be held at the Southwestern Louisiana Institute, Lafayette.

The Belgian Academy of Sciences has awarded its prize for a contribution to the resolution of problems with axial symmetry in general relativity (see this Bulletin, vol. 32, p. 402) to Maurice Nuyens, of the University of Brussels, for his memoir entitled *Champ gravifique dû à un électron animé d'un mouvement de rotation.*

A prize in honor of U. Dini will be conferred by the University of Pisa for a manuscript or printed work in pure mathematics, mathematical physics, celestial mechanics, or theoretical geodesics; only graduates or students of the University of Pisa, and graduates of other universities who have studied at the University of Pisa since their graduation, are eligible. The competition closes in December, 1928.

Competition for the first award of the Lazzario Fubini prize of the Unione Matematica Italiana (see this Bulletin, vol. 31, p. 282) closes December 31, 1928; memoirs should be sent to Professor E. Bortolotti, of the University of Bologna, before that date. This prize is to be awarded every two years and is restricted to Italian graduates of Italian universities who received their degrees within six years of the closing date.

Dr. Paul Koebe, professor of mathematics at the University of Leipzig, has been awarded the international mathematics prize of the King of Sweden, on recommendation of the Swedish Academy of Sciences, for his memoir *Allgemeine Theorie der Riemannschen Mannigfaltigkeiten (konforme Abbildung und Uniformisierung).*

Cambridge University has awarded Smith prizes to W. L. Edge, of Trinity College, for an essay on *Ruled surfaces of the fourth, fifth and sixth orders,* and to A. H. Wilson, of Emmanuel College, for an essay on *The* *two-centre problem in wave mechanics.* Rayleigh prizes have been awarded to J. A. Gaunt, of Trinity College, for an essay on *The foundation of the Debye-Huckel ionization theory with application to gases,* and to W. H. M'Crae, of Trinity College, for an essay on *The quantum theory and the specific heat of gases.*

Dr. W. Nernst, professor of physical chemistry in the University of Berlin, has been awarded the Franklin medal of the Franklin Institute.

The Louis Edward Levy medal of the Franklin Institute has been awarded to Dr. Vannevar Bush, professor of electric power transmission at the Massachusetts Institute of Technology, for two papers on the product integraph, an instrument developed under Dr. Bush's direction.

Dr. W. H. Wright, astronomer of the Lick Observatory, has been awarded the Henry Draper medal of the National Academy of Sciences, for his researches on nebulae, new stars, and planetary atmospheres.

Professor Paul Alexandroff, of the University of Moscow, has been elected a corresponding member of the Göttingen Scientific Society.

Professor Harry Bateman, of the California Institute of Technology, has been elected a fellow of the Royal Society of London.

The Montanistische Hochschule in Leoben has conferred an honorary doctorate on Professor E. Doležal, of Vienna.

Professor G. Kowalewski has been elected a member of the Saxon Academy of Sciences.

Professor Louis Roy, of the Faculty of Sciences of Toulouse, has been elected a correspondent of the Paris Academy of Sciences in the section of mechanics.

Professor Albert Einstein has been elected an honorary member of the British Physical Society.

Dublin University has conferred an honorary doctorate on Professor A. S. Eddington, of Cambridge.

The University of Liverpool has conferred an honorary doctorate of science on Professor J. E. Littlewood, of Cambridge.

Professor J. D. Tamarkin, of Brown University, has been elected a member of the American Academy of Arts and Sciences.

Dr. K. Menger, of the University of Vienna, has been promoted to an associate professorship.

Dr. Leopold Vietoris, of the University of Vienna, has been appointed to an associate professorship at the University of Innsbruck.

At the University of Paris, the chair of the theory of groups and the calculus of variations has been changed to a chair of the theory of functions and the theory of transformations, to which Professor E. Vessiot has been appointed; Professor R. Garnier succeeds him in his former chair.

At the University of Besançon, Professor J. Haag, of the University of Clermont, has been appointed to the chair of rational and applied mechanics, as successor to Professor J. Andrade, retired.

At the University of Dijon, the chair of pure mathematics has been changed to a chair of the differential and integral calculus to which Dr. Lagrange, of the faculty of sciences of Aix-Marseille, has been appointed.

Professor Georges Cerf, of the University of Dijon, has been appointed to the professorship of general mathematics at the University of Strasbourg.

Dr. Adolph Fraenkel, professor of mathematics at the University of Marburg, has been called to the University of Kiel.

Dr. A. Hammerstein, of the University of Berlin, has been promoted to an associate professorship.

Associate Professor Friedrich Hartogs, of the University of Munich, has been promoted to a full professorship.

Professor Otto Holder, of the University of Greifswald, has retired.

Dr. K. Reinhardt, of the University of Leipzig, has been promoted to a professorship.

Dr. G. Pólya has been appointed to a professorship at the Zurich Technical School, as successor to Professor F. Rudio, retired.

Dr. R. K. Butchart, lecturer in mathematics at the University College, Dundee (the University of St. Andrews) has been appointed professor of mathematics at Raffles College, Singapore.

The electors of the newly established Rouse Ball professorship of mathematics at the University of Cambridge have elected J. E. Littlewood, fellow of Trinity College and Cayley lecturer, to the professorship.

Professor Constantin Carathéodory of the University of Munich, Professor G. C. Evans of the Rice Institute, and Professor H. B. Phillips of the Massachussetts Institute of Technology, will lecture at the University of California during the summer of 1928.

Professor Herman Weyl, of the Zurich Technical School, has been appointed to the Thomas D. Jones research professorship of mathematical physics at Princeton University.

At Princeton University, Associate Professors J. W. Alexander, S. Lefschetz, C. R. MacInnes, and J. H. M. Wedderburn have been promoted to full professorships of mathematics.

Professor Daniel Buchanan, head of the department of mathematics at the University of British Columbia, has been appointed Dean of the Faculty of Arts and Science at that university.

At the University of Syracuse, Associate Professor Alan D.Campbell has been promoted to a professorship of mathematics.

Professor G. W. Gorrell has been made head of the department of mathematics at the University of Denver.

Assistant Professor Charles Duncan Gregory, of William and Mary College, has been promoted to an associate professorship of mathematics.

Dr. C. E. Hille of Princeton, and Dr. Alfred Errera of Belgium, will lecture at Stanford University during the summer session.

At Yale University, Professor W. R. Longley has been appointed to the Colgate professorship, as successor to Professor Luquiens; and Dr. L. T. Moore has been promoted to an assistant professorship of mathematics.

On account of blindness, Professor T. E. McKinney, of the department of mathematics at the University of South Dakota, will retire from active service at the end of the present academic year.

Dr. Max Mason, president of the University of Chicago and formerly professor of mathematical physics at the University of Wisconsin, has resigned to accept an appointment as director of the newly created division of sciences of the Rockefeller Foundation.

At Columbia University, Associate Professor G. W. Mullins, of Barnard College, has been promoted to a full professorship of mathematics; Dr. R. G. Archibald and Dr. B. O. Koopman have been promoted to assistant professorships of mathematics; and Dr. Lulu Hofmann has been appointed to a lecturership in mathematics.

Dr. C. H. Richardson, of Georgetown College, Kentucky, has been appointed head of the department of mathematics at Bucknell University, as successor to Professor W. C. Bartol, who has retired after forty-seven years of service.

Professor J. E. Rowe, of the College of William and Mary, has resigned to accept the presidency of the Clarkson Memorial Institute of Technology at Potsdam, N. Y.

At the University of California at Los Angeles, Assistant Professor H. M. Showman has been made Recorder of the University but will remain a member of the department of mathematics, and Assistant Professor Myrtie Collier has withdrawn; Dr. Raymond Garver of the University of Rochester, and Professor W. M. Whyburn, formerly of the Texas Technological College, and recently National Research Fellow, have been appointed to assistant professorships of mathematics.

Dr. D. J. Struik has been appointed assistant professor of mathematics at the Massachusetts Institute of Technology.

Assistant Professor J. D. Tamarkin, of Brown University, has been promoted to a full professorship of mathematics.

The following appointments to instructorships are announced:

Albion College, Mr. Watson M. Davis.

The University of Arkansas, Mr. Harvey A. Wright.

Columbia University, Mr. M. S. Demos.

The University of Chicago, Dr. B. W. Jones.

Dartmouth College, Dr. F. W. Perkins.

Harvard University, Dr. M. M. Slotnick (National Research Fellow), A. C. Berry, A. B. Brown, A. H. Fox, G. A. Hedlund, J. J. Hinrichsen, J. K. Peterson, G. B. Price, G. Saute, and Charles Wexler.

St. Olaf's College, Mr. Clarence Carlson.

Professor Henri Bosmans, of the Jesuit College of Saint-Michel, Brussels, died February 3, 1928, at the age of seventy-six. He was known for his work in the history of mathematics.

Professor L. Schleiermacher, of the Technical School at Darmstadt, died November 10, 1927.

Professor Emil Waelsch, of the German Technical School at Brunn, died June 5, 1927.

Professor Giuseppe Sforza died December 17, 1927.

Professor Emilio Veneroni, president of the Royal Technical School at Piacenza, and docent in geometry at the University of Padova, died June 18, 1927.

Professor W. S. Aldis, of Auckland College, New Zealand, died March 7, 1928, at the age of eighty-nine.

Mr. W. B. Croft, formerly of the department of mathematics at Winchester College, died March 23, 1928, at the age of seventy-six.

Dr. J. B. Chittenden, professor of mathematics at the Brooklyn Polytechnic Institute, died March 20, 1928, at the age of sixty-four.

Professor Gaetano Lanza, emeritus professor of theoretical and applied mechanics at the Massachusetts Institute of Technology, died March 21, 1928, at the age of seventy-nine.

Dr. F. S. Luther, emeritus president and formerly professor of mathematics at Trinity College, Hartford, died January 4, 1928, at the age of seventy-seven.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

ACKERMANN (W.). See HILBERT (D.).

ANDRADE (E. N. DA C.). See LUCRETIUS.

APPELL (P.). See POINCARÉ (H.).

BOREL (E.). Leçons sur les séries divergentes. 2e édition revue et entièrement remaniée avec le concours de Georges Bouligand. (Borel Series.) Paris, Gauthier-Villars, 1928. 260 pp.

—— Leçons sur la théorie des fonctions. 3e édition. Paris, Gauthier-Villars, 1927. 290 pp.

BOULIGAND (G.). See BOREL (E.).

BROGGI (U.). Análisis matemático. Volumen II: Teorías generales. Funciones de más de una variable. La Plata, Universidad Nacional de La Plata, 1927. 211 pp.

CARSLAW (H. S.). Introduction to the theory of Fourier's series and integrals. (In Japanese.) Translated by G. Takemae. Tokyo, Uchida Rokakuho, 1927. 12+482 pp.

CARTAN (E.). See DELENS (P. C.).

CASSINIS (G.). Calcolo numerici, grafici e meccanici. Pisa, Mariotti-Paccini, 1928. 20+672 pp.

CORRAL (J. I.). Nuevas formulas de transformacion de coordenadas. Habana, 1928. 91 pp.

CURTISS (D. R.) and MOULTON (E. J.). Trigonometry, plane and spherical. New York, Heath, 1927. 11+264 pp.

DELENS (P. C.). Méthodes et problèmes de géométries différentielles, euclidienne et conforme. Avec une préface de M. E. Cartan. Paris, Gauthier-Villars, 1927. 10+184 pp.

DRACH (J.). See POINCARÉ (H.).

DURELL (C.) and WRIGHT (R. M.). Elementary trigonometry. London, Bell, 1927. 18+288+31+24 pp.

FLADT (K.). Euklid. Berlin, Salle, 1927. 72 pp.

FRANZ (W. E.). $X^n+Y^n\gtrless Z^n$. Berlin, Verein für Soziale Ethik und Kunstpflege, 1927.

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FIFTY-FIFTH REGULAR MEETING OF THE SAN FRANCISCO SECTION

The fifty-fifth regular meeting of the San Francisco Section of the Society was held at Reed College on Saturday, June 2, 1928. In the absence of Chairman Bell, Professor E. E. DeCov acted as temporary chairman. The total attendance was 26, including the following 19 members of the Society:

J. R. Ballantine, Bernstein, Butler, J. W. Campbell, A. F. Carpenter, D. R. Davis, Donahue, Gavett, Griffin, Hanawalt, Jerbert, Kent, Langford, McAlister, W. E. Milne, Moritz, Mullemeister, Neikirk, Smail.

The Secretary announced that, by invitation of the Program Committee, Professor E. T. Bell will address the Section at its October meeting on the topic *The theory of algebraic numbers in the light of Kronecker's program.*

On motion of Professor Moritz, it was resolved: *That the Section express its sense of loss in the death on May* 29 *of Professor R. E. Allardice, one of the organizers, a former Chairman, and a contributor to the program of the Section.*

The Secretary announced that Chairman Bell appointed the following committee to nominate officers at the October meeting of the Section: Bernstein, Griffin, E. R. Hedrick.

Professor Kent invited the Section to hold its next Summer meeting at the Oregon State Agricultural College. The invitation was accepted.

Following the meeting, the members and their friends were greeted by President Colman of Reed College at a luncheon given them by the College. A drive up the beautiful Columbia River Highway followed the luncheon.

Titles and abstracts of the papers presented at the meeting follow below. The papers by Ayres, Neikirk, Roberts, and Whyburn were read by title. Miss Haycraft and Miss Nelson were introduced by Professor Griffin. Mrs. Williamson was introduced by Professor Carpenter.

1. Professor B. A. Bernstein: *Irredundant sets of postulates for the logic of propositions.*

Alonzo Church has defined a set of postulates to be *irredundant* if the postulates are independent and no postulate can be "weakened" with respect to the set (Transactions of this Society, vol. 27, p. 318). In the present paper the author gives a number of irredundant sets of postulates for the logic of propositions (see Transactions of this Society, vol. 28, p. 472). The sets are free from the objections usually to be raised against a set of postulates made irredundant by Church's "mechanical" method.

2. Professor J. W. Campbell: *Catenary problems in mechanics.*

The solutions of catenary problems in mechanics are greatly facilitated when in addition to the functions sinh x, cosh x, and tanh x there are available tables of values of (sinh $x)/x$, (cosh) $x-1))/x$, and (cosh $x)/x$. Tables of these functions from $x=0.000$ to $x=3.000$ have been compiled and tabulated by the writer and their use in connection with various types of catenary problems shown.

3. Professor A. F. Carpenter: *Tetrads of ruled surfaces.*

In this paper the projective differential properties of four ruled surfaces whose line elements are in correspondence in sets of four, are determined by means of the invariants and covariants of a system of eight ordinary linear homogeneous differential equations of the first order in eight dependent variables, to which are adjoined four linear homogeneous relations between these variables.

4. Professor F. L. Griffin: *Points of minimum travel for a distributed population,* II.

In this paper Professor Griffin discusses further cases of the problem considered at the Vancouver meeting of the Section.

5. Professor F. L. Griffin: *Note on an investment problem involving mean values and probability.*

This note calls attention to an interesting elementary illustration of mean values and probability which occurs in connection with long-term leases containing optional building clauses and accompanying adjustment of rentals, when it is a question of purchasing the lessor's rights at a price to yield a specified rate of interest.

6. Miss Alace Haycraft: *Concerning geodesics on certain surfaces of revolution.*

Miss Haycraft studies in some detail the geodesics on the circular paraboloid and hyperboloid of one sheet, and she contrasts these geodesics with corresponding curves on the cylinder, cone, and sphere.

7. Dr. A. R. Jerbert: *Quadruples of space curves.*

A system of four linear, homogeneous, differential equations of the first order is interpreted as defining, to within a projectivity, four space curves. By virtue of their expression in terms of a common parameter, these curves are in one-to-one point correspondence. If corresponding points are joined by straight lines, a tetrahedron is formed. The study of the four associated curves is accordingly equivalent to the study of a one-parameter family of tetrahedrons. The latter point of view derives some interest from the fact that the tetrahedron configuration occurs frequently in projective differential geometry.

8. Professor W. E. Milne: *The behavior of a boundary value problem as the interval becomes infinite.*

The investigation applies to the linear differential equation $d^2u/dx^2 + G(x, \lambda) = 0$ with linear homogeneous boundary conditions at the ends of an interval $a \leqq x \leqq b$. The object is to determine the behavior of the characteristic numbers, characteristic solutions, and oscillation properties as a and b approach $-\infty$ and $+\infty$ respectively. It is assumed that $G(x, \lambda)$ increases from $-\infty$ to $+\infty$ as λ increases from $-\infty$ to $+\infty$, and also that as x becomes positively or negatively infinite, $G(x, \lambda)$ becomes negatively infinite. Then as the interval becomes infinite, the characteristic numbers approach a set of critical values $\lambda_0, \lambda_1, \lambda_2, \cdots$, which have a limit point at plus infinity only, and which are entirely independent of the boundary conditions. The corresponding characteristic solutions approach critical solutions $u_0, u_1, u_2, \cdots$, which are independent of the boundary conditions. The solution, u, vanishes exactly n times in the infinite interval. Furthermore the integrals from minus infinity to plus infinity of $(du/dx)^2$ and of $G(x, \lambda)u^2$ exist if u is any critical solution, but do not exist for any other solution of the differential equation.

9. Professor R. E. Moritz: *An elementary derivation of the probability function.*

The view-point in the derivation is purely geometric. The procedure is the gradual deformation of the frequency polygon corresponding to binomial distribution $(p+q)^n$. By a proper choice of coordinates and repeated application of Stirling's formula it is shown that as n is indefinitely increased the frequency polygon goes over into the curve $y = \exp(-\pi x^2)$.

10. Professor R. E. Moritz: *A new theory of depreciation.* Preliminary report.

All known methods for the determination of the depreciation of physical assets proceed on the assumption that the life time of the asset is known. In this paper the writer assumes that the repair function is known and attempts to determine the life of the asset so as to make the annual charge to production a minimum.

11. Professor L. I. Neikirk: *A class of totally discontinuous functions.* Second paper.

This paper contains a generalization of the results of the first paper presented to the Section last Spring. The problem is to construct a function $u(x)$ with the following properties: it is equal to some one of the functions $f(x), f_1(x), f_2(x), \cdots, f_r(x)$ at every point of the interval $(0, 1)$, so that if $u(a_1)=f(a_1)$ and $u(a_2)=f(a_2)$, where $0 \leqq a_1 < a_2 \leqq 1$, then at some number, a', where $a_1 < a' < a_2$, $u(a')=f_i(a')$, $1 \leqq i \leqq r$; and if $u(b_1)=f_i(b_1)$ and $u(b_2)=f_i(b_2)$, where $0 \leqq b_1 < b_2 \leqq 1$, then there is a b', $b_1 < b' < b_2$, at which $u(b')=f(b')$; and also if $u(c_1)=f_2(c_1)$ and $u(c_2)=f_2(c_2)$, where $0 \leqq c_1 < c_2 \leqq 1$, then there are numbers c'_s, $s=1, 2, \cdots, i-1, i+1, \cdots, r$ and $c_1 < c'_s < c_2$ such that $u(c'_s)=f_s(c'_s)$. The function $u(x)$ is constructed and is defined explicitly.

12. Miss Mildred Nelson: *A further generalization of the circular and hyperbolic functions of sectorial areas.*

The generalization of circular and hyperbolic functions presented by Miss Pennock at the Vancouver meeting of the Section is extended by Miss Nelson. She substitutes an ellipse for the circle and an hyperbola of unequal semi-axes for the usual rectangular hyperbola; also she considers sectional areas from points asymmetrically located as well as from points on the transverse axis.

13. Mr. J. H. Roberts and Mr. J. L. Dorroh: *On a problem by G. T. Whyburn.*

In his recent paper *Concerning irreducible cuttings of continua*, G. T. Whyburn gives an example of a continuum M which contains no indecomposable continuum, but which contains two points A and B, such that no cutting of M between A and B is irreducible. He proves that if M is an indecomposable continuum, then *every* two points A and B of M are such that no irreducible cutting of M between A and B exists. He raises the following questions: (1) If a continuum M is such that no subset of M is an irreducible cutting of M between any two points A and B of M, then is M necessarily indecomposable? (2) If every cutting of a continuum M is reducible, is M necessarily indecomposable? In the present paper it is shown by an example that the answer to both of these questions is in the negative. Also, it is proved that a necessary and sufficient condition that every subcontinuum of a continuum M be indecomposable is that no subcontinuum of M contain an irreducible cutting of itself.

14. Mr. J. H. Roberts: *Concerning closed sets which contain no domain in n-dimensional euclidean space.*

The author has recently shown that if M denotes a plane continuum containing no domain, then there exists a continuous one-to-one transformation of the plane into itself such that in the transformed plane there exists a rectangular coordinate system with the property that if l denotes any line parallel to one of the coordinate axes, then the set of points common to l and M is either vacuous or totally disconnected. This result is included in the following more general theorem, proved in the present paper: If S is an n-dimensional euclidean space, and M is a closed subset of S containing no n-dimensional domain, then there exists a continuous one-to-one transformation of S into itself, such that in the transformed space every straight line contains at most a totally disconnected subset of M.

15. Professor L. L. Smail: *On the effect of the operation of summability on the interval of oscillation of an infinite sequence.* Preliminary report.

Practically all previous studies of the summability of divergent series have been concerned with the unique limit of an auxiliary sequence associated with the given series. This paper treats of the relation between the upper and lower limits of the auxiliary sequence whose unique limit, when it exists, gives the usual generalized sum of the given sequence. This relation is investigated for the general methods of summability, and also for the Cesàro method when the order of summability is increased.

16. Professor G. T. Whyburn: *Concerning cuttings of continua.*

In this paper it is shown that in a locally compact metric and separable space: (1) no continuum M contains an uncountable collection of mutually exclusive connected subsets each containing a compact proper set which cuts M; (2) the author's theorem (previously established in the plane) that all save a countable number of the cut points of any continuum M are points of order two (Menger-Urysohn) of M holds true; and indeed that if G is any collection of mutually exclusive connected cuttings of M, then all save a countable number of elements of G can be ϵ-separated by two sets of G; (3) all save a countable number of the *cut points im kleinen* of any continuous curve M are points of order two of M; (4) the junction points of every continuous curve are countable; and (5) if every subcontinuum of a continuous curve M contains uncountably many cut points im kleinen of M, then M is a Menger regular curve. Also the notion of a *junction continuum*, an extension of the notion of a *junction point* due to R. L. Moore, is discussed, and it is pointed out that no *plane continuum* contains an uncountable collection of mutually exclusive junction continua of itself.

17. Professor E. H. Mc Alister: *On the dynamical form of Hooke's law.* Preliminary report.

Hooke's law leads to results not in agreement with experiment when the velocity of strain is not negligible. In particular, the rapid decay of elastic vibrations finds no explanation in Hooke's law, and is accounted for only in small part by the resistance of the air. Kelvin conjectured some internal resistance, and coined the phrase "viscosity of metals," but concluded that the resistance is not merely proportional to the velocity, as in viscous fluids. This paper attempts to formulate a law suitable for dynamical conditions, as Hooke's law is for statical conditions. The point of mathematical interest is the search for an unknown function by mathematical means, guided by experiment. A form is found which agrees with certain experiments on torsional vibrations of a steel wire. The mathematical investigation indicates that other forms are possible, and the results of this paper are merely tentative, awaiting further research. The paper is therefore presented as a preliminary report. The tentative law admits of isochronous vibrations, in accordance with Rayleigh's dictum, and reduces to Hooke's law when the velocity is null or negligible.

18. Professor D. R. Davis: *The inverse problem of the calculus of variations in space of* $(n+1)$ *dimensions.*

In this paper is considered a system of n differential equations of the form $H_j(x, y_i(x), y_i'(x), y_i''(x))=0$, $(i, j=1, \cdots, n)$, whose solutions are $y_i=y_i(x)$, with the derivatives y_i', y_i''. The conditions required for the given system to have equations of variation which are self-adjoint are found. It is then proved that these conditions are sufficient to insure that a function $f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$ can be determined such that the given equations are the differential equations of the solutions for the problem of minimizing the integral

$$I=\int_{x_1}^{x_2} f(x, ,y_1, \cdot\ \cdot\ y_n, y_1', \ \cdot\cdot, y_n')dx$$

Incidentally, in this paper, are also found necessary and sufficient conditions that there exist a solution for a system of linear partial differential equations of the first order of the form $B_{jy_i}-B_{iy_j}=\phi_{ij}$, $B_{ix}-A_{y_i}=\phi_i$, $(i, j=1, \cdots, n)$, where $A, B_1, \cdots, B_n, \phi_{ij}, \phi_i$ are functions of $x, y_1, \cdots, y_n$, and $\phi_{ij}=-\phi_{ji}$.

19. Professor G. T. Whyburn: *Concerning certain points of continuous curves.*

In this paper a study is made of the *im kleinen cut points*, *im kleinen cycle points*, and ramification points (i.e., points of Menger order>2) of a bounded continuous curve M in n dimensions. The point P of M is called an *im kleinen cycle point* of M if for each $\epsilon>0$, M contains a simple closed curve of diameter $<\epsilon$ containing P. Some of the results proved are as follows. Let K, N, H, and W denote the set of all *im kleinen cut points, im kleinen cycle points*, end points, and ramification points respectively of M. Then (1) $K+H+N=M$, (2) $K\cdot N$ and $K\ \ W$ are countable, (3) $K.\ N\subset W$, (4) if W is dense on an arc t of M, both N and $M-K$ are uncountably dense on t, (5) if M is cyclicly connected and $M-K$ is not uncountably dense on any subcontinuum of M, then (a) W is totally disconnected, and (b) every component of $M-W$ is an arc segment, (6) if $M=K$, then for each $\epsilon>0$, M is the sum of a finite number of ϵ-continua no two having more than one common point.

20. Professor G. T. Whyburn: *Concerning a certain type of regular curve.*

A continuous curve M will be called a *node curve* provided that for each $\epsilon>0$, M is the sum of a finite number of ϵ-continua each having at most two points in common with the rest of M. In this paper a study is made of node curves, and some of the results found are as follows. Let M denote any bounded continuous curve in n dimensions. Then (1) M is a node curve if and only if it is true that if C is any maximal cyclic curve of M then (a) C is a node curve and (b) every point of C which is a limit point of some component of $M-C$ is an *im kleinen cut point* of C; (2) if M is cyclicly connected, M is a node curve without points of order w if and only if no ramification point of M is an *im kleinen cycle point* of M; (3) if no point of M is an *im kleinen cycle point*, then M is a node curve; (4) the ramification points of a node curve are countable and each of them is an *im kleinen cut point.*

21. Professor G. T. Whyburn: *A continuous curve in 3-space.*

In this paper a simple example is given in 3 dimensions of a continuous curve ψ every subcontinuum of which is a continuous curve which has the following properties: (1) ψ contains two points A and B and infinitely many arcs AX_iB ($i=1, 2, 3, \cdots$) from A to B no two of which have any common points except A and B, (2) ψ contains an arcwise connected subset which is not arcwise connected *im kleinen* (in fact, not connected *im kleinen* at all) and which has an inaccessible boundary point, and (3) ψ contains a connected subset which is not arcwise connected. It is already known that ψ could not lie in the plane and have properties (1) or (2). It is also pointed out in this paper that no Menger regular curve can have properties (1) or (2), and a number of related questions are settled in n dimensions for regular curves and for continuous curves all of whose subcontinua are continuous curves.

22. Mrs. E. N. Williamson: *Two plane curves in point correspondence.*

This paper deals with the projective differential properties of two plane curves in one-to-one reciprocal point correspondence, making use of a defining system of three ordinary linear homogeneous differential equations in two dependent variables, two of the equations being of the second order and one of the first. Geometric interpretations are obtained for certain of the invariants and covariants of the system.

23. Dr. L. Ayres, (National Research Fellow): *Concerning the arc-curves and the basic sets of a continuous curve.* Second Paper.

This paper studies the relations between the cyclic elements of a continuous curve and the arc-curves, and develops some new properties of the basic sets of a continuous curve. It is shown that every arc-curve of more than one point is a collection of cyclic elements, and every connected collection of cyclic elements is an arc-curve. Every arc-curve of two points is a simple cyclic chain and conversely. A subset K of a continuous curve M, which is not cyclicly connected, is an irreducible basic set of M if and only if (1) no point of K is a cut point of M, (2) every point of K belongs to some node of M and each node of M contains exactly one point of K. An irreducible basic set of a continuous curve M contains the set of all end points of M, and an irreducible basic set has many of the properties of the set of all end points.

B. A. BERNSTEIN.
Secretary of the Section

ON A CERTAIN SYSTEM OF ∞^{r-2} LINES IN r-SPACE

BY B. C. WONG

This paper deals with the following theorem.

The locus of ∞^{r-2} *lines incident with* r *given* $(r-2)$*-spaces in* S_r *is an* $(r-1)$*-dimensional manifold* V_{r-1}^{r-1} *of order* $r-1$.

To prove this, we note that for $r=2$ there is one line joining two given points in a plane and that for $r=3$ the lines meeting three given lines in an S_3 form a quadric surface. If $r=4$, that is, if four planes are given in S_4, the locus of the ∞^2 lines incident with them is a V_3^3. This cubic hypersurface with its many interesting properties has been studied by a number of writers.*

If $r=5$, that is, if five three-spaces are given in S_5, pass an S_4 through one of them, say R_3. This S_4 meets the other four 3-spaces in four planes and the lines incident with these four planes are also incident with R_3. These lines form a V_3^3 in S_4. The manifold of the ∞^3 lines incident with the five given 3-spaces is intersected by the S_4 through R_3 in R_3 and a V_3^3 and is therefore of order 4. If we apply this process of reasoning to the cases $r=6$, 7, etc., we soon arrive at the general theorem stated above.

Consider another proof. Let the r given $(r-2)$-spaces in S_r be $S'_{r-2}, S''_{r-2}, \cdots, S^{(r)}_{r-2}$, and further let a general line l be given. The points of l determine with $S'_{r-2}, S''_{r-2}, \cdots, S^{(r-1)}_{r-2}$ $r-1$ projective pencils of hyperplanes. The rth $(r-2)$-space, $S^{(r)}_{r-2}$, intersects these pencils in $r-1$ pencils of $(r-3)$-spaces. As there are $r-1$ sets of corresponding $(r-3)$-spaces each intersecting in a point, there are $r-1$ lines of intersection of corresponding hyperplanes of the pencils in S_r which meet $S^{(r)}_{r-2}$. Hence, the general line l meets $r-1$ of the ∞^{r-2} lines incident with the r given $S^{(i)}_{r-2}[i=1, 2, \cdots, r]$, and therefore the manifold of the ∞^{r-2} lines is of order $r-1$.

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* See Bertini, *Projecktive Geometrie Mehrdimensionaler Räume*, 1924, Chap. 8, §§25–36, where references are given.

It is of interest to note that the V^{r-1}_{r-1} in question contains the r given $(r-2)$-spaces $S^{(i)}_{r-2}$ $[i=1, 2, \cdots, r]$. These $S^{(i)}_{r-2}$ intersect ν by ν in $\binom{r}{\nu}$ $(r-2\nu)$-spaces $S^{(j)}_{r-2\nu}$ $[\nu=1, 2, \cdots, r/2$ if r is even, $\nu=1, 2, \cdots, (r-1)/2$ if r is odd; $j=1, 2, \cdots, \binom{r}{\nu}]$. The V^{r-2}_{r-1} contains all these $S^{(j)}_{r-2\nu}$ ν-ply. Any V^{r-1}_{r-1} in S_r containing r $(r-2)$-spaces is a hypersurface of this type, for any line meeting these r $(r-2)$-spaces lies entirely on the hypersurface and there are ∞^{r-2} such. For the case $r=4$, there are additional planes and conical points. The $V_3{}^3$, besides containing the four given planes and the six conical points in which they intersect two by two, contains eleven other planes and four other conical points. These fifteen planes and ten nodes are such that each plane contains four nodes and each node is on six planes.

The equation of the V^{r-1}_{r-1} we are considering is not yet known. The writer has derived the following equation for $V_3{}^3$ in S_4:

$$x_0x_1x_4 + x_1x_2x_3 + x_2x_3x_4 - x_0x_2x_3 - x_1x_2x_4 - x_1x_3x_4 = 0.$$

The ten conical points are the vertices of the coordinate simplex, the unit point and the following:

$(1:1:1:0:0)$, $(1:0:1:0:1)$, $(1:0:0:1:1)$, $(1:1:0:1:0)$.

By the linear transformation

$$\begin{array}{rllllll}
\rho x_0 = & & & - y_2 & + y_3, & \\
\rho x_1 = & y_0 & + y_1 & & + y_3 & + y_4, \\
\rho x_2 = & & y_1 & & + y_3, & \\
\rho x_3 = & y_0 & & & + y_3, & \\
\rho x_4 = & & & - y_2 & & - y_4
\end{array}$$

the above equation is transformed into

$$y_0^3 + y_1^3 + y_2^3 + y_3^3 + y_4^3 + y_5^3 = 0,$$

where

$$y_0 + y_1 + y_2 + y_3 + y_4 + y_5 = 0.$$

THE UNIVERSITY OF CALIFORNIA

NEW DIVISION ALGEBRAS

BY L. E. DICKSON

1. *Introduction.* No technical acquaintance with linear algebras is presupposed in this note. We consider only linear algebras for which multiplication is associative. As with quaternions, an algebra A is called a *division* algebra if every element $\neq 0$ of A has an inverse in A. A division algebra A over a field F is called *normal* if the numbers of F are the only elements of A which are commutative with every element of A.

In a paper recently offered to the Transactions of this Society, A. A. Albert determined all normal division algebras of order 16 and found a new type. The object of this note is to derive from mild assumptions the corresponding type of normal division algebras A of order $4p^2$, where p is a prime. We shall first draw simple conclusions from an initial assumption.*

Assumption 1. Let A contain an element i_1 satisfying an equation $f(\omega^2)=0$ of degree $2p$ with only even powers of ω, whose coefficients are in F, that of ω^{2p} being unity, and which is irreducible in F, such that the polynomials in i_1 are the only elements of A which are commutative with every element of A.

2. LEMMA 1. *Let an element i_2 of A be commutative with $I=i_1^2$, but not with i_1 itself. The algebra S generated by i_1 and i_2 is of order $4p$. It may be regarded as an algebra of order 4 with the basis $1, i_1, i_2, i_1i_2$ over $F(I)$; this algebra is normal. In other words, the polynomials in I are the only elements of S which are commutative with every element of S.*

Let K denote the field composed of all those elements of

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[*] Except for the requirement concerning even powers of ω, Assumption 1 is proved in the writer's *Algebren und ihre Zahlentheorie*, Zürich, 1927, pp. 262–3.

S which are commutative with every element of S. If K is of order k and S is of order s over F, then S is a normal division algebra of order n^2 over K, where $s=n^2k$. Since K contains the root I of an equation of degree p irreducible in F, the subfield $F(I)$ is of order p, whence k is a multiple of p.

Since i_2 is not commutative with i_1, i_2 is not a polynomial in i_1 and hence is not a rational function of i_1. Thus

$$i_1^j, \; i_1^j i_2, \qquad (j = 0, 1, \cdots, 2p-1), \tag{1}$$

are linearly independent with respect to F. Hence $s \geqq 4p$. Since S and A are normal over different fields K and F, $S \neq A$. Thus s is a divisor $<4p^2$ of $4p^2$. First, let $p>2$. If s is not divisible by p^2, then $s=4p$. But if s is divisible by p^2, either $s=2p^2$, or $s=p^2$ and $p>4$. If $p=2$, evidently $s=8=4p$.

If either $s=p^2$, $p>4$, or $s=2p^2$, $p>2$, then $s=n^2k$ and the divisibility of k by p show that $n=1$, $S=K$, contrary to the fact that i_2 is not commutative with i_1.

Hence $s=4p=n^2k$, whence $n=2$, $k=p$. Thus $K=F(I)$ and S is a normal algebra of order 4 over $F(I)$. The $4p$ elements (1) form a basis of S over F.

3. LEMMA 2. *Any element of A which is commutative with $I=i_1^2$ belongs to S.*

Any element not in S extends S to a division subalgebra whose order exceeds $4p$, is a multiple of $4p$, and is a divisor of $4p^2$. Hence it extends S to A itself (of order $4p^2$).

Suppose that e is commutative with I and is not in S. Since I is commutative with every element of S and with e, which extends S to A, I is commutative with every element of A. Since I is not in F, this contradicts the hypothesis that A is normal over F.

4. *Assumption* 2. Let A contain elements i_1 and z such that i_1 satisfies Assumption 1 and such that

$$i_2 = z i_1 z^{-1}, \; i_3 = z i_2 z^{-1}, \; \cdots, \; i_p = z i_{p-1} z^{-1} \tag{2}$$

are all commutative with $I=i_1^2$, while i_2 is not commutative with i_1, and $i_2^2 \neq I$.

Since $zIz^{-1}=i_2^2 \neq I$, z is not commutative with I and hence is not in S. By §3, z extends S to A. Since (1) gives a basis of S, every element of S is of the form

$$G = p(i_1) + q(i_1)i_2. \tag{3}$$

Then

$$G' = zGz^{-1} = p(i_2) + q(i_2)i_3. \tag{4}$$

For $p \geqq 3$, i_3 is commutative with i_1^2 and hence is in S. Thus

$$zG = G'z,\ G' \text{ in } S. \tag{5}$$

5. Lemma 3. *$i_1^2, \cdots, i_p^2$ are all distinct.*

Suppose that $i_{r+1}^2=i_1^2$, where r is one of $2, 3, \cdots, p-1$. Then

$$z^r i_1^2 z^{-r} = i_{r+1}^2 = i_1^3,$$

whence z^r is commutative with i_1^2 and is in S. Using also (5), we see that every element of the algebra A obtained by extending S by z is of the form

$$H_0 + H_1 z + \cdots + H_{r-1}z^{r-1},$$

where each H is in S. Since S is of order $4p$, the order of A is $\leqq 4p \cdot r < 4p^2$. But A is of order $4p^2$.

Suppose that $i_{r+s}^2=i_s^2$ $(r>0,\ s>1)$. These are the transforms of i_{r+s-1}^2 and i_{s-1}^2 by z. Hence the latter are equal. After $s-1$ such steps, we get $i_{r+1}^2=i_1^2$, just proved impossible.

6. Lemma 4. *We have the following identity:*

$$f(\epsilon) \equiv (\epsilon - i_p^2) \cdots (\epsilon - i_2^2)(\epsilon - i_1^2). \tag{6}$$

Note that

$$i_r \text{ is commutative with } i_{r+1}, \cdots, i_p, \quad (r=1, \cdots, p-1). \tag{7}$$

This is true by Assumption 2 if $r=1$. To proceed by induction, let (7) hold when $r=j$, whence i_j^2 is commutative with i_k for $k \geqq j+1$. Transformation by z shows that i_{j+1}^2 is commutative with i_{k+1}, whence (7) holds when $r=j+1$.

Write v_j for i_j. As a special case of (7), $v_1, \cdots, v_p$ are commutative. The indeterminate ϵ is commutative with every quantity of A. Thus z transforms $f(\epsilon)$ into itself. But $f(v_1)=0$. Hence by (2), $f(v_2)=0, \cdots, f(v_p)=0$. Let

$$f(\epsilon) = \sum_{j=0}^{p} a_j \epsilon^{p-j}, \quad q(\epsilon) = \sum_{j=0}^{p-1} c_j \epsilon^{p-1-j}, \quad a_0 = c_0 = 1,$$

$$c_j = a_j + c_{j-1} v_1, \qquad (j = 1, \cdots, p).$$

Then, since v_1 is commutative with ϵ,

$$f(\epsilon) \equiv q(\epsilon)(\epsilon - v_1) + c_p. \tag{8}$$

By induction on r,

$$c_r = \sum_{j=0}^{r} a_j v_1^{r-j}, \qquad c_p = f(v_1) = 0.$$

Since v_i is commutative with v_1, we obtain a true equality from (8) by replacing ϵ by v_i. Thus $0 = q(v_i)(v_i - v_1)$. The second factor is not zero if $i \geqq 2$. In our division algebra we therefore have $q(v_i) = 0$ when $i \geqq 2$.

We may repeat this argument with f and v_1 replaced by q and v_2. Hence $q(\epsilon) \equiv r(\epsilon)(\epsilon - v_2)$, in which the coefficients of $r(\epsilon)$ are polynomials in v_1 and v_2. Since they are commutative with v_j, $0 = r(v_j)(v_j - v_2)$. Hence $r(v_j) = 0$ when $j \geqq 3$.

Proceeding similarly, we ultimately obtain

$$f(\epsilon) \equiv (\epsilon - v_p) \cdots (\epsilon - v_2)(\epsilon - v_1).$$

7. THEOREM 1. *$f(\epsilon) = 0$ is a cyclic equation.*

By (6), $i_1^2 + \cdots + i_p^2$ is a number of F and hence is transformed into itself by z. But z transforms i_1^2 into i_2^2, $\cdots$, i_{p-1}^2 into i_p^2. Hence z must transform i_p^2 into i_1^2. Since z^{p-2} transforms i_2^2 into i_p^2, z^{p-1} transforms i_2^2 into i_1^2 and evidently transforms i_1 into i_p. Hence z^{p-1} transforms

$i_2^2 i_1$ and $i_1 i_2^2$ into $i_1^2 i_p$ and $i_p i_1^2$. The latter are equal by by Assumption 2. Hence the former are equal. Since i_2^2 is therefore commutative with both generators i_1 and i_2 of S, it is commutative with every element of S. By Lemma 1, $i_2^2 = \theta(i_1^2)$, where θ is a polynomial with coefficients in F. Transformation by z gives

$$i_3^2 = \theta(i_2^2) = \theta[\theta(i_1^2)] = \theta^2(i_1^2),$$

if $\theta^r(k)$ denotes the rth iterative of $\theta(k)$ and not its rth power. By induction,

$$i_{r+1}^2 = \theta^r(i_1^2). \tag{9}$$

Take $r = p-1$ and transform by z. Hence

$$i_1^2 = \theta^{p-1}(i_2^2) = \theta^p(i_1^2). \tag{10}$$

Since $f(\epsilon) = 0$ has these properties, it is cyclic.

8. THEOREM 2. *Every element of A can be expressed in one and only one way in the form*

$$A_0 + A_1 z + \cdots + A_{p-1} z^{p-1}, \tag{11}$$

where each A_j is in S. The product any two sums (11) *can be expressed as a third such sum by means of*

$$zG = G'z, \quad z^p = s, \tag{12}$$

where G, G', s are all in S and are defined in (4), (5).

Since z^{p-1} transforms i_1^2 into i_p^2, and z transforms the latter into the former, z^p is commutative with i_1^2 and hence is in S. By means of (12), every element of A (to which z extends S) can be expressed in the form (11). Since S and A are of orders $4p$ and $4p^2$, two polynomials (11) are distinct unless identical.

9. THEOREM 3. *S is an algebra of generalized quaternions over $F(I)$ with the basis* 1, i_1, y, $i_1 y$, *where* $y = i_1 i_2 - i_2 i_1$.

Since i_2 is not commutative with i_1, $y \neq 0$. Since i_2 is commutative with i_1^2,

$$yi_1 = -\, i_1 y. \tag{13}$$

Thus y is not commutative with i_1 and hence is not a polynomial in i_1. We may therefore replace the basis (1) of S over F by $i_1{}^j$, $i_1{}^j\, y$. Thus S has the basis in Theorem 3.

By §7, i_2^2 is commutative with i_1. Hence

$$r = i_1 i_2 + i_2 i_1$$

is commutative with i_2. Since i_2 is commutative with $I = i_1^2$, $ri_1 = i_1 r$. Hence r is commutative with every element of S. Thus r is a polynomial $P(I)$ in I. We have

$$2i_1 i_2 = P(I) + y,\ 2i_2 i_1 = P(I) - y.$$

But y is commutative with I. Hence

$$4i_1 i_2^2 i_1 = P^2 - y^2.$$

Since i_2^2 is commutative with i_1,

$$y^2 = [P(I)]^2 - 4I\, \theta(I).$$

This fact that y^2 is a polynomial in I and relation (13) together show that S is an algebra of generalized quaternions over $F(I)$.

THE UNIVERSITY OF CHICAGO

NOTE ON THE DERIVATIVE CIRCULAR CONGRUENCE OF A POLYGENIC FUNCTION*

BY EDWARD KASNER

In the first paper† I published on polygenic functions I stated the following theorem.

If

$$w = \phi(x,y) + i\psi(x,y)$$

is a polygenic function of

$$z = x + iy$$

then the first derivative of w with regard to z, $dw/dz = \gamma = \alpha + i\beta$, is represented in the γ-plane by the congruence of circles

$$(\alpha - H)^2 + (\beta - K)^2 = R^2 \tag{1}$$

where

$$\left\{\begin{array}{ll} 2H = \phi_x + \psi_y, & 2K = -\phi_y + \psi_x, \\ R^2 = h^2 + k^2, & \\ 2h = \phi_x - \psi_y, & 2k = \phi_y + \psi_x. \end{array}\right. \tag{2}$$

To every point (x, y) of the z-plane corresponds by means of dw/dz that circle of (1) determined by the particular pair of values (x, y).

In this paper I wish to study the converse problem. Let $H(x, y)$, $K(x, y)$, and $R(x, y)$ be three arbitrary functions of (x, y)‡ and for every pair of values (x, y) construct in the γ-plane the circle whose center is (H, K)

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† *A new theory of polygenic functions*, Science, vol. 66 (Dec., 1927), pp. 581–582. See also a previous paper by Hedrick, Ingold, and Westfall, Journal de Mathématiques, *Theory of non-analytic functions of a complex variable*, (6), vol. 2 (1923), pp. 327–342.

‡ We require, however, that these functions be continuous and have continuous first, second, and third derivatives.

and whose radius is R. What are the necessary and sufficient conditions to be fulfilled by H, K, and R, in order that the congruence of circles so obtained may map the derivative of a polygenic function? In other words, when can we find two functions $\phi(x, y)$ and $\psi(x, y)$, such that the relations (2) expressing the correspondence between the circles of the congruence and the points of the z-plane are fulfilled?

The required conditions are found by eliminating ϕ and ψ from the three equations

$$(3) \quad \left\{ \begin{array}{l} 2H = \phi_x + \psi_y, \qquad 2K = -\phi_y + \psi_x, \\ 4R^2 = (\phi_x - \psi_y)^2 + (\phi_y + \psi_x)^2. \end{array} \right.$$

This is accomplished by forming all the first derivatives, then all the second derivatives, then all the third derivatives etc., of system (3), till at one step the total number of equations obtained is larger than the number of derivatives $\phi_x, \phi_y, \psi_x, \cdots$ occurring in them. The necessary and sufficient conditions on H, K, R are then furnished by the eliminants with regard to $\phi_x, \phi_y, \psi_x, \cdots$ of the entire system of equations.

Calculation shows that the process of differentiation has to be carried up to the third derivatives of (3); the total number of equations obtained is then thirty, while the number of derivatives $\phi_x, \phi_y, \psi_x, \cdots, \cdots \psi_{yyyy}$ is only twenty-eight. The system will therefore have two eliminants, that is, *H, K, and R have to obey two conditions in order that their congruence may represent the derivative of a polygenic function.*

We will not carry out the elimination directly from (3) but from another system of three independent functions of H, K and R, namely,

$$M = \tfrac{1}{2}[H_x - K_y + i(H_y + K_x)] = \tfrac{1}{2}[h_x + ik_x - i(h_y + ik_y)],$$

$$N = \tfrac{1}{2}[H_x - K_y - i(H_y + K_x)] = \tfrac{1}{2}[h_x - ik_x + i(h_y - ik_y)],$$

$$O = R^2 = h^2 + k^2.$$

We simplify this system by the following changes.

1. We use

$$u = x + iy, \quad v = x - iy^*$$

as independent variables instead of x and y; in order to be able to substitute

$$x = \frac{u+v}{2}, \quad y = \frac{u-v}{2i}$$

into ϕ and ψ, we assume these functions to be analytic, that is, developable into power series, in x and y in the region considered.†

2. We use

$$h + ik = a, \quad h - ik = b$$

instead of ϕ and ψ as the functions to be eliminated. Then O, M, N assume the simple forms

$$O = a \cdot b, \qquad M = a_u, \qquad N = b_v.$$

To carry through the elimination of a and b we start out from the following expressions:

$$(4) \quad \left\{\begin{array}{ll} M = a_u, & O = ab, \\ N = b_v, & O_{uv} = a_{uv}b + ab_{uv} + a_ub_v + a_vb_u, \\ O_u = a_ub + ab_u, & M_v = a_{uv}, \\ O_v = a_vb + ab_v, & N_u = b_{uv}. \end{array}\right.$$

From these eight equations we eliminate the six first and second derivatives of a and b which occur in them. The resulting equations are

$$(5) \qquad ab - O = 0, \quad Aa + Bb + C = 0,$$

where

$$(6) \qquad \left\{\begin{array}{l} A = ON_u - O_uN, \\ B = OM_v - O_vM, \\ C = O_uO_v + O(2MN - O_{uv}). \end{array}\right.$$

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[*] We use u and v instead of z and $\bar{z}$, merely to simplify the writing when these letters occur as subscripts.

[†] In my previous papers, ϕ and ψ were only assumed to be continuous differentiable functions of x and y.

Differentiating the second equation of (5) with regard to u and with regard to v, and replacing the first derivatives of a and b in the two expressions obtained by their values in M, N, O_u, O_v, a, b according to (4), we find

$$(7)\qquad \begin{cases} A_u a^2 + (AM + C_u)a - BMb + (BO)_u = 0, \\ B_v b^2 + (BN + C_v)b - ANa + (AO)_v = 0. \end{cases}$$

The elimination of a and b from (5) and (7) furnishes two equations in O and its derivatives up to the third order, and in M and N and their derivatives up to the second order. If in these equations we return from the complex quantities u, v, M, N to the original real quantities x, y, H, K, and from O to R, we obtain two equations containing R and the derivatives of H, K, and R up to the third order. According to what was said above, they represent the necessary and sufficient conditions that must be fulfilled by H, K, and R.

These two equations will be either mutually conjugate or each self-conjugate. This follows from the fact that u and v, a and b, M and N, A and B, are mutually conjugate, while O and C are self-conjugate, so that equations (5) are self-conjugate, while equations (7) are mutually conjugate.

The elimination of a and b from (5) and (7) can be carried through as follows. By multiplying the linear equation of (5) first by $A_u \cdot a/A$, then by $B_v \cdot b/B$, and subtracting it first from the first, then from the second, equation of (7), we reduce (5) and (7) to the simpler system

$$(8)\qquad \begin{cases} ab - O = 0, & Aa + Bb + C = 0, \\ A_1 a + B_1 b + C_1 = 0, & A_2 a + B_2 b + C_2 = 0, \end{cases}$$

where

$$\begin{aligned} A_1 &= A(AM + C_u) - A_u C, & A_2 &= -ABN, \\ B_1 &= -ABM, & B_2 &= B(BN + C_v) - B_v C, \\ C_1 &= A(BO)_u - BA_u O, & C_2 &= B(AO)_v - AB_v O. \end{aligned}$$

Then

$$(9)\qquad \begin{vmatrix} A & B & C \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix} = 0, \quad \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} \cdot \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}^2 \cdot O$$

represent the two eliminants of the system (8).

The two necessary and sufficient conditions to be fulfilled by three arbitrary functions $H(x, y)$, $K(x, y)$, and $R(x, y)$ of x and y in order that the congruence of circles

$$(\alpha - H)^2 + (\beta - K)^2 = R^2$$

which they determine in the (α, β)-plane may represent the derivative of a polygenic function are obtained by retransforming the two equations (9) *from the quantities u, v, M, N, O into the quantities x, y, H, K, R; the two final conditions contain R and the derivatives of H, K and R up to the third order.*

COLUMBIA UNIVERSITY

ON THE INVERSION OF ANALYTIC TRANSFORMATIONS*

BY B. O. KOOPMAN†

We wish to consider the transformation

$$(1)\qquad x_i = f_i(y_1, \cdots, y_n), \qquad (i = 1, \cdots, n),$$

in the neighborhood of the origin $(y) = (0)$, at which point the functions f_i are analytic, and vanish simultaneously. We are interested in the case in which the jacobian

$$J = \frac{\partial(f_1, \cdots, f_n)}{\partial(y_1, \cdots, y_n)}$$

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* Presented to the Society under the title *Some theorems on inversion*, October 29, 1927.

The methods and point of view of the second chapter of W. F. Osgood's *Lehrbuch der Funktionentheorie* are assumed throughout this paper. Our results may be regarded as the extension of §20 of that work.

† National Research Fellow, 1926-1927.

is zero at the origin, without vanishing identically. We shall assume throughout that $(y)=(0)$ is a non-specialized point of the locus $J=0$; that is, it will be sufficient for our results to apply only when $(y)=(0)$ does not belong to certain exceptional $(2n-4)$-dimensional loci satisfying a further equation $E(y_1, \cdots, y_n)=0$, where E is analytic and relatively prime to J.

What we wish to accomplish is to obtain, with the aid of non-singular analytic transformations of the variables (x) and the variables (y), a standard form for the transformation (1), having particularly in view the investigation of its inverse.

Since $(y)=(0)$ is a non-specialized point of $J=0$, we shall have in a neighborhood of this point

$$J(y_1, \cdots, y_n) \equiv S(y_1, \cdots, y_n)^s \cdot L(y_1, \cdots, y_n),$$

where S and L are analytic at the origin, L does not vanish there, while S has there the value zero. Moreover, at least one derivative, say $\partial S/\partial y_1$, will be distinct from zero for $(y)=(0)$.

After the non-singular transformation

$$y_1' = S(y_1, \cdots, y_n), \qquad (y_2' = y_2, \cdots, y_n' = y_n),$$

the equations (1) retain the form

$$x_i = f_i'(y_1', \cdots, y_n'), \qquad (i = 1, \cdots, n), \tag{1'}$$

while their jacobian is readily found to be

$$J'(y_1', \cdots, y_n') \equiv y_1'^{s} \cdot L'(y_1', \cdots, y_n'),$$

L' being of the same nature as L. Hence we may and we shall assume that (1) has this form, that is, we shall drop the accents.

We now introduce, with Osgood, the matrix

$$\left\| \begin{matrix} \dfrac{\partial f_1}{\partial y_2}, & \cdots, & \dfrac{\partial f_n}{\partial y_2} \\ \cdot & \cdots & \cdot \\ \dfrac{\partial f_1}{\partial y_n}, & \cdots, & \dfrac{\partial f_n}{\partial y_n} \end{matrix} \right\| . \tag{2}$$

Suppose, in the first place, that when $y_1=0$, every element vanished identically. Considering the Taylor expansions of the f's, and observing that they are zero for $(y)=(0)$, we see that

$$f_i = y_1^{p_i}F_i(y_1, \cdots, y_n), \qquad (p_i > 0 \,;\, i = 1, \cdots, n).$$

Here the F's are analytic at the origin, and are not identically zero when $y_1=0$. Since the origin is a non-specialized point, we assume that $F_1\neq 0$ there. Then the transformation

$$\begin{cases} y_1' = y_1[F_1(y_1, \cdots, y_n)]^{1/p_1}, & \\ y_i' = y_i, & (i = 1, \cdots, n), \end{cases}$$

is analytic and non-singular at $(y)=(0)$; and we have

$$\begin{cases} f_1(y) = f_1'(y') = y_1'^{p_1}, & \\ f_i(y) = f_i'(y') = y_1'^{p_i}F_i(y_1', \cdots, y_n'), & (i = 2, \cdots, n). \end{cases}$$

Hence, dropping accents, we may assume (1) to have the form

$$(3) \qquad \begin{cases} x_1 = y_1^{p_1}, & \\ x_i = y_1^{p_i}F_i(y_1, \cdots, y_n), & (i = 2, \cdots, n). \end{cases}$$

In the case where the elements of (1) do not behave in the above manner, there will be a determinant j of (2) of order $r(0<r<n)$, which does not vanish identically when $y_1=0$, while every determinant in (2) of greater order is zero identically for $y_1=0$. Let us assume, as we may, that

$$j = \frac{\partial(f_{n-r+1}, \cdots, f_n)}{\partial(y_{n-r+1}, \cdots, y_n)}.$$

The origin being non-specialized, we assume that $j\neq 0$ when $(y)=(0)$.

The last r of the equations (1) admit the unique solutions

$$\begin{cases} y_{n-r+1} = g_{n-r+1}(y_1, \cdots, y_{n-r}\,;\, x_{n-r+1}, \cdots, x_n), \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \qquad y_n = g_n(y_1, \cdots, y_{n-r}\,;\, x_{n-r+1}, \cdots, x_n), \end{cases}$$

where the g's are analytic for zero values of the arguments, and vanish for these values. When these expressions are substituted into the first $n-r$ equations (1), we obtain

$$
\begin{aligned}
x_i = {} & f_i[y_1, \cdots, y_{n-r}; g_{n-r+1}(y_1, \cdots, y_{n-r}; x_{n-r+1}, \\
& \cdots, x_n), \cdots, g_n(y_1, \cdots, y_{n-r}; x_{n-r+1}, \\
& \cdots, x_n)], \\
= {} & H_i(y_1, \cdots, y_{n-r}; x_{n-r+1}, \cdots, x_n), \\
& (i = 1, \cdots, n-r).
\end{aligned} \tag{4}
$$

I say that, when $y_1 = 0$, the H's do not contain the y's:

$$
\frac{\partial H_i}{\partial y_j} \equiv 0 \text{ for } y_1 = 0; \quad (i = 1, \cdots, n-r; \; j = 1, \cdots, n-r).
$$

This follows, as in the proof of the theorem of functional dependence, from the fact that, when $y_1 = 0$, all determinants of order $r+1$ of (2) are identically zero.* It follows that the equations (4) have the form

$$
x_i = G_i(x_{n-r+1}, \cdots, x_n) + y_1^{p_i}\Gamma_i(y_1, \cdots, y_{n-r}; x_{n-r+1}, \cdots, x_n),
$$

for $p_i > 0$; $i = 1, \cdots, n-r$, where G and Γ are analytic, and $G_i(0) = 0$.

Now perform the non-singular analytic transformation

$$
\begin{cases}
x_i' = x_i - G_i(x_{n-r+1}, \cdots, x_n), & (i = 1, \cdots, n-r), \\
x_j' = x_j, & (j = n-r+1, \cdots, n).
\end{cases}
$$

The equations (1) retain their form, becoming

$$
x_i' = f_i'(y_1, \cdots, y_n), \qquad (i = 1, \cdots, n); \tag{1''}
$$

and

$$
J' = \frac{\partial(x_1', \cdots, x_n')}{\partial(y_1, \cdots, y_n)} = \frac{\partial(x_1', \cdots, x_n')}{\partial(x_1, \cdots, x_n)} \frac{\partial(x_1, \cdots, x_n)}{\partial(y_1, \cdots, y_n)},
$$

---

* See Osgood, *Funktionentheorie*, vol. 2, Chap. 2, §23.

and hence $J' = J = y_1^s L$; also, when $y_1 = 0$, $j' = j$. Indeed, the properties which we have demonstrated for equation (1) are retained. But it is seen at once that $f_1', \cdots, f_{n-r}'$ all vanish identically when $y_1 = 0$. Hence, dropping the accents, we may write them in the form

$$f_i = y_1^{p_i} F_i(y_1, \cdots, y_n), \qquad (p_i > 0\,;\, i = 1, \cdots, n - r).$$

Reasoning as before, we may make the non-singular transformation

$$\begin{cases} y_1' = y_1[F_1(y_1, \cdots, y_n)]^{1/p_1}, & \\ y_i' = y_i, & (i = 2, \cdots, n), \end{cases}$$

and, dropping the accents, we may write the equations in the form

$$\begin{cases} x_1 = y_1^{p_1}, \\ x_2 = y_1^{p_2} F_2(y_1, \cdots, y_n), \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ x_{n-r} = y_1^{p_{n-r}} F_{n-r}(y_1, \cdots, y_n), \\ x_i = f_i(y_1, \cdots, y_n), \\ \qquad\qquad (i = n - r + 1, \cdots, n). \end{cases}$$

A little computation shows that the new determinant j does not vanish for $(y) = (0)$. Hence we may apply the following non-singular transformation

$$\begin{cases} y_i' = y_i, & (i = 1, \cdots, n - r), \\ y_j' = f_j(y_1, \cdots, y_n), & (j = n - r + 1, \cdots, n), \end{cases}$$

and, dropping the accents, we may write the equations in the form

$$(5)\qquad \begin{cases} x_1 = y_1^{p_1}, \\ x_2 = y_1^{p_2} \phi_2(y_1, \cdots, y_n), \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ x_{n-r} = y_1^{p_{n-r}} \phi_{n-r}(y_1, \cdots, y_n), \\ x_i = y_i, \qquad (i = n - r + 1, \cdots, n). \end{cases}$$

These equations, which include (3) as a special case ($r=0$), will be made the basis of the rest of our work.

To begin with, we form the square matrix

$$(6)\qquad \left\| \begin{matrix} \dfrac{\partial \phi_2}{\partial y_2} & \cdots & \dfrac{\partial \phi_{n-r}}{\partial y_2} \\ \cdot & \cdots & \cdot\ \cdot \\ \dfrac{\partial \phi_2}{\partial y_{n-r}} & \cdots & \dfrac{\partial \phi_{n-r}}{\partial y_{n-r}} \end{matrix} \right\| .$$

Consider first the case where all the elements are identically zero when $y_1=0$. Then the ϕ's have the form

$$\phi_i = \psi_i(y_{n-r+1}, \cdots, y_n) + y_1\phi_i'(y_1, \cdots, y_n), \qquad (i = 2, \cdots, n-r).$$

In this case we form the matrix (6′), in which ϕ_i' replaces ϕ_i in (6). If all the elements of (6′) are identically zero for $y_1=0$, we repeat the process. Since the right-hand members of (5) actually contain *all* the y's (for $J\neq 0$), we are eventually stopped; that is, we have, for a certain integer λ,

$$\begin{cases} \phi_i = \psi_i + \psi_i' y_1 + \cdots + \psi_i^{(\lambda-1)} y_1^{\lambda-1} \\ \qquad + y_1^{\lambda}\phi_i^{(\lambda)}(y_1, \cdots, y_n), \\ \psi_i^{(\kappa)} = \psi_i^{(\kappa)}(y_{n-r+1}, \cdots, y_n), \quad (i = 2, \cdots, n-r), \end{cases}$$

where the matrix $(6^{(\lambda)})$ of the $\phi^{(\lambda)}$'s does not have all its elements identically zero when $y_1=0$.

In all cases, we are led to a matrix, (6) or $(6^{(\lambda)})$, in which at least one determinant δ of order $\rho(0<\rho<n-r)$ does not vanish identically for $y_1=0$, whereas every determinant of higher order is identically zero for that value. Let us assume, as we may, that

$$\delta = \frac{\partial(\phi_{n-r-\rho+1}, \cdots, \phi_{n-r})}{\partial(y_{n-r-\rho+1}, \cdots, y_{n-r})},$$

and that $\delta\neq 0$ at the non-specialized point $(y)=(0)$. We perform the non-singular substitution

$$\left\{\begin{aligned} y_1' &= y_1, \cdots, y_{n-r-\rho}' = y_{n-r-\rho}, \\ y_{n-r-\rho+1}' &= \phi_{n-r-\rho+1}(y) - \phi_{n-r-\rho+1}(0), \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ y_{n-r}' &= \phi_{n-r}(y) - \phi_{n-r}(0), \\ y_{n-r+1}' &= y_{n-r+1}, \cdots, y_n' = y_n. \end{aligned}\right.$$

This reduces (5) to a transformation which, dropping the accents, we may write in the form

$$\left\{\begin{aligned} x_1 &= y_1^{p_1}; \\ x_i &= \omega_{i0}y_1^{p_i} + \omega_{i1}y_1^{p_i+1} + \cdots + \omega_{i,\lambda+\mu}y_1^{p_i+\lambda+\mu} + y_1^{p_i+\lambda+\mu}\Phi_i, \\ &\qquad\qquad\qquad\qquad (i = 2, \cdots, n-r-\rho), \\ x_j &= \omega_{j0}y_1^{p_j} + \omega_{j1}y_1^{p_j+1} + \cdots + \omega_{j\lambda}y_1^{p_j+\lambda}y_j, \\ &\qquad\qquad\qquad\qquad (j = n-r-\rho+1, \cdots, n-r), \\ x_k &= y_k, \qquad\qquad\qquad (k = n-r+1, \cdots, n). \end{aligned}\right.$$

Here $\omega_{i\beta} = \omega_{i\beta}(y_{n-r-\rho+1}, \cdots, y_n)$, $\omega_{j\beta} = \omega_{j\beta}(y_{n-r+1}, \cdots, y_n)$, when $\beta > 0$; otherwise, the ω's are constants. We repeat the discussion of the last two paragraphs, applying it to $\Phi_2, \cdots, \Phi_{n-r-\rho}$, and the matrix

$$\left\| \begin{matrix} \dfrac{\partial\Phi_2}{\partial y_2} & \cdots & \dfrac{\partial\Phi_{n-r-\rho}}{\partial y_2} \\ \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\ \dfrac{\partial\Phi_2}{\partial y_{n-r-\rho}} & \cdots & \dfrac{\partial\Phi_{n-r-\rho}}{\partial y_{n-r-\rho}} \end{matrix} \right\|,$$

in place of $\phi_2, \cdots, \phi_{n-r}$ and (6) or $(6^{(\lambda)})$. Continuing in this manner, we find, at the end of a finite number of steps, the final standard form for the equations (1) of the transformation we are considering:

$$(7)\qquad \left\{\begin{aligned} x_1 &= y_1^{p_1}, \\ x_i &= \sum_{q=0}^{\lambda_i} y_1^{p_i+q}\omega_{iq}(y_{i+1}, \cdots, y_n) + y_1^{p_i+\lambda_i}y_i, \\ &\qquad\qquad\qquad\qquad (i = 2, \cdots, n-r), \\ x_j &= y_j, \qquad\qquad\qquad (j = n-r+1, \cdots, n). \end{aligned}\right.$$

Here r may have any value from 0 to $n-1$. When $r=n-1$, the second group of equations is lacking in (7), and, when $r<n-1$,

$$\lambda_2 \geqq \lambda_3 \geqq \cdots \geqq \lambda_{n-r} \geqq 0.$$

The method of inversion of the transformation is obvious. After the first equation and the last r equations have been solved for y in terms of x, we can solve the last of the remainder, then the next to the last, and so on. We clearly find that the y's are functions of $(x_1{}^{1/p_1}, x_2, \cdots, x_n)$, meromorphic at the point $(0, 0, \cdots, 0)$.

It will be seen that the case treated by Osgood (loc. cit., "Case I") is the case where $r=n-1$. But the usefulness of the result for $r<n-1$ is in general less than when $r=n-1$, for the neighborhood of $(x)=(0)$ may be transformed by (7) into a portion of the $(y_1 \cdots y_n)$-space not lying in the neighborhood of $(y)=(0)$, (compare the transformation $x_1=y_1$, $x_2=y_1y_2$). When this is the case, the fact that the *neighborhoods* of the origins of the given $(y_1 \cdots y_n)$-space of (1) and the new $(y_1 \cdots y_n)$-space of (7) correspond is insufficient to furnish complete information regarding the inversion of (1). It may be necessary to apply these results to an infinite set of neighborhoods. Consider, for example, the transformation $x_1=y_1$, $x_2=y_1\tan y_2$.

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CONCERNING LOGICAL PRINCIPLES*

BY C. H. LANGFORD

1. *Introduction.* In a recent number of this Bulletin, Dr. Alonzo Church discusses certain questions concerning the nature of the principles of logic, and advances certain views with regard to the nature and status of these principles, in particular, with regard to the status of the so-called principle of excluded middle.† It would seem that the views which Dr. Church is holding are views that are often held concerning the principles of logic, or similar to views that are often held; but it is clear that they are incompatible with certain tenets of ordinary logic that are commonly accepted; and I think it possible that those who adopt positions similar to the one Dr. Church appears to be adopting have not considered in detail the bearing of such positions on more ordinary logical conceptions, and have not assured themselves that the views they are holding are in fact compatible with other views which they would be equally inclined to accept. For this reason I wish to present, as clearly and as briefly as I can, some points concerning the nature and status of the principles of logic, and to suggest an account of these principles which is in accordance with commonly accepted tenets of ordinary logic, and which is incompatible in many respects with the interpretation of logical principles suggested by Dr. Church. In giving this account I shall be concerned on occasion to point out explicitly the bearing of the views I shall be advocating on views advanced by Dr. Church, but for the most part I shall be confined simply to presenting a different interpretation.

2. *Alternative Logics.* We may begin by considering the way in which logical principles are exemplified in the relation-

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* Presented to the Society, San Francisco Section, October 29, 1927.

† *On the law of excluded middle*, this Bulletin, vol. 34 (1928), pp. 75-78.

ships in which propositions stand, one to another. We may consider, in particular, the way in which such principles are exemplified in a pair of propositions having respectively the forms $(x).fx$ and $(\exists x).\sim fx$, and we may use these expressions to denote the propositions in question. These propositions exemplify the principle of excluded middle, in that $(x).fx.\vee.(\exists x).\sim fx$ could not be false; they exemplify the principle of contradiction, in that $(x).fx.(\exists x).\sim fx$ could not be true; they exemplify logical necessitation or entailing, in that $\sim(x).fx$ entails $(\exists x).\sim fx$; they exemplify logical equivalence, in that $\sim(x).fx.\equiv.(\exists x).\sim fx$ could not be false; and, no doubt, they exemplify other logical principles and relations, which are perhaps nameless, but which we might discover and name. Now one point which is brought out by an examination of this example, and upon which I wish to place emphasis, is that the logical principles and logical relations which these propositions exemplify in the relationship in which they stand, are found there as a matter of discovery; so that the occurrence of these properties is not at all an occasion for the exercise of choice or preference on our part. Another point of equal importance, connected with the first, is that these logical properties are what may be called necessary properties, in that their occurrence is dependent upon characteristics essential to the being of the propositions themselves, and upon nothing else; but in order to bring out this point, it will be necessary to describe a distinction which is commonly known as the distinction between internal and external relations.

When we consider the relations which hold among entities of various sorts, there appears to be a fundamental division among these relations, which we can describe by saying that the occurrences of some relations are dependent solely upon intrinsic features of the terms related, whereas other relationships are fortuitous so far as the intrinsic features of the terms related are concerned; or by saying, as is often done, that some relationships are grounded in the nature of their terms and others are not. Thus, to use a simple illustration, this pen, with which I am writing, is related to the paper upon which I am writing in a way that we can describe by saying that the pen is in contact with the paper. But it is clear that there is nothing in our conception of the terms having this relation from which the fact that they are so related could be inferred; and we can express this circumstance with regard to the relationship in which these things stand by saying that their being so related is conceptually fortuitous, or by saying that they stand in external relationship in this respect. On the other hand, there are two facts connected with this pen which are related in a way that is not conceptually fortuitous, namely, the fact that this pen is green, and the fact that it is colored. These facts are related in such a way that the first necessitates the second; and this is an internal relationship.*

Now we are often interested in making suppositions that are contrary to fact; and such suppositions are sometimes possible and sometimes not, and whether they are possible depends upon the nature of the facts in question. Thus we can make a supposition contrary to fact by supposing that this pen is not in contact with this paper, and this supposition is intelligible, and might be of interest; but when, for example, we attempt to suppose that this pen is both green and not colored, we find that our assent to this attempted supposition is merely verbal, that our words cannot retain their meanings, since being green involves being colored; and we are implicated in a species of self-contradiction, and consequent unintelligibility. Of course we can *say* that this pen is both green and not colored, and understand that what we say could not be true, that is, that there is no intelligible supposition to the effect that it is true. Now wherever internal relationships occur, there facts occur which do not admit of suppositions contrary to them; and such facts are known as analytic facts. On the other hand, wherever ex-

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*The best discussion of internal and external relations with which I am acquainted is that of Professor G. E. Moore, in his *Philosophical Studies*, Chap. 9.

ternal relationships occur, there facts occur which do admit of suppositions contrary to them; and such facts are known as contingent or empirical facts.

This brings me to a point that I want to make concerning the possibility of alternatives to ordinary logical principles that shall be in some sense incompatible with these principles. It is clear that all of the facts pointed out above, with regard to the relationship in which the two propositions $(x).fx$ and $(\exists x).\sim fx$ stand, are analytic facts; and generally, it is clear, I think, that all logical facts are analytic, and thus that logical principles, which are based on logical facts, do not admit of intelligible alternatives. This means that we cannot have alternative logics; for logic is the system of all propositions expressing analytic facts of a certain kind, namely formal analytic facts, and there cannot in the nature of the case be more than one such system, actual or conceivable.

3. *Deduction.* We may now examine in some detail the nature of the connections existing between propositions, and between properties, in virtue of which we are able to argue validly from one proposition to another, or from one property to another. When we consider a pair of properties, p, q, it often happens that they are related in a way which we can describe by saying that there could not be an instance of the first that is not also an instance of the second; and when this is the case, we say that the first entails or necessitates the second, that the second is deducible from the first.* As an example of this relation, we may take a case of entailing that occurs in connection with certain properties which are similar to properties commonly used in the definition of serial order, but which differ from properties of serial order in that reference to a class K, that is, to a function fx, is omitted. It is clear that the conjunction of the property of transitivity with the contrary of the reflexive

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*See Lewis, *A Survey of Symbolic Logic*, Chap. 5.

property necessitates the property "$\sim Rx_1x_2 \vee \sim Rx_2x_1$ for every x_1, x_2"; that is, that

$$(x_1, x_2, x_3) : Rx_1x_2 . Rx_2x_3 . \supset . Rx_1x_3 : . (x) . \sim Rxx$$

entails*

$$(x_1, x_2) . \sim Rx_1x_2 \vee \sim Rx_2x_1 .$$

As soon as we have envisaged clearly the meanings of these two properties, we see that the first involves the second, and that this relation is immediate and direct, that no mediating principles are required. This case is relatively simple, but in more complicated cases the situation is precisely the same; in complicated cases we often require an elaborate technique of proof, for the purpose of exhibiting to ourselves the relatedness of the properties in question, but this technique of proof is employed solely for the purpose of displaying the facts, and does not in any way condition them. Moreover, the fact that the first of the above properties necessitates the second, so far from being dependent upon some logical principle, is itself of such a kind that it can be taken as the ground of a logical principle, which we can express by saying that there could not be an instance of the propositional function

$$(x_1, x_2, x_3) : Rx_1x_2 . Rx_2x_3 . \supset . Rx_1x_3 : . (x) . \sim Rxx : . \supset . \\ (x_1, x_2) . \sim Rx_1x_2 \vee \sim Rx_2x_1$$

that is false, just as we can express the principle of excluded middle by saying that there could not be an instance of the propositional function $p \vee \sim p$ that is false. This new principle differs from the principle of excluded middle in being more determinate; but that is a matter of degree, not an essential difference. In general, then, whatever properties $p_1, \cdots, p_n$ may be, if q is a logical consequence of $p_1 \cdots p_n$,

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*It is to be noted that R is not an undefined term, but a logical function, $f(x, y)$; or perhaps it would be more accurate to say that R and similar entities, which are often called undefined terms, are simply logical functions or other items of logical structure. See Russell, *Introduction to Mathematical Philosophy*, Chap. 3.

we can say that q's being a consequence of $p_1 \cdots p_n$ does not depend upon anything outside the properties themselves, and especially that it does not depend upon anything that could be described as the logic which we assume, since the fact of q's following from $p_1 \cdots p_n$ is itself an analytic fact, namely, the fact that there could not be an instance of $p_1 \cdots p_n \cdot \sim q$ that is true.

I think it possible that the view that logical principles are in some sense premises in the deduction of one property from another, or of one proposition from another, is responsible for Dr. Church's view that we can have logics in which some ordinary logical principle is said to be not assumed, or in which some logical principle is invalid; for of course it often happens that a proposition $p_1 . p_2$ entails a proposition q, whereas p_1 alone does not entail q; and if p_2 were thought of as being some logical principle, then it might still be held that q could not be inferred if p_2 were not asserted, and that if p_1 were asserted and q denied, p_2 would have to be denied. In opposition to this view, I am maintaining that if we attempt to deny some logical principle, which is in fact valid, and on the basis of this deny certain inferences from one proposition to another, we do not get a new logic—we simply make a mistake; and similarly, that if we refuse to recognize an inference from one proposition to another, on the ground that we do not admit the logical principle which these propositions exemplify in the relationship in which they stand, then we simply make a mistake of another kind.

And I should like to introduce a remark at this point with regard to another view suggested by Dr. Church, a view to the effect that in logic we are not concerned with questions as to the truth and falsity of propositions. We do not raise questions concerning the truth and falsity of premises at the basis of abstract deductive systems, for the sufficient reason that, being properties, not propositions, these premises are neither true nor false, and we do not raise questions concerning the truth and falsity of propositions that are instances of properties at the basis of abstract deductive systems, because these propositions, when they are true, are not logical truths; but we do raise questions concerning the truth of propositions which we take to be logical principles, since these propositions must be true if we are not to be mistaken in supposing that they are logical principles.

In what has been said up to this point, and especially in what has been said concerning analytic and empirical facts, and internal and external relations, there is implicit a fundamental division of properties into two classes,—the class of contingent properties on the one hand, and the class of necessary and impossible properties on the other. An ordinary property, such as $f(x, y)$, is a contingent property if and only if, in view of its structure, it could have an instance that is true, and could have an instance that is false; whereas, a property is necessary if and only if it is such that it could not have an instance that is false, and a property is impossible if and only if it is such that it could not have an instance that is true. Thus, if we form the property $f(x, y) \vee {\sim}f(x, y)$, we have a necessary property, which is a specific form of the property involved in the principle of excluded middle, whereas if we form the property $f(x, y) . {\sim}f(x, y)$, we have an impossible property, which is a specific form of the property involved in the principle of contradiction. In this connection I wish to suggest, as a possible extension of the analysis given of logical and mathematical propositions of the form "p implies q," that any proposition of logic or mathematics can be expressed as an assertion with regard to some property that that property could not have an instance that is false (and, alternatively, as an assertion with regard to a property that that property could not have an instance that is true), or as an assertion with regard to some property that that property could have an instance that is true (and, alternatively, as an assertion with regard to a property that that property could have an instance that is false,—as in theorems on non-deducibility), or as a combination of such assertions. In sets of properties at the basis of deductive systems, and often elsewhere, contingent properties are given in isolation from the propositions into which they enter; but necessary properties do not as a rule occur in isolation. Thus, in an arithmetical proposition, say $2+3=5$, we have a necessary property whose instances express relationships of classes, and we have no conventional way of expressing this property as distinguished from the logically necessary proposition into which it enters; but if we allow $f(2, 3, 5)$ to express the arithmetical property in question, then we can, so I am holding, express the arithmetical proposition by saying that there could not be an instance of $f(2, 3, 5)$ that is false. As illustrations of contingent logical properties, we may take fx, Rxy, the cardinal number 10, the order-type ω, and the system of abstract euclidean geometry. Of course a property such as either of the last two is not in practice dealt with directly, but through a logically equivalent property, called a postulate-set or set of defining properties.

Now some properties are species of other properties, in the sense in which *green* is a species of *color*, or in which *roundness* is a species of *shape*; and this relation of species to genus is relevant to an account of the way in which propositions that are known as logical principles are related to other logical and mathematical propositions. I am holding that propositions known as logical principles do not differ in any essential respect from logical and mathematical propositions generally; but they are, as a rule, relatively simple propositions, so that their truth can be apprehended more immediately, and this is no doubt a reason for their being selected as principles. There is, however, another reason for their being selected, connected with their simplicity: the properties upon which logical principles are based, being simple, are more generic, and have many species among less generic properties, so that logical principles have many consequences among less generic propositions. Thus $(x)fx \vee (\exists x){\sim}fx$ is a species of $p \vee {\sim}p$, and is sufficient for $p \vee {\sim}p$; but we do not formulate principles such as

"$(x)fx \vee (\exists x)\sim fx$ could not have an instance that is false" (or "$(\exists x)\sim fx \cdot (x)fx$ could not have an instance that is true"), because a single more generic principle will do.

It is to be noted, however, that whether we consider logical principles at all is ultimately a matter of preference, and is dependent upon whether we wish to organize logical and mathematical propositions into a deductive order—as is done, for example, in *Principia Mathematica.* One who is not interested in such organization can insist that any logical or mathematical proposition, say the proposition to the effect that π is transcendental, is just as much a logical principle as the principle of excluded middle, since the evidence for the truth of the proposition is found within the proposition itself. Of course this does not mean that relationships of deducibility which occur among logical propositions are in any sense arbitrary; it means that they are not logically prior to relationships of deducibility which occur among other propositions, and thus that it is quite unnecessary to infer the implication of one contingent proposition by another from the implication of one logical proposition by another. For when we consider the logical propositions to which the relationships of necessary properties give rise, we see that these propositions are species of ordinary logical principles, and are thus the same in kind as the logical propositions which arise from relationships of contingent properties. On the other hand, we have noted that properties which are logically impossible can be used equally well in the formulation of logical principles, and here too it is clear that the relations of an impossible property to another property—whether this other property is impossible, necessary, or contingent—simply give rise to species of ordinary principles.

We may now revert briefly to the point originally made concerning the impossibility of alternatives to logical principles, in order to connect that point with the subsequent discussion. It is clear that if the account given of logical principles is a right account, then, in view of the way in which these principles are formulated, what precisely distinguishes a logical principle from an ordinary contingent proposition is the absence of alternative possibilities, this impossibility of alternatives being explicitly stated in the formulation of the principle itself; so that whoever holds with regard to an assigned proposition that there could be circumstances under which that proposition would fail is holding that the proposition in question is not a logical principle.

THE UNIVERSITY OF WASHINGTON

NOTE ON EINSTEIN'S EQUATION OF AN ORBIT

BY JAMES PIERPONT

In a paper* bearing the above title Morley has given an extremely elegant solution of Einstein's equation

$$(1)\qquad \left(\frac{dx}{d\theta}\right)^2 = 2x^3 - x^2 + 2\lambda x - \lambda^2(1 - e^2),$$

which defines the motion of a single planet about the sun. Here r, θ are the polar coordinates of the planet, a the major semi-axis, e the eccentricity of the orbit, M the mass of the sun, and

$$(2)\qquad x = \frac{M}{r}, \qquad \lambda = \frac{M}{a(1 - e^2)}\cdot$$

In Eddington units, $M = 1.45$. For Mercury, the values are

$$a = 5.8\cdot 10^{-7}, \qquad e = 0.206, \qquad \lambda = 2.6\cdot 10^{-8}.$$

The roots of the right side of (1) are thus, to a high degree of approximation,

$$(1 - e)\lambda, \qquad (1 + e)\lambda, \qquad \tfrac{1}{2} - 2\lambda.$$

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* American Journal of Mathematics, vol. 43 (1921), p. 29. I notice two obvious typographical errors in this paper. In the last term of (2) α should be α^2; also just below, x_1 should read $x_1 = \frac{1}{2} - 2\alpha$.

We reduce (1) to Weierstrass' canonical form, on setting

$$x = p + \tfrac{1}{6}, \qquad d\theta = 2^{1/2} du.$$

This gives

$$\left(\frac{dp}{du}\right)^2 = 4(p - e_1)(p - e_2)(p - e_3),$$

where

$$(3)\ e_1 = \tfrac{1}{3} - 2\lambda,\ e_2 = -\tfrac{1}{6} + \lambda(1 - e),\ e_3 = -\tfrac{1}{6} + \lambda(1 + e),$$

the indices being chosen so that $e_2 < e_3 < e_1$.

We find now

$$\theta = 2^{1/2} \cdot t\omega_1, \qquad u = \omega_2 + t\omega_1.$$

To $t = 0$ corresponds maximum r, while to $t = 1$ corresponds minimum r. Thus as t increases from 0 to 1, the planet passes from aphelion to perihelion, and θ increases by $\Delta\theta = 2^{1/2}\omega_1$.

It only remains to calculate this quantity. Morley now remarks that "the appropriate formulas are given in works on the elliptic functions. But as the proofs of the full formulas are necessarily complicated, I shall interpolate a proof of the approximate formulas of a kind that is at once intelligible."

I wish to submit an alternate form of proof which requires only the simplest and most commonplace formulas.*

We have

$$\omega_1 = \frac{K}{(e_1 - e_2)^{1/2}}, \qquad (4),\ \text{p. } 448,$$

$$k'^2 = \frac{e_1 - e_3}{e_1 - e_2}, \qquad (3),\ \text{p. } 449,$$

$$q = \frac{1}{2}\,\frac{1 - k'^{1/2}}{1 + k'^{1/2}}, \qquad (3),\ \text{p. } 437,$$

$$\left(\frac{2K}{\pi}\right)^{1/2} = 1 + 2q + 2q^4 + \cdots, \qquad (6),\ \text{p. } 438.$$

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* The numbers on the right refer to my book *Functions of a Complex Variable.* Unfortunately the most distressing confusion exists in the notations employed by different authors. For this reason I may be excused for referring to my book which contains only the briefest account of the elliptic functions. I take this occasion to correct formula (5), p. 448. The numerator in this formula should be $e_1 - e_2$ instead of $e_2 - e_1$.

Now, from (3),

$$e_1 - e_3 = \tfrac{1}{2}[1 - 2\lambda(3 + e)], \quad e_1 - e_2 = \tfrac{1}{2}[1 - 2\lambda(3 - e)],$$

$$k'^2 = [1 - 2\lambda(3 + e)][1 + 2\lambda(3 - e)] = 1 - 4\lambda e,$$

$$k'^{1/2} = 1 - \lambda e, \qquad q = \frac{1}{2}\,\frac{\lambda e}{2 - \lambda e} = \frac{1}{4}\lambda e,$$

$$\frac{2K}{\pi} = 1 + 4q, \qquad K = \pi\frac{1 + \lambda e}{2},$$

$$(e_1 - e_2)^{-1/2} = 2^{1/2}[1 + \lambda(3 - e)],$$

$$\omega_1 = \pi(1 + \lambda e)2^{1/2}\frac{1 + \lambda(3 - e)}{2} = \pi\frac{1 + 3\lambda}{2^{1/2}}.$$

Thus $\Delta\theta = \pi + 3\pi\lambda$. Hence the advance in the radius vector from perihelion to perihelion over 360° is $\phi = 6\pi\lambda$. Now in the Newtonian theory of elliptic motion $M = 4\pi^2a^3/T^2$. Substituting in (2) and using one second as the unit of time, we find

$$\phi = \frac{24\pi^3a^2}{c^2(1 - e^2)T^2}.$$

This is Einstein's celebrated formula for the advance of the perihelion of a planet.

YALE UNIVERSITY

ON LEVI-CIVITA'S THEORY OF PARALLELISM

BY C. E. WEATHERBURN

1. *Parallel Displacements.* The use of two-parametric differential invariants* (grad, div etc.) for a surface not only leads very neatly and quickly to the condition of parallelism and the known theorems, but also puts in evidence other results hitherto unnoticed. In particular, some interesting properties of Tchebychef nets are discovered.

Consider a curve C drawn on a surface S in ordinary space. Let $\boldsymbol{R}$ be a vector of constant length (which may be taken as unity), and a point-function along C, being everywhere tangential to the surface. Then $\boldsymbol{R}$ is said to undergo a parallel displacement along C, or to remain parallel to itself for displacement along this curve, provided the derivative of $\boldsymbol{R}$ along C is everywhere normal to the surface.† We lose no generality by regarding C as a member of a family T of curves, with unit tangent $\boldsymbol{t}$ and arc-length s, and $\boldsymbol{R}$ as the unit tangent to a second family cutting the former at a variable angle θ, so that θ is the angle of rotation from $\boldsymbol{t}$ to $\boldsymbol{R}$ in the positive sense about the unit normal $\boldsymbol{n}$. If then $\boldsymbol{m}$ is the unit surface vector $\boldsymbol{n}\times\boldsymbol{t}$, perpendicular to $\boldsymbol{t}$, we may write

$$\boldsymbol{R} = \boldsymbol{t}\cos\theta + \boldsymbol{m}\sin\theta.$$

Let a prime denote differentiation with respect to s, that is to say, in the direction of $\boldsymbol{t}$. Then in order that $\boldsymbol{R}$ may be normal to S we must have

$$0 = \boldsymbol{t}\cdot\boldsymbol{R}' = \sin\theta[\boldsymbol{t}'\boldsymbol{m}\cdot - \theta'],$$

and

$$0 = \boldsymbol{m}\cdot\boldsymbol{R}' = \cos\theta[\boldsymbol{m}\cdot\boldsymbol{t}' + \theta'],$$

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* All the differential invariants of this article are two-parametric invariants. See the writer's *Differential Geometry*, Chap. 12, Cambridge University Press, or Quarterly Journal of Mathematics, vol. 50, pp. 230–269.

† See Levi-Civita, *The Absolute Differential Calculus*, pp. 101–119.

from either of which it follows that

$$(1) \qquad \frac{d\theta}{ds} = \operatorname{div} \boldsymbol{m} = -\gamma$$

where γ denotes* the geodesic curvature of C. This is Levi-Civita's formula† for the arc-rate of increase of θ along C. From it follow the theorems of the constancy of the angle between two vectors undergoing parallel displacements along C, and of the constancy of θ for parallel displacement along a geodesic.

Since the derivative of $\boldsymbol{R}$ along C is normal to S, its *magnitude* is given by

$$(2) \qquad \boldsymbol{n}\cdot\boldsymbol{R}' = \boldsymbol{n}\cdot(\boldsymbol{t}'\cos\theta + \boldsymbol{m}'\sin\theta) = \kappa_n\cos\theta + \tau\sin\theta,$$

where κ_n is the normal curvature of the surface in the direction of $\boldsymbol{t}$, and τ is the torsion of the geodesic tangent to C, or the moment‡ of the family of curves T. This magnitude is clearly the resolved part of the vector $\kappa_n\boldsymbol{t}+\tau\boldsymbol{m}$ in the direction of $\boldsymbol{R}$.

Again, consider a family of curves A with unit tangent $\boldsymbol{a}$, and their orthogonal trajectories B with unit tangent $\boldsymbol{b}$, so that $\boldsymbol{n}=\boldsymbol{a}\times\boldsymbol{b}$. Let a third family T cut the first at a variable angle $-\theta$, so that the unit tangent $\boldsymbol{t}$ to this family is given by $\boldsymbol{t} = \boldsymbol{a}\cos\theta - \boldsymbol{b}\sin\theta$. Then the geodesic curvature γ of the last family is found from§

$$(3) \qquad \begin{aligned} -\gamma &= \operatorname{div}(\boldsymbol{a}\sin\theta + \boldsymbol{b}\cos\theta) \\ &= \sin\theta\operatorname{div}\boldsymbol{a} + \cos\theta\operatorname{div}\boldsymbol{b} + \boldsymbol{t}\cdot\nabla\theta, \end{aligned}$$

where $\nabla\theta$ denotes the gradient of θ. If now the vector $\boldsymbol{a}$ is such that it suffers a parallel displacement along a curve of the family T, we have $\boldsymbol{t}\cdot\nabla\theta = -\gamma$, and (3) becomes

$$(3') \qquad \sin\theta\operatorname{div}\boldsymbol{a} + \cos\theta\operatorname{div}\boldsymbol{b} = 0,$$

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* See the writer's *Differential Geometry*, p. 234.

† Levi-Civita, Palermo Rendiconti, vol. 42 (1917), p. 185.

‡ *Differential Geometry*, §130.

§ Ibid., §121.

from which two theorems may be deduced. First suppose that the curves of the family A constitute a family of parallels. Then div $\boldsymbol{a}$ vanishes* identically; and the last equation shows that either $\cos\theta=0$ or div $\boldsymbol{b}=0$. The second of these alternatives can be satisfied only where the geodesic curvature of the parallel curves is zero, that is to say, on their line of normal curvature. Hence we have the following theorem.

THEOREM 1. *If the tangent to a family of parallel curves undergoes a parallel displacement along a transversal, either this transversal is an orthogonal trajectory (and therefore a geodesic), or else it is a line of normal curvature of the family of parallels.*

Secondly, let the family A be one of geodesics. Then div $\boldsymbol{b}$ vanishes identically, and the previous equation (3′) takes the form $\sin\theta$ div $\mathbf{a}=0$. Since then $\sin\theta$ is not zero, this requires div $\boldsymbol{a}=0$, which holds only at the line of striction† of the family of geodesics. Hence we have Bianchi's theorem‡ that if the tangent to a family of geodesics on a surface undergoes a parallel displacement along a transversal this transversal is a line of striction of the family of geodesics.

From (3) follows also another result which, though not connected with the parallelism of Levi-Civita, may be mentioned here and will be used in §2. Suppose that the two families T and A are geodesics, and that the angle θ is constant along each member of the former. Then γ, div $\boldsymbol{b}$ and $\boldsymbol{t}\cdot\nabla\theta$ all vanish identically, and therefore by (3), so also does div $\boldsymbol{a}$. The family A is thus also a family of parallels; and the surface is developable. For the second (or Gaussian) curvature K is given by§

$$(4)\qquad K = -(\operatorname{div}\boldsymbol{a}\operatorname{div}\boldsymbol{a}+\boldsymbol{b}\operatorname{div}\boldsymbol{b}) = 0.$$

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* *Differential Geometry,* p. 258.

† Ibid., §126.

‡ *Geometria Differenziale,* vol. 2, p. 802.

§ *Differential Geometry,* §125.

Hence we have the following theorem:

THEOREM 2. *If a family of geodesics cut a family of geodesics (or parallels) at an oblique angle which is constant along each member of the former, the latter are both geodesics and parallels, and the surface is developable.*

2. *Tchebychef Nets.* A Tchebychef system (or net) on a given surface consists of two families of curves, U, V, such that the tangent to each family undergoes a parallel displacement along each curve of the other family.* If $\boldsymbol{u}$, $\boldsymbol{v}$, are the unit tangents to the two families U, V, respectively, the derivative of $\boldsymbol{v}$ in the direction of $\boldsymbol{u}$ is $\boldsymbol{u}\cdot\nabla\boldsymbol{v}$; and since this is normal to the surface we must have $\boldsymbol{u}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{u}=0$. Similarly it follows that $\boldsymbol{v}\cdot\Delta\boldsymbol{u}\cdot\boldsymbol{v}=0$. Conversely, if these relations hold, both derivatives are normal to the surface, and we have the following theorem.

THEOREM 3. *Necessary and sufficient conditions that the two families of curves with unit tangents* $\boldsymbol{u}$, $\boldsymbol{v}$ *constitute a Tchebychef system are expressed by the equations*

$$\boldsymbol{u}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{u} = 0, \qquad \boldsymbol{v}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{v} = 0. \tag{5}$$

Now the tendency of a family of curves in any direction is the resolved part in that direction of the derivative of the unit tangent in that direction.† The equations (5) therefore express that, at any point of the surface, the tendency of each family in the direction of the other is zero. If then we introduce the conics‡

$$\boldsymbol{r}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{r} = 1, \tag{6}$$

and

$$\boldsymbol{r}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{r} = 1, \tag{7}$$

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* Bianchi, loc. cit., p. 809.

† See §1 of a paper by the author, *On some properties of a family of curves on a surface*, recently communicated to the Edinburgh Mathematical Society.

‡ For this form of the equation of a quadric see the author's *Advanced Vector Analysis*, G. Bell and Sons, §66.

whose centers are at P, the origin of the vector $\boldsymbol{r}$, and whose plane is the tangent plane at P, it follows from (5) that $\boldsymbol{v}$ has the direction of one of the asymptotes of (6). But the other asymptote has the direction of $\boldsymbol{u}$; so that the tangents to the two families of curves are the asymptotes of (6). The same is true of the conic (7), and the two conics are therefore similar. The tendency of the family U in any direction varies inversely as the square of the radius of the conic (6) in that direction.* Thus the ratio of the tendencies of the two families in any direction at P is invariant. But the sum of the tendencies of a family in two directions at right angles is equal to the divergence of the family. Hence we may state the following theorem.

THEOREM 4. *The ratio of the tendencies of the two families of a Tchebychef system in any direction at a given point is invariant, and equal to the ratio of their divergences at that point.*

Also, at points where the two families U and V cut orthogonally the asymptotes of the conics (6) and (7) are at right angles. The sum of the tendencies in two perpendicular directions is zero for each family, so that the divergence of each is zero. Hence we have the following theorem.

THEOREM 5. *The locus of the points at which the two families of a Tchebychef system cut orthogonally is a common line of striction for the two families.*

Let $\boldsymbol{p}$, $\boldsymbol{q}$ denote the unit tangential vectors $\boldsymbol{n}\times\boldsymbol{u}$, $\boldsymbol{n}\times\boldsymbol{v}$ perpendicular to $\boldsymbol{u}, \boldsymbol{v}$ respectively. Then the geodesic curvatures γ_1, γ_2 of the two curves U, V are given by the equations

$$-\gamma_1 = \operatorname{div} \boldsymbol{p}, \qquad -\gamma_2 = \operatorname{div} \boldsymbol{q}.$$

Let θ be the angle of rotation from $\boldsymbol{u}$ to $\boldsymbol{v}$ in the positive sense. Then, since the vector $\boldsymbol{u}$ undergoes a parallel displacement along the curve V, it follows from (3) and (3′) that

$$\operatorname{div} \boldsymbol{u} \tan\theta = \operatorname{div} \boldsymbol{p} = -\gamma_1. \tag{8}$$

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* Edinburgh Mathematical Society, loc. cit., §1.

From equation (8), $\gamma_1=0$ implies div $u=0$ since tan $\theta\neq 0$. We may therefore state the following theorem.

THEOREM 6. *A line of normal curvature of either family of a Tchebychef system is also a line of striction of that family.*

The *gradient* of θ may be expressed in terms of u and v as follows. Taking the rot of both members of the identity

$$u \times v = n \sin \theta, \tag{9}$$

we have*

$$v\cdot\nabla u - u\cdot\nabla v + u \operatorname{div} v - v \operatorname{div} u = \cos\theta\nabla\theta \times n. \tag{10}$$

The derivatives represented by the first two terms are parallel to n. Therefore, on forming the vector product of both sides with n, we have*

$$\cos\theta\nabla\theta = n \times (u \operatorname{div} v - v \operatorname{div} u). \tag{11}$$

Since $\cos\theta = u\cdot v$, this formula expresses $\nabla\theta$ in terms of u and v. Where $\cos\theta$ is zero, both div u and div v must vanish, and we see again that this is a common line of striction of the two families U and V.

If the two families of a Tchebychef net cut at a constant angle, $\nabla\theta$ is everywhere zero, so that div u and div v must vanish identically. Thus both families are families of parallel curves; and, as they cut at a constant angle, it follows from the last theorem of §1 that they are also geodesics, and the surface is developable. Hence we have the following theorem.

THEOREM 7. *If the two families of a Tchebychef net cut at a constant angle, each is a family of parallel geodesics, and the surface is developable.*

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* *Differential Geometry*, p. 237, Ex. 10.

PLANE NETS WHOSE FIRST AND MINUS FIRST LAPLACIAN TRANSFORMS EACH DEGENERATE INTO A STRAIGHT LINE*

BY J. O. HASSLER

1. *Introduction.* In the projective theory of plane nets, as developed by Wilczynski,† the members $y_1(u, v)$, $y_2(u, v)$, $y_3(u, v)$ of any fundamental system of solutions of a completely integrable system of partial differential equations of the form

$$(1) \qquad \begin{cases} y_{uu} = ay_u + by_v + cy, \\ y_{uv} = a'y_u + b'y_v + c'y, \\ y_{vv} = a''y_u + b''y_v + c''y \end{cases}$$

are interpreted as the homogeneous coordinates of a point P_y in a plane, defining a non-degenerate net of plane curves consisting of two one-parameter families. The invariants and covariants of the system (1) under the transformations

$$(2) \qquad y = \lambda(u,v)\bar{y},$$

and

$$(3) \qquad \bar{u} = U(u), \text{ and } \bar{v} = V(v)$$

may be interpreted geometrically by certain projective properties of the net. The two covariants

$$(4) \qquad \rho = y_u - b'y, \quad \sigma = y_v - a'y$$

may be taken as the homogeneous coordinates of two points P_ρ and P_σ when we substitute successively for y the values y_1, y_2, and y_3. As u and v vary P_σ and P_ρ describe nets called the first and minus first Laplacian transforms of the original net.

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* Presented to the Society, April 6, 1928.

† E. J. Wilczynski, *One-parameter families and nets of plane curves*, Transactions of this Society, vol. 12 (1911), pp. 473–510.

In this paper we consider nets such that the curves $u=\text{const.}$ of the first Laplacian transform all degenerate into one straight line while the curves $v=\text{const.}$ go into points on the line, and the curves $v=\text{const.}$ of the minus first Laplacian transformd egenerate into one straight line while the curves $u=\text{const.}$ go into points on the line. The invariants and covariants of the net are computed and the net determined by certain boundary conditions.

2. *The Canonical Form of the Equations and the Invariants.* If we apply the transformation (2) to the system of equations (1), we obtain

$$(1') \qquad \left\{ \begin{aligned} y_{uu} &= -B'y_u + By_v + Cy, \\ y_{uv} &= A'y_u + B'y_v + C'y, \\ y_{vv} &= A''y_u - A'y_v + C''y, \end{aligned} \right.$$

where the coefficients B', B, C, etc., are functions of a, b, c, a', b', c', a'', b'', c'', λ and their derivatives.* The system is then said to be in its canonical form. The coefficients B and A'' are relative invariants of the system under the transformations (2) and (3). The five remaining fundamental invariants are

$$(5) \qquad \begin{aligned} &\mathfrak{A}' = A' + \frac{A_v''}{6A''}, \quad \mathfrak{B}' = B' + \frac{B_u}{6B}, \quad \mathfrak{C}' = C' + A'B', \\ &\mathfrak{C} = C - B_u' - 2B'^2 + A'B, \\ &\mathfrak{C}'' = C'' - A_v' - 2A'^2 + A''B'. \end{aligned}$$

Two other invariants,†

$$(6) \qquad H = C' + A'B' - A_u', \quad K = C' + A'B' - B_v',$$

have a special significance in connection with the Laplacian transforms.

Wilczynski has shown‡ that if $H=0$, the curves $v=\text{const.}$ of the first Laplacian transform degenerate into points and the curves $u = \text{const.}$ all coincide with the locus of these points, making the degenerate transformed net consist of a single curve. Similarly, if $K = 0$, the minus first Laplacian transform becomes a single curve $v = \text{const.}$ Furthermore, if*

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* Wilczynski, loc. cit., equations (13).

† Wilczynski, loc. cit., equations (21) and (29).

‡ Wilczynski, loc. cit., pp. 489–490.

$$A''^2K + A''\mathfrak{C}_v' - \mathfrak{C}''A_v'' = 0, \tag{7}$$

the curves $u = \text{const.}$ of the first Laplacian transform become straight lines, and if

$$B^2H + B\mathfrak{C}_u - \mathfrak{C}B_u = 0, \tag{8}$$

the curves $v = \text{const.}$ of the minus first Laplacian transform become straight lines. We shall consider the case where equations (7) and (8) are satisfied simultaneously with

$$H = K = 0. \tag{9}$$

From (7) and (8), by means of (9), we obtain, after integrating,

$$\log\frac{\mathfrak{C}''}{A''} = \phi(u), \qquad \log\frac{\mathfrak{C}}{B} = \psi(v),$$

where ϕ and ψ are arbitrary functions of u and v, respectively. The quotients $\mathfrak{E}''/A''$ and $\mathfrak{E}/B$ are relative invariants. If we apply the transformation (3), they are transformed into

$$\frac{\overline{\mathfrak{C}}''}{\overline{A}''} = \frac{1}{U'}\,\frac{\mathfrak{C}''}{A''}, \qquad \frac{\mathfrak{C}}{\overline{B}} = \frac{1}{V'}\,\frac{\mathfrak{C}}{B}.$$

If we choose U and V so that $U' = e^{-\phi(u)}$ and $V' = e^{-\psi(v)}$, we have

$$\mathfrak{C}'' = A'', \qquad \mathfrak{C} = B. \tag{10}$$

The system (1) is subject to certain integrability conditions† which, if used with (5), (6), (9), and (10), enable us to express all of the coefficients of the canonical form in terms of B and A''. In fact, we find that

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* Wilczynski, loc. cit., equation (54a).

† Wilczynski, loc. cit., equations (5) and (14).

$$(11)\quad \begin{cases} B = B, \quad C = \dfrac{2}{9}(1 + B + B^2) - \dfrac{1}{3}(B_u + B_v), \\ A' = \dfrac{1-A''}{3}, \; B' = \dfrac{1-B}{3}, \; C' = \dfrac{1}{9}(2A''B + A'' + B - 1), \\ A'' = A'', \quad C'' = \dfrac{2}{9}(1 + A'' + A''^2) - \dfrac{1}{3}(A_u'' + A_v''), \end{cases}$$

with the further conditions that the invariants A'' and B must satisfy the partial differential equations

$$(12)\qquad A_u'' = B_v = - A''B.$$

The general solution of (12) is*

$$(13)\qquad A'' = \frac{\psi'(v)}{\phi(u) + \psi(v)}, \quad B = \frac{\phi'(u)}{\phi(u) + \psi(v)},$$

where ϕ and ψ are arbitrary functions of u and v, respectively. It is easy to verify that the integrability conditions are identically satisfied if a system of partial differential equations of the form (1′) has coefficients defined by (11) and (13). Hence we may state the following conclusion.

If the coefficients of a system of partial differential equations of the form (1′) *satisfy conditions* (11) *and* (13), *any fundamental system of solutions* y_1, y_2, y_3 *of* (1′) *defines a net whose first and minus first Laplacian transforms each degenerate into a straight line. Conversely, any net whose first and minus first Laplacian transforms each degenerate into a straight line gives rise to a system of partial differential equations of the form* (1′) *whose coefficients satisfy conditions* (11) *and* (13).

3. *Determination of the Net by Boundary Conditions.* In the projective differential geometry of plane curves† three linearly independent solutions $y_1(x)$, $y_2(x)$, $y_3(x)$, of a linear homogeneous differential equation of the third order,

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* Acknowledgement is hereby given to G. E. Raynor, of the University of Oklahoma, for the solution of these equations.

† E. J. Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, Leipzig, B. G. Teubner, 1916, pp. 58–61.

$$(14)\qquad \frac{d^3y}{dx^3} + 3p_1\frac{d^2y}{dx^2} + 3p_2\frac{dy}{dx} + p_3y = 0,$$

are interpreted as the homogeneous coordinates of a point on a plane.

Two functions of the coefficients of (14), called θ_3 and θ_8 by Wilczynski, which remain invariant under the transformations

$$y = \lambda(x)\eta, \qquad x = f(\xi),$$

are sufficient to determine the projective properties of the curve. If, then, two arbitrary functions $\omega_3(x)$ and $\omega_8(x)$ be given there exists a curve whose invariants θ_3 and θ_8 are respectively equal to these given functions and this curve will be uniquely determined except for projective transformations. In particular, if $\theta_3 = 0$, the corresponding curve is a conic.

If we differentiate both members of the first equation of (1′) with respect to u and then eliminate y_v and y_{uv}, we obtain an equation of the form (14). We may then compute the invariants θ_3 and θ_8 for a curve v=const. of the net under consideration. In fact, if we employ the equations derived by Wilczynski, making use of equations (11) and (13), we find

$$\begin{aligned} 54\theta_3 &= -9\frac{\phi^{\text{iv}}}{\phi'} + 9\frac{\phi'''}{\phi'} + 6\frac{\phi''}{\phi'} + 45\frac{\phi''\phi'''}{(\phi')^2} \\ (15)\qquad &\qquad\qquad - 15\frac{\phi''^2}{\phi'^2} - 40\frac{\phi''^3}{\phi'^3} + 4, \\ \theta_8 &= 6\theta_3\theta_3'' - 7\theta_3'^2 - 27P_2\theta_3^2, \end{aligned}$$

where the accents indicate derivatives with respect to u and where

$$P_2 = p_2 - p_1^2 - p_1' = \frac{\phi'''}{3\phi'} - 4\frac{\phi''^2}{9\phi'^2} - \frac{\phi''}{9\phi'} - \frac{1}{9}.$$

We notice that the invariants are independent of v, since ϕ is a function of u alone. All the curves v=const. are projectively equivalent.

Let C be any analytic curve, not a straight line, and let equation (14) be its differential equation with respect to any independent variable x to which it may be referred parametrically. Let $\theta_3=\omega_3(x)$ and $\theta_8=\omega_8(x)$ be the values of its invariants as functions of x. If the curve $v=v_0$ of the net coincides with C it must be possible to determine u as a function of x so that the equations

$$
\left\{\begin{aligned} \theta_3(u) &= \omega_3(x), \\ \theta_8(u) &= \omega_8(x) \end{aligned}\right. \tag{16}
$$

are identically satisfied. On the other hand, if equations (16) are identically satisfied the curve C will be projectively equivalent to the curve $v=v_0$ of the net.

To reduce the order of the differential equations we make the substitution

$$
w = \frac{\phi''}{\phi'} \tag{17}
$$

in equations (15). If we substitute the values of θ_3 and θ_8 from (15) in (16) we find that the differential equations to be satisfied by w and u as functions of x have the form

$$
\left\{\begin{aligned} & -9\frac{d^2w}{du^2}+9(1+2w)\frac{dw}{du}-4w^3-6w^2 \\ & \qquad\qquad\qquad +6w+4 = 54\omega_3(x), \\ & 6\omega_3\frac{d^2\omega_3}{du^2}-7\left(\frac{d\omega_3}{du}\right)^2-3\left(\frac{dw}{du}-w^2-w-1\right)\omega_3^2 = \omega_8(x). \end{aligned}\right. \tag{18}
$$

If C be not a conic, so that $\omega_3(x)\neq 0$, we can by a suitably chosen transformation of the independent variable x make $\omega_3=1$.* We may then, after using x as the independent variable in all the differentiation, write the equations in the form

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* Wilczynski, *Projective Differential Geometry*, p. 61.

$$
(19)\quad \begin{cases} -9\dfrac{d}{dx}\left[\dfrac{\dfrac{dw}{dx}}{\dfrac{du}{dx}}\right]\dfrac{1}{\dfrac{du}{dx}} + 9(1+2w)\dfrac{\dfrac{dw}{dx}}{\dfrac{du}{dx}} - 4w^3 \\ \qquad\qquad\qquad\qquad - 6w^2 + 6w - 50 = 0, \\ -3\dfrac{\dfrac{dw}{dx}}{\dfrac{du}{dx}} + 3w^2 + 3w + 3 = \omega_8(x). \end{cases}
$$

If we use the second equation to simplify the first, we obtain the form

$$
(20)\quad \begin{cases} -9(1+2w)\dfrac{dw}{dx} + (14w^3 + 21w^2 + 33w - 41 \\ \quad - 6w\,\omega_8(x) - 3\omega_8(x))\dfrac{du}{dx} + 3\omega_8' = 0, \\ 3\dfrac{dw}{dx} - (3w^2 + 3w + 3 - \omega_8(x))\dfrac{du}{dx} = 0. \end{cases}
$$

We have in (20) two differential equations of the first order with two dependent variables w and u and one independent variable x. There exist then two integrals involving the variables and two arbitrary constants.

From (17), ϕ can be determined by two quadratures, which brings the total number of arbitrary constants up to four. The variable u enters only through its derivative du/dx. Hence $u+k$, where k is an arbitrary constant, will satisfy the same system as u. By fixing arbitrarily the origin of the u-scale k may be made equal to zero. Thus we determine ϕ as a function of u except for three arbitrary constants.

If C be a conic, $\omega_8(x)=0$, and equations (18) reduce to the single equation

$$
(21)\quad -9\frac{d^2w}{du^2} + 9(1+2w)\frac{dw}{du} - 4w^3 - 6w^2 + 6w + 4 = 0,
$$

for which there exists an integral involving w as a function of u and two arbitrary constants. As in the preceding case ϕ may be determined as a function of u except for three arbitrary constants.

If we differentiate the third equation of (1′) with respect to v and eliminate y_u and y_{uv}, we obtain an equation in the form (14) where v is the independent variable. By means of this we study the projective properties of the curves $u=$const. and by choosing an arbitrary curve C' we can determine the function $\psi(v)$ except for three arbitrary constants.

Equations (13) show that the invariants A'' and B are determined except for six arbitrary constants. We shall now proceed to determine these six constants by choosing the straight lines of the Laplacian transforms.

Let $f(y_1, y_2, y_3)=0$ be the equation of the curve $C(v=v_0)$, where it is to be understood that y_1, y_2, and y_3 form a fundamental system of solutions of equations (1). Let $F(\rho_1, \rho_2, \rho_3)=0$ be the equation of the corresponding curve C_{-1}, generated by the point P_ρ, of the minus first Laplacian transform, where each ρ is defined as a function of y by the first of equations (4) which, on account of (11), become

$$(4') \qquad \rho = y_u - \frac{1-B}{3}y, \quad \sigma = y_v - \frac{1-A''}{3}y.$$

$F(\rho_1, \rho_2, \rho_3)$ becomes a function of y_1, y_2, y_3, and B. According to the conditions set forth in this paper C_{-1} is a straight line, the degenerate minus first Laplacian transform. If we choose this line arbitrarily we will have two relations between the arbitrary constants in B.

Furthermore, equations (4′) show that the point P_ρ lies on the tangent to the curve C at P_y and the point P_σ on the tangent to C'. If we choose a fixed point $u=u_0$ on C the point P_ρ can occupy only a single infinity of positions. Since the line C_{-1} has been prescribed, the position of P_ρ will be determined, thus giving us a third relation between the arbitrary constants.

In a similar manner we may secure three relations between the constants by choosing a straight line C_1 to be the degenerate first Laplacian transform and a point $v=v_0$ on the curve C'. The six relations are sufficient to determine the six arbitrary constants. We may therefore state the following conclusion.

Choose two non-rectilinear but otherwise arbitrary analytic curves C and C' intersecting in a point P and having distinct tangents T and T' at P. Choose an arbitrary straight line C_{-1} intersecting T and another line C_1 intersecting T'. There exists one and only one net which contains the curves C and C' and which moreover has C_1 and C_{-1} for its degenerate first and minus first Laplacian transforms.

The family of curves $v=$ const. may be obtained from the curve C by projective transformations. Similarly, the family $u=$ const. may be obtained from C'.

THE UNIVERSITY OF OKLAHOMA

SOME PROPERTIES OF UPPER SEMI-CONTINUOUS COLLECTIONS OF BOUNDED CONTINUA*

BY W. A. WILSON

1. *Introduction.* If $T=\{t\}$ denotes a closed set of points and with each point t there is associated a unique bounded continuum X (or X_t) in such a way that (a) $X_t \cdot X_{t'}=0$ if $t\neq t'$, (b) at each point $t=\tau$ of T the upper closed limit of X_t as $t\rightarrow\tau$ is a part of X_τ, we say that $X=f(t)$ is an upper semi-continuous function in T. The collection of continua $\{X\}$ is also known as an upper semi-continuous collection of continua. These aggregates have been discussed by various writers here and abroad and enjoy numerous interesting properties.

R. L. Moore, in particular, has given an extensive treatment of the subject[†] and among other things has shown that, if the continua $\{X\}$ all lie in a plane but none of them cuts the plane, then the complement of $M=\Sigma[X]$ is a simply connected region whose frontier is a part of M if T is a simple arc and the complement of M is two simply connected regions whose frontiers are parts of M if T is a circumference. It is easily seen by examples that the frontiers of the complementary regions need not coincide with M in either case; it is the purpose of this note to give the conditions under which they do.[‡]

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* Presented to the Society, February 25, 1928.

To this end we define $X=f(t)$ as a *minimal* upper semi-continuous function in T if there exists no upper semi-continuous function $Y=g(t)$ such that at every point t, $Y\subset X$, and at some point $Y\neq X$. If T denotes the interval $(-1\leqq t\leqq 1)$ and in some plane we let X_t be the point $(t, \sin(1/t))$ when $t\neq 0$ and the point set $x=0$, $-1\leqq y\leqq 2$ when $t=0$, then $X=f(t)$ is an upper semi-continuous function which is not a minimal function, but becomes so if we replace $X_0=f(0)$ by the set $x=0$, $-1\leqq y\leqq 1$. An example of a minimal upper semi-continuous function where no X is a point is given by the author in this Bulletin, vol. 32, p. 679.

2. *Notation.* The following notation will be convenient. If $X=f(t)$ in T and $M=\Sigma[X]$ we write $M=F(T)$. If T is a bounded continuum and $f(t)$ is upper semi-continuous, it is obvious that M is a bounded continuum; in this case we say that X is an *element* of M.

If T is a simple arc ab, $M=F(ab)$ will be called a generalized arc, or simply an arc if no confusion is caused. This may be denoted by X_aX_b and the elements X_a and X_b will be called the ends. Likewise, $M-(X_a+X_b)$ is called a *(generalized) open arc* and denoted by $X_a{}^*X_b{}^*$.

[†] R. L. Moore, *Concerning upper semi-continuous collections of continua*, Transactions of this Society, vol. 27, pp. 416–428.

[‡] The attention of the reader is directed to an article by C. Kuratowski, *Sur la structure des frontières communes à deux régions*, Fundamenta Mathematicae, vol. 12 (1928), pp. 20–42, of which an advance copy was received while this paper was in press. Although Kuratowski's article is concerned chiefly with the converse problem, the reader will note a certain degree of similarity between the two papers.

If T is a circumference C, $M=F(C)$ is called a *(generalized) simple closed curve.* Obviously any two elements X_1 and X_2 divide M into two arcs having X_1 and X_2 as end elements and no other common points.

The plane will be denoted throughout by Z.

3. *Certain Corollaries.* Certain properties of the sets under consideration are either corollaries of Moore's work or are so easily demonstrated that their proofs are omitted.

(a) *If $M=F(ab)$ is a generalized arc in a plane Z, no sub-continuum of M separates X_a from X_b unless some element of M does.*

(b) *If $M=F(C)$ is a generalized simple closed curve in a plane, no two elements of M are separated by a sub-continuum of M.*

These are readily proved with the aid of a theorem of Janiszewski.[†]

(c) *In a plane Z let $M=F(ab)$ be a generalized arc and no element of M separate X_a from X_b, or let $M=F(C)$ be a generalized simple closed curve. Then there is not more than one element X such that a bounded component of $Z-X$ contains $M-X$.*

(d) *In a plane Z let $M=F(ab)$ be a generalized arc and no element of M separate X_a from X_b, or let $M=F(C)$ be a generalized simple closed curve. For each element X let the component of $Z-X$ containing $M-X$ be unbounded. Let Y be the union of X and the components of $Z-X$ containing no points of M. Then $Y=g(t)$ is upper semi-continuous in ab or C, respectively.*

4. Lemma. *Let $M=F(C)$ be a generalized simple closed curve in a plane Z. Then $Z-M$ has two components whose frontiers have points in every element of M and every other component has a frontier which is a part of some element.*

[†] Z. Janiszewski, *Sur les coupures du plan faites par des continus*, Prace Matematyczno-Fizyczne, vol. 26, Theorem A. See also R. L. Moore, *Concerning the prime parts of certain continua which separate the plane*, Proceedings of the National Academy of Sciences, vol. 10, p. 173.

PROOF. This is really a corollary of R. L. Moore's work. By §3 (c) there is at most one element X such that a bounded component of $Z-X$ contains $M-X$. Since inversion with respect to a point within this component will make its image unbounded, there is no loss in generality if we assume that for every element X the unbounded component of $Z-X$ contains $M-X$.

Defining $Y=g(t)$ as in §3 (d), it follows from this reference that $N=\Sigma[Y]$ is a generalized simple closed curve no element of which separates Z. In this case Moore has shown (loc. cit., Theorem 11) that $Z-N$ consists of two components R and S, and the frontier of each of these has at least one point on every element Y. It is readily seen that these are also components of $Z-M$. Hence the first part of the lemma is proved.

That the frontier of each of the other components of $Z-M$ is a part of some element X is a consequence of the definition of $Y=g(t)$.

DEFINITION. The components of $Z-M$ which have frontier points on every element of M will be called *principal* components.

COROLLARY. *Let* $C=\{t\}$ *be a circumference, let* $X=f(t)$ *be an upper semi-continuous function defined over* C, *and let* $M=\Sigma[X]$ *lie in the plane* Z. *If* M *is the common frontier of two components of* $Z-M$, *then* $f(t)$ *is a minimal upper semi-continuous function.*

PROOF. Let R and S be the components of $Z-M$ having the frontier M. If the theorem is not true, let $Y=g(t)$ be upper semi-continuous over C, let $Y\subset X$ for every t, and let $Y\neq X$ for some t. Let $N=\Sigma[Y]$.

By the above lemma, $Z-N$ has two principal components R' and S'. Since M is an irreducible cut of Z between points of R and S and N is a proper part of M, R and S lie in the same component of $Z-N$, and this must contain all the points of $M-N$. Suppose that $S'\ (R+S)=0$. Then the frontier of S' is a part of some element of M, as S' is a component of $Z-M$. This is a contradiction, since N contains points of every element of M.

5. LEMMA. *Let $M=F(C)$ be a generalized simple closed curve in a plane Z, and let R be one of the principal components of $Z-M$. Let a and b be points of M accessible from R and lying on different elements A and B of M. Let A and B divide M into the arcs M_1 and M_2. If F is the frontier of R, $F=H+K$, where H and K are sub-continua of M_1 and M_2, respectively, joining a and b.*

PROOF. Let m be a point of R and ma and mb be simple arcs lying in R except for the points a and b and having only m in common. R. L. Moore has shown (loc. cit., p. 423) that the arc $ab=ma+mb$ divides R into two simply connected regions R_1 and R_2 such that their frontiers are parts of $M_1+a^*b^*$ and $M_2+a^*b^*$, respectively.

Let H and K, respectively, denote those points of these frontiers not on a^*b^*; then $H\subset M_1$, $K\subset M_2$. Since the frontier of R_1 is a continuum, every point of H can be joined to a or b by a sub-continuum of H. If H is not a continuum, $H=H_1+H_2$, where H_1 and H_2 are continua containing a and b, respectively, and $H_1\cdot H_2=0$. This hypothesis would give a contradiction by the theorem of Janiszewski referred to earlier, for neither ab, H_1, nor H_2 separates R_1 from R_2, while $(ab)\cdot H_1=a$, $(ab)\cdot H_2=b$, and $(ab+H_1)\cdot(ab+H_2)=ab$. Thus H, and in like manner K, is a continuum.

Now $F\supset H+K$. On the other hand every frontier point of R is necessarily one of R_1, or of R_2, or of both. Hence $F=H+K$.

6. THEOREM. *Let $C=\{t\}$ be a circumference, let $X=f(t)$ be a minimal upper semi-continuous function defined over C, and let $M=\Sigma[X]$ lie in the plane Z. Then M is the frontier of two components of $Z-M$ and the frontier of each of the remaining components is a part of some element of M.*

PROOF. The last assertion is a restatement of a previous result. (See §4.) Let R and S be the principal components of $Z-M$, and let F be the frontier of one of them, say R. Since accessible points are everywhere dense in F and F contains at least one point in every element X, there is an everywhere dense set of points t each of whose corresponding elements contains an accessible point of F.

Orient the points of C, let τ be a fixed point t, and let $\{t_i\}$ and $\{t_i'\}$ be sequences of points $\{t\}$ such that $t_1<t_2<\cdots\to\tau$, $t_1'>t_2'>\cdots\to\tau$, and for each t_i and t_i' the corresponding X_i or X_i' contains an accessible point of F.

Let M_i be the arc of M joining X_i and X_i' and containing $X_\tau=f(\tau)$. Obviously $X_\tau=\Pi_1^\infty[M_i]$. By the previous lemma M_i contains a sub-continuum F_i of F joining X_i and X_i', and $(F-F_i)\cdot X_\tau=0$. Moreover, $\Pi_1^\infty[F_i]$ is a continuum.

But $F_i\subset M_i$; hence $\Pi[F_i]\subset X_\tau$. As $(F-F_i)\ X_\tau=0$, $F\cdot X_\tau\subset F_i$, whence $F\cdot X_\tau=\Pi[F_i]$. Thus we have shown that for each t, $Y_t=F\cdot X_t$ is a continuum.

On the other hand, $\overline{\lim}_{t\to\tau}\ Y_t\subset F\cdot X_\tau\subset Y_\tau$. For $Y_t\subset F$ $Y_t\subset X_t$, and $X_t=f(t)$ is upper semi-continuous. Thus $Y_t=g(t)$ is upper semi-continuous. But $f(t)$ is a minimal upper semi-continuous function. Hence for every t, $Y_t=X_t$ and so $F=M$.

COROLLARY. *Let M satisfy the hypotheses of the above theorem, let A and B be any two elements of M, and M_1 and M_2 the complementary arcs of M thus determined. Then $M=H_1+H_2$, where $H_1\subset M_1$, $H_2\subset M_2$, $H_1\cdot H_2=\alpha+\beta$, where $\alpha\subset A$, $\beta\subset B$, and both H_1 and H_2 are continua irreducible between α and β.*

PROOF. Let $M=H_1+H_2$ be an irreducible decomposition* of M such that $H_1\subset M_1$ and $H_2\subset M_2$. Then $H_1\supset M_1-(A+B)$ and $H_2\supset M_2-(A+B)$. By the above theorem M is an irreducible cut of the plane between a point of R and one of S. By a theorem proved elsewhere† H_1 and H_2 are both irreducible between $\alpha=A\cdot H_1\cdot H_2$ and $\beta=B\cdot H_1\cdot H_2$.

---

* The decomposition $M=H_1+H_2$ is called irreducible if H_1 and H_2 are continua and there exists no proper sub-continuum K of H_1 or H_2 such that $M=K+H_2$ or $M=H_1+K$, respectively.

† W. A. Wilson, *On irreducible cuts of the plane between two points*, Annals of Mathematics, vol. 29, §9.

7. Lemma. *Let $M=F(ab)$ be a generalized arc lying in a plane Z and let no element of M separate X_a from X_b. Then $Z-M$ has one component whose frontier has points on every element of M and every other component has a frontier which is a part of some elements.*

Proof. This is a corollary of a theorem by R. L. Moore (loc. cit., Theorem 9). It is proved in the same manner as the lemma of §4.

8. Theorem. *Let the aggregate $\{t\}$ be a simple arc ab, let $X=f(t)$ be a minimal upper semi-continuous function defined over ab, and let $M=\Sigma[X]$ lie in the plane Z, while no element of M separates X_a from X_b. Then M is a continuum irreducible between X_a and X_b.*

Proof. By §7 there is one component R of $Z-M$ which has frontier points on every element of M. Since accessible points are everywhere dense, there is a decreasing sequence $\{t_i\}$ where $t_i \to a$ and an increasing sequence $\{t_i'\}$ where $t_i' \to b$, such that $X_i=f(t_i)$ and $X_i'=f(t_i')$ contain accessible points x_i and x_i', respectively.

Let x_ix_i' be a simple arc lying in R except for the end points. Let u run over a circumference C. Let the segment t_it_i' of ab be homeomorphic with an arc cd of C in such a way that t_i corresponds to c and t_i' to d. Let the arc x_ix_i' be homeomorphic to the complementary arc $dc=\overline{C-cd}$ in such a way that x_i corresponds to c and x_i' to d. Now define the function $Y=g(u)$ as follows. If $u=c$, Y is a sub-continuum of X_i irreducible about x_i and $\overline{\lim}\ X_t$ as $t \to t_i$ in t_it_i'; if $u=d$, Y is a sub-continuum of X_i' irreducible about x_i' and $\overline{\lim}\ X_t$ as $t \to t_i'$ in t_it_i'; if u is any other point of cd and t is the corresponding point of t_it_i', $Y=X_t=f(t)$; if u is a point of $dc-(c+d)$, Y is the corresponding point x of the simple arc x_ix_i'. It is readily seen that $Y=g(u)$ is a minimal upper semi-continuous function and that $N=\Sigma[Y]=G(C)$ is a generalized simple closed curve.

Now Y_c and Y_d determine two generalized arcs $N_1=G(dc)$ and $N_2=G(cd)$ having these elements as ends and no other common points. If we set H_1 equal to the simple arc x_ix_i' and $H_2=N-x_ix_i'+x_i+x_i'$, it is evident that $H_1\subset N_1$, $H_2\subset N_2$, and that $N=H_1+H_2$ is an irreducible partition of N. Then by §6, Corollary, H_2 is irreducible between x_i and x_i'. As $H_2\supset N_2-(Y_c+Y_d)$, $H_2\supset X_iX_i'-(X_i+X_i')$. That is, any sub-continuum of the arc X_iX_i' joining the end elements contains all elements between them. Hence any sub-continuum of M joining X_a and X_b contains every point of all the elements between X_i and X_i'. As $t_i\to a$ and $t_i'\to b$, this means that it contains every point of $M-(X_a+X_b)$. But, since $X=f(t)$ is a minimal upper semi-continuous function, $M=\overline{M-(X_a+X_b)}$. Hence M is irreducible between X_a and X_b.

COROLLARY. *Let the aggregate* $\{t\}$ *be a simple arc ab, let* $X=f(t)$ *be a minimal upper semi-continuous function defined over ab, and let* $M=\Sigma[X]$ *lie in the plane* Z, *while no element of* M *separates* X_a *from* X_b. *Then* M *is the frontier of one component of* $Z-M$ *and the frontier of each of the remaining components is a part of some element of* M.

PROOF. By §7 there is one component of $Z-M$ whose frontier has points on every element of M. Since this frontier is a continuum joining X_a and X_b and is a part of M, it must coincide with M by the above theorem. The last part of the corollary is merely restated from §7.

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AN UNNECESSARY CONDITION IN TWO THEOREMS OF ANALYSIS SITUS*

BY P. M. SWINGLE

1. *Introduction.* In this paper K will be a closed point set such that, if S refers to the space under consideration, $S-K=S_1+S_2$. One or more of the following conditions will hold.

CONDITION 1. *Every simple continuous arc which joins a point of S_1 to a point of S_2 contains at least one point of K.*

CONDITION 2. *Every point P of S_i ($i=1$, and 2) can be joined to any point Q of K by a simple continuous arc PQ such that $PQ-Q$ is contained in S_i.*

CONDITION 3. *Every two points of S_i ($i=1$, and 2) can be joined by a simple continuous arc contained in S_i.*

Schoenflies,† using the definition of a simple closed curve given by Jordan, stated and proved the following theorem for a plane S.

THEOREM A. *If K is bounded and if Conditions* 1, 2, *and* 3 *hold, then K is a simple closed curve.*

This theorem was later proved by Lennes‡, using his own definition of a simple closed curve. In his paper *On the foundations of plane analysis situs*, R. L. Moore§ points out that this theorem is true in a space S satisfying his set of Axioms Σ_3. J. R. Kline* stated and proved the following theorem for a space S satisfying Σ_3.

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* Presented to the Society, December 29, 1925.

† A. Schoenflies, *Ueber einen grundlegenden Satz der Analysis Situs*, Göttinger Nachrichten, 1902, p. 185.

‡ N. J. Lennes, *Curves in non-metrical analysis situs with an application in the calculus of variations*, American Journal, vol. 33 (1911), pp. 287–326, Lennes defines a *simple continuous arc* from A to B as a closed, bounded, connected point set containing $A+B$ but containing no proper connected subset that contains $A+B$. He defines a *simple closed curve* as the sum of two simple continuous arcs which have their end points and only their end points in common.

§ Transactions of this Society, vol. 17 (1916), p. 131–164, Theorem 48. This paper will be referred to hereafter as F. A.

THEOREM B. *If K is such that Conditions* 1, 2, *and* 3 *hold, and if S_i ($i=1$, and 2) is unbounded, then K is an open curve.*†

In this paper it is proposed to show that Condition 3 is unnecessary in both Theorem A and Theorem B. The proof will be given on the basis of R. L. Moore's set of Axioms Σ_3. The letter S will be used to represent a space satisfying these axioms. Proofs of these theorems will first be given, using only Conditions 1 and 2, which for Theorem A is independent both of the proof by Schoenflies and of that by Lennes and which for Theorem B is independent of that by Kline. It will then be shown in a very simple manner, independent of these proofs, that Condition 3 is unnecessary in the presence of the other two conditions. I wish to thank Professor R. L. Wilder for his valuable suggestions.

2. *Definitions and Proofs.*

DEFINITION A.‡ *A simple continuous arc from A to B (A and B distinct points) is a closed, bounded,§ connected point set, M, containing $A+B$ such that if X is any point of M distinct from A and B, then $M-X$ is disconnected.*

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* Transactions of this Society, vol. 18 (1917), p. 177–184. Lennes' definition of a simple continuous arc is used.

† An *open curve* is a closed, connected set of points, M, such that if P is a point of M, then $M-P$ is disconnected. A point set is *closed* if it contains all of its own limit points. A set N is *disconnected* (or *not connected*) if there exist point sets U and V such that $N=U+V$ separate. A point set which cannot be so divided is *connected*. By $N=U+V$ *separate* is meant that N is the sum of two non-vacuous mutually exclusive point sets U and V such that neither contains a limit point of the other.

‡ R. L. Moore, *Concerning simple continuous curves*, Transactions of this Society, vol. 21 (1920), pp. 313–320, Def. 2.

§ A point set M is *bounded* if there exists a finite set of regions R_1, $R_2, \cdots, R_n$ such that M is contained in $R_1'+R_2'+\cdots+R_n'$, where $R_i'=R_i+$(the limit points of R_i which are not contained in R_i).

DEFINITION B.* *If AXB and AYB are two simple continuous arcs having only A and B in common, then the point set composed of these two arcs is a simple closed (Jordan) curve.*

DEFINITION C.† *If M is a closed, connected point set which lies in a space S and consists entirely of non-cut points, but which is disconnected by the omission of any two of its points, then M is a simple closed (Jordan) curve.*

DEFINITION. *A region Z will be a region bounded by a simple closed curve C such that $C\times K$‡$=A_1+A_2$, where A_1 and A_2 are any two points of K, where $C=A_1X_1A_2+A_2X_2A_1$, and where $A_1X_iA_2$ $(i=1, 2)$ is an arc contained in $S_i+(A_1+A_2)$; and K_1 will represent $K\times Z$ and K_2 will represent $K\times(S-Z')$.*§

THEOREMS 1–47. *These theorems are given in F. A.*

LEMMA I. *If K is such that Condition 1 holds, then there exist at least two distinct points of K.*

Since S is connected, by Theorem 22, and is then a domain, there exists a simple continuous arc, by Theorem 15, joining a point of S_1 to a point of S_2. This arc contains at least one point of K, by Condition 1. Let N be a region containing this point such that $S-N'$ contains a point P of S_1 and a point Q of S_2. Since N' is a closed point set and since $S-N'$ is connected, by Axiom 3, $S-N'$ is a domain and there exists an arc PQ contained in $S-N'$, by Theorem 15. This arc contains at least one point of K, by Condition 1. Therefore there exist at least two distinct points of K.

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*N. J. Lennes, loc. cit.

† This definition is due to J. R. Kline, this Bulletin, vol. 27 (1921), p. 399. R. L. Wilder has proved that this definition is equivalent to Definition B in our space S; see this Bulletin, vol. 31 (1925), p. 224.

‡ If C and K are point sets $C\times K$ is the point set common to C and K.

§ If N is a point set, N' will represent the point set composed of N and its limit points.

Lemma II. *If K is such that Conditions* 1 *and* 2 *hold, and if A_1 and A_2 are any two points of K, then there exists a region Z, and A_i* ($i=1$, *and* 2) *is a limit point of both K_1 and K_2.*

Let W be a point of S_1. There exists an arc A_1W, by Condition 2. such that A_1W-A_1 is contained in S_1 and an arc A_2W such that A_2W-A_2 is contained in S_1. There exists then an arc $A_1Y_1A_2$* such that $A_1Y_1A_2-(A_1+A_2)$ is contained in S_1. Similarly there exists an arc $A_1Y_2A_2$ such that $A_1Y_2A_2-(A_1+A_2)$ is contained in S_2. Let C be the simple closed curve $A_1Y_1A_2+A_2Y_2A_1$. The region bounded by C is then a region Z.

In any region, L, containing A_i ($i=1$, or 2) there exists by Theorem 28 an arc $W_1B_1W_2$ contained in Z (or in $S-Z'$) such that W_j ($j=1$, or 2) is contained in S_j. This arc contains at least one point of K by Condition 1. Therefore A_i is a limit point of K_1 (or of K_2).

Lemma III. *If there exists a region Z, and if $K_1' = K_{11}+K_{12}$ separate, then there exists a simple closed curve J_{1i}* ($i=1$, *and* 2) *enclosing points of K_{1i} and containing at most A_1+A_2 of K and such that $J_{11}\times J_{12}=0$ and J_{1i} is contained in Z'.*

Since $K_1+(A_1+A_2)=K_1'$ is closed and K_{11} and K_{12} are separate, K_{1i} ($i=1$, and 2) is closed. Let P be any point of K_{11}, except A_1 and A_2. P is not a limit point of K_{12} for K_{12} and K_{11} are mutually exclusive closed point sets. There then exists a region, W say, containing P but not containing a point of K_{12}. Then there exists a region, Y say, contained in $Z\times W$, which contains P. Therefore there exists a region, L, by Theorem 5, which contains P and is such that L' is contained in Y. Therefore $L'\times K_{12}=0$. Thus for each point P of $K_{11}-(A_1+A_2)$ there exists a region L containing P such that L' is contained in Z and does not contain a point of K_{12}.

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* Use is here made of the fact: if AB is an arc and K is a closed point set having at least one point in common with AB, then there exists on AB a point belonging to K which precedes every other point of K on AB.

If A_j ($j=1$, or 2) is contained in K_{11} there exists a region, Q_j say, containing the one A_j but not the other and such that $Q_j'\times K_{12}=0$. Then there exists an arc $B_{j1}X_jB_{j2}$ contained in Z', by Theorem 28, such that it has B_{ji} ($i=1$, and 2) and only these points common with C, and such that this arc and the arc $B_{j1}A_jB_{j2}$ of C is contained in Q_j. Let T_{j1} be the region bounded by $B_{j1}X_jB_{j2}+B_{j2}A_jB_{j1}$. It is contained in Z, by Theorem 21. Similarly there exists an arc $C_{j1}Y_jC_{j2}$ contained in $S-Z$, by Theorem 28, but otherwise having the properties of the above arc. Let T_{j2} be the region bounded by $C_{j1}Y_jC_{j2}+C_{j2}A_jC_{j1}$. It is contained in $S-Z'$. Take now for example, the case where $C_{j1}A_jC_{j2}$ contains $B_{j1}A_jB_{j2}$. There exists then, by Theorem 27, a region $T_j=T_{j1}+T_{j2}+(B_{j1}A_jB_{j2}-(B_{j1}+B_{j2}))$. The set T_j' does not contain points of K_{12}.

Since K_{11} is bounded and closed it has the Heine-Borel property,* by Theorem 12. There exists then a finite number of regions, $L_1, L_2, \cdots, L_n$ say, of the set $(L)+(T_i)$ containing all the points of K_{11}. All the above regions can be taken with simple closed curves as boundaries, by Theorem 36. There then exists a subset, G say, of $L_1+L_2+\cdots+L_n$ such that each region of G has a simple closed curve as boundary and such that their sum forms a maximal connected subset of $L_1+L_2+\cdots+L_n$. Replace T_j by T_{j1} in case T_j is a region of G. There exists by Theorem 42 a simple closed curve, J_{11}, which is a subset of the boundaries of the regions of G and which encloses G. The curve J_{11} contains at most A_1+A_2 of K, for only these points of K are on the boundaries of (L) and T_{j1} and not contained in a region of (L).

Since K_{12} and $L_1'+L_2'+\cdots+L_n'$ are mutually exclusive closed point sets, there exists a region, M, for each point, except A_1 and A_2, of K_{12} such that M' is contained in Z and such that it does not contain points of $L_1'+L_2'+\cdots+L_n'$. If A_i ($i=1$, or 2) is contained in K_{12}, take Q_i such that

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* A point set is said to have the *Heine-Borel property* if it is true that for every infinite collection H of regions covering it, there exists a finite collection of regions belonging to H and covering it.

Q_i' does not contain points of $L_1'+L_2'+\cdots+L_n'$ and so obtain T_i as above. There then exists a simple closed curve J_{12} enclosing a maximal connected subset, H say, of regions of the set $(M)+(T_{i1})$, where H is composed of a finite number of regions, and J_{12} is a subset of the boundaries of the regions of H.

J_{12} contains at most A_1+A_2 of K and $J_{11}\times J_{12}=0$.

LEMMA IV. *If there exists a region Z, if K_2' is a bounded point set, and if $K_2'=K_{21}+K_{22}$ separate, then there exists a simple closed curve J_{2i} $(i=1$, and $2)$ enclosing points of K_{2i}, containing at most A_1+A_2 of K, and such that J_{2i} is contained in $S-Z$ and $J_{21}\times J_{22}=0$.*

The proof is the same as in Lemma III.

LEMMA V. *If K is such that Condition 1 holds, and if J is a simple closed curve contaning at most one point, A say, of K, then $J-A$ is contained either in S_1 or in S_2.*

Assume that J contains a point B_i $(i=1, 2)$ of S_i. Then J is the sum of two arcs, B_1AB_2 and B_1B_2. Each of these arcs contains at least one point of K, by Condition 1, and therefore J contains at least two distinct points of K, contrary to hypothesis.

LEMMA VI. *If K is such that Condition 2 holds, if J is a simple closed curve contained in S_c $(c=1$, or $2)$ except for the point set $(A)=J\times K$ which is either vacuous or contains but one point, A, and if J encloses at least one point of K, then J encloses $K-(A)$ and S_i $(i=2$, or $1)$.*

Let R be the region enclosed by J. Say $S_c=S_1$. Assume that there exists a point, Y say, of S_2 in $S-R'$. Since by hypothesis R contains a point of K, P say, there exists an arc YP, by Condition 2, such that $YP-P$ is contained in S_2. But this is impossible* since J does not contain a point of S_2. Therefore S_2 is contained in R. Assume that there exists a point P of K in $S-R'$. Let W be a point of S_2. There exists then an arc PW, by Condition 2, such that $PW-W$ is contained in S_2 which is impossible. Therefore R contains $K-(A)$.

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* Use is here made of the fact: a connected point set which contains a point of a domain, D, and a point not in D, contains a boundary point of D.

LEMMA VII. *If there exists a region Z, and if K is such that Conditions* 1 *and* 2 *hold, then K_1' is a connected point set.*

Assume that $K_1'=K_{11}+K_{12}$ separate. Then, by Lemma III, there exists a simple closed curve J_{1i} $(i=1$, and $2)$ enclosing points of K_{1i} and containing at most A_1+A_2 of K and such that $J_{11}\times J_{12}=0$. J_{11} may contain A_1, in which case J_{12} will contain at most A_2 of K. Therefore in every possible case one of these curves contains at most one point of K. Say J_{11} does. Then J_{11}, by Lemma V, is contained, except for possibly one point, in S_i $(i=1$, or $2)$ and so, by Lemma VI, encloses all except possibly one point of K. But it is shown in Lemma II that K_2 is non-vacuous and hence there are points of K in $S-Z'$. But J_{11} is contained in Z' and so K is. Therefore our assumption is incorrect and so K_1' is a connected point set.

LEMMA VIII. *If there exists a region Z, and if K is such that Conditions* 1 *and* 2 *hold, and if K_2' is bounded, then K_2' is a connected point set.*

Assume that $K_2'=K_{21}+K_{22}$ separate. Then there exists a simple closed curve J_{2i} $(i=1$, and $2)$ enclosing points of K_{2i}, contained in $S-Z$, containing at most A_1+A_2 of K, and such that $J_{21}\times J_{22}=0$, by Lemma IV. If one curve contains A_1+A_2 the other one does not contain a point of K; if one contains at most one point of K the other contains at most one point of K. Consider, for example, the case where J_{21} contains A_1+A_2, and let R_{21} be the region bounded by this curve. Consider now how J_{21} was obtained in Lemma IV. If it contains A_i it must contain the arc $C_{i1}A_iC_{i2}$ which is common to the boundaries of the region T_{i2} and the region Z, for only the boundaries of these two regions contained A_i in Lemma IV. Thus there exists a simple closed curve,

J_{21}'' say, which is a subset of C and J_{21} and which encloses Z and R_{21}, and also encloses T_1 and T_2. Therefore J_{21}'' does not contain A_i $(i=1$, or $2)$. Thus in each possible case we obtain two curves each contained in S_j $(j=1$, or $2)$ except for possibly one point, by Lemma V. Then each curve encloses all except possibly one point of K, by Lemma VI. Consider, for example, the case where J_{21} and J_{22} have this property. Since they have no points in common and since each encloses all except possibly one point of K, one must enclose the other. Say J_{21} encloses J_{22}. Consider again how J_{21} was obtained in Lemma IV. It is composed of the boundaries of regions each of which, among other properties, contains points of K_{21}. Say N is one of these regions and that it contains the point P of K_{21}. N' does not contain a point of J_{22} for N was so selected in the proof of Lemma IV. Let R_{22} be the region bounded by J_{22}. Since N has boundary points in $S-R_{22}'$ and since it does not contain boundary points of R_{22}, it, and so points of K, are contained in $S-R_{22}'$. Thus J_{22} does not enclose all except possibly one point of K on J_{22}. Therefore K_2' is a connected point set.

LEMMA IX. *If there exists a region* Z, *if* K_2' *is bounded, and if* K *is such that Conditions* 1 *and* 2 *hold, then the omission of any point* P_i $(i=1$, *and* $2)$ *of* K_i', $P_i \neq A_1$ *or* A_2, *disconnects* K_i'.

There exists an arc $Q_{i1}P_iQ_{i2}$ which is, except for its end points, contained in Z or in $S-Z'$ according as $i=1$ or 2, where $Q_{ij}P_i-P_j$ $(j=1$, or $2)$ is contained in S_j, by Condition 2. The arc $Q_{11}P_1Q_{12}$, by Theorem 25, divides Z into two regions, Z_{11} and Z_{12} say, where Z_{ij}' $(j=1$, and $2)$ contains A_j. Let $KX(Z_{1n}+A_n)=K_{1n}$. The sets K_{1n} are non-vacuous by Lemma II. Therefore $K_1'-P_1=K_{11}+K_{12}$ separate. Also $Q_{21}Q_{22}$ of C and $Q_{21}P^2Q_{22}$ enclose a region, Z'' say. Take, for example, the case where $Q_{21}Q_{22}$ contains A_1. Let $Z'' \times K_2 + A_1 = K_{21}$ and $(S-Z'') \times K_2 + A_2 = K_{22}$. Then $K_2'-P_2 = K_{21}+K_{22}$ separate.

Lemma X. *If there exists a region Z, and if K is bounded and such that Conditions* 1 *and* 2 *hold, then K_i' ($i=1$, or 2) is a simple continuous arc.*

K_i' is bounded and closed by hypothesis, connected by Lemma VII and Lemma VIII, and disconnected by the omission of any point other than A_1 or A_2, by Lemma IX. Therefore, by Definition A, K_i' is a simple continuous arc with end points A_1 and A_2.

Theorem I. *This is Theorem* A *with Condition* 3 *omitted.*

By Lemma I, there exist two points of K. Then, by Lemma II, there exists a region Z. Thus, by Lemma X, K is the sum of two simple continuous arcs having only their end points in common. Therefore, by Definition B, K is a simple closed curve.

Lemma XI. *If K is such that Condition* 1 *holds, and if S_i ($i=1$, and 2) is unbounded, then K is not a simple closed curve.*

Assume that K is a simple closed curve, enclosing the region R. There exists a point of S_1 and a point of S_2 in $S-R'$, for S_i ($i=1$, and 2) is unbounded. Then there exists an arc joining these two points and contained in $S-R'$. This arc does not contain a point of K, contrary to hypothesis. Therefore K is not a simple closed curve.

Theorem II. *This is the same as Theorem* B *with Condition* 3 *omitted.*

There exist at least two distinct points of K by Lemma I, P_1 and P_2 say. Assume that $K=N_1+N_2$ separate and let A_i ($i=1$, and 2) be a point of N_i. There exists a region Z by Lemma II. Then, by Lemma VII, there exists a connected subset of K, contained in Z', and containing A_1+A_2. But that this connected subset of K should contain points of both of the separate point sets N_1 and N_2 is impossible. Therefore our assumption is incorrect and so K is a connected point set.

K is disconnected by the omission of any two points, A_1 and A_2. For there exists a region Z, by Lemma II, having these two points and only these two points of K on its boundary and such that Z and $S-Z'$ each contain points of K. Thus the omission of these two points divides K into two separate point sets $K\times Z$ and $K\times(S-Z')$. K is a closed point set by hypothesis. Thus by Definition C, if K consists entirely of non-cut points, K is a simple closed curve, contrary to Lemma XI. Thus there exists a cut point, P say, of K. Let $K-P=M_1+M_2$.

Let A_i $(i=1$, and $2)$ be any point of M_i. Then there exists a region Z by Lemma II, which, by Lemma VII, contains a connected subset of K joining A_1 and A_2. That is, this connected point set joins points of the two separate point sets M_1 and M_2. This is impossible unless it contains P. Just as in Lemma X, K_1' being bounded here is an arc. Both M_1 and M_2 have points in Z as P is clearly a limit point of both. Now $K_1'-P$ is the sum of two connected sets, each of which can lie, obviously, in only one of the sets M_1, M_2. Thus it is seen that one of these connected sets must be contained in M_1 and one in M_2. Let $M_i\times(Z+A_i)=M_{i1}$ and $M_i\times(S-Z')=M_{i2}$. These point sets are non-vacuous by Lemma II.

The sets M_{12} and M_2 are separate since M_1 and M_2 are separate. Also $(M_{11}-A_1)$ and M_{12} are separate for Z and $S-Z'$ are separate. P is not a limit point of $S-Z$ and so is not a limit point of M_{12}. Therefore $K-A_1=M_{12}+(P+(M_{11}-A_1)+M_2)$ separate. Therefore the omission of any point of M_1 disconnects K. Similarly for any point of M_2. Therefore the omission of any point of K disconnects K.

Let P be any point of K and let $K-P=N_1+N_2$ separate. Assume that $N_1=N_{11}+N_{12}$ separate. Let A_i $(i=1$, and $2)$ be any point of N_{1i}. There exists by Lemma II a region Z such that $K\times Z'=A_1A_2$ is a simple continuous arc as in Lemma X. This arc joins points of the two separate point sets N_{11} and N_{12}. This is impossible unless it contains P. Thus this arc is the sum of two arcs PA_1 and PA_2. The arc

PA_1 cannot contain points of N_2 for it would then contain a connected point set containing points of the two separate point sets N_1 and N_2. Similarly, it cannot contain points of N_{12}. Therefore PA_1-P must be contained in N_{11}. Similarly, PA_2-P must be contained in N_{12}. Therefore the arc A_1A_2-P must be contained in N_1. Therefore the region Z does not contain points of N_2. Thus P is not a limit point of N_2. But it is well known that N_2+P is a connected point set and so P is a limit point of N_2. Therefore our assumption is incorrect and so N_i ($i=1$, and 2) is a connected point set.

K is closed by hypothesis. Therefore K is an open curve.

A simple proof that Condition 3 is unnecessary in Theorems A and B may also be obtained from Lemma I, Lemma II, and the following theorem.

THEOREM III. *If* L *is a subset of a point set* M, *and if* $M'-L=M_1+M_2$ *such that* (2) *every point of* M_i ($i=1$ *and* 2) *can be joined to any point* Q *of* L *by a simple continuous arc* PQ *such that* $PQ-Q$ *is contained in* M_i, *and* (3) M_j ($j=1$, *or* 2) *is not arc-wise connected,* L *contains at most two points.*

Take for example the case where M_1 is not arc-wise connected, there being no arc AB of M_1 joining the two points A and B of M_1. Assume that L contains at least three points, E_i ($i=1$, 2, and 3) say. Then by Condition 2 there exist three arcs AE_i and three arcs BE_i contained in $M_1+(E_1+E_2+E_3)$. Then there exist three arcs, A_1E_i, distinct except for A_1, contained in $AE_1+AE_2+AE_3$ and three arcs B_1E_i, distinct except for B_1, contained in $BE_1+BE_2+BE_3$. These two sets of three arcs have but $E_1+E_2+E_3$ in common for otherwise there exists an arc AB in M_1. There exist then three arcs $A_1E_iB_1$ ($i=1$, 2, 3) having only A_1 and B_1 in common. Thus there exist three simple closed curves $A_1E_1B_1+A_1E_2B_1=J_1$ say, $A_1E_2B_1+A_1E_3B_1=J_2$ say, and $A_1E_3B_1+A_1E_1B_1=J_3$ say. Let R_i ($i=1$, 2, and 3) be the region bounded by J_i. Consider for example the case where $R_1=R_2+R_3+A_1E_3B_1-(A_1+B_1)$. J_1 is contained in M_1+L.

Assume that $S-R_1'$ contains a point, say U, of M_2. Then by Condition 2 there exists an arc UE_3 contained in M_2+E_3. This is impossible for such an arc must contain a point of J_1. Therefore M_2 is contained in R_1. Similarly it can be proved that M_2 is contained in $S-R_2'$ and so in R_3. Similarly it can be proved that M_2 is contained in $S-R_3'$. Thus a contradiction is obtained. Therefore there do not exist three points of L.

CORNELL UNIVERSITY

TWO CLASSES OF PERIODIC ORBITS WITH REPELLING FORCES*

BY T. H. RAWLES

1. *Introduction.* The purpose of this paper is to obtain certain classes of periodic orbits† of a system consisting of one body of very great mass which attracts two mutually repellent bodies of very small mass. The forces of attraction and repulsion in the system are assumed to vary inversely as the squares of the distances. It will be shown that if certain conditions of symmetry are imposed the orbits of the two small bodies must lie either in two parallel planes or else be coplanar. In the former case they are circles about a line perpendicular to the two planes and passing through the large body. For the second type of orbit a development in powers of a parameter will be obtained.

2. *The Equation of Motion.* By a proper choice of the units of time and space we can make the coefficient of the intensity of the attraction equal to 1. Then if K^2 is the

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† The orbits referred to were first calculated by Langmuir, who employed numerical integration. (Physical Review, vol. 17 (1921), p. 339.) They have been discussed by Van Vleck in his monograph on *Quantum Principles and Line Spectra.* (Bulletin of the National Research Council, Number 54, p. 89.)

ratio of the repulsion to the attraction the force function of the system is

$$U = \frac{1}{r_1} + \frac{1}{r_2} - \frac{K^2}{\Delta}, \tag{1}$$

where r_1 and r_2 represent the distances between the two small bodies and the large body while Δ is the distance between the small bodies.

The unit of mass may be so chosen that the mass of each of the small bodies is unity. Then let m denote the mass of the large body. The system will be referred to the rectangular axes ξ, η, ζ, and we shall indicate the coordinates of the large body by the subscript 0, and the coordinates of the small bodies by the subscripts 1 and 2. With this choice of units and coordinates, the equations of motion are

$$m\ddot{\xi}_0 = \frac{\partial U}{\partial \xi_0}, \qquad m\ddot{\eta}_0 = \frac{\partial U}{\partial \eta_0}, \qquad m\ddot{\zeta}_0 = \frac{\partial U}{\partial \zeta_0}; \tag{2}$$

$$\ddot{\xi}_i = \frac{\partial U}{\partial \xi_i}, \qquad \ddot{\eta}_i = \frac{\partial U}{\partial \eta_i}, \qquad \ddot{\zeta}_i = \frac{\partial U}{\partial \zeta_i}, \qquad (i = 1,2). \tag{3}$$

If m is very large, the effect of the attraction of the other bodies on the large body is very small. We shall assume that this can be neglected altogether. In other words we assume that the mass of the large body is infinite as compared to that of the other two. Under this assumption we can neglect the right members of (2). These equations then show that the large body moves through space with uniform velocity in a straight line.

We now take this body as the origin, and denote the coordinates of the other two by x_i, y_i, z_i, $(i=1, 2.)$ The equations of motion of the two small bodies then become

$$\ddot{x}_i = \frac{\partial U}{\partial x_i}, \qquad \ddot{y}_i = \frac{\partial U}{\partial y_i}, \qquad \ddot{z}_i = \frac{\partial U}{\partial z_i}, \tag{4}$$

where

$$U = \frac{1}{r_1} + \frac{1}{r_2} - \frac{K^2}{\Delta}, \tag{5}$$

and

$$(6)\quad \begin{cases} r_i^2 = x_i^2 + y_i^2 + z_i^2, & (i = 1,2), \\ \Delta^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2. \end{cases}$$

3. *Periodic Orbits.* The periodic orbits with which we are concerned are those for which

$$(7)\qquad y_1 - y_2 = 0, \quad z_1 - z_2 = 0, \quad x_1 - x_2 = 0.$$

With these restrictions equations (4) now become

$$(8)\quad \begin{cases} \dot{x}_1 = \dfrac{-x_1}{r_1^3} + \dfrac{k^2(x_1 - x_2)}{((x_1 - x_2)^2)^{3/2}} = \dfrac{-x_1}{r_1^3} + \dfrac{k^2}{(2x_1)^2}, \\ \dot{y}_1 = \dfrac{-y_1}{r_1^3}, \quad \dot{z}_1 = \dfrac{-z_1}{r_1^3}, \end{cases}$$

with three similar equations for the coordinates of the other particle. The variables are now separated and we need only consider equations (8). When these are solved the motion of the other particle can be obtained from (7).

Let us now put $x_1 = x$, $y_1 = y$, $z_1 = z$, $r_1 = r$, and $k^2/4 = 1 - e^2$. Our equations may then be written in the form

$$(9)\qquad x = \frac{-x}{r^3} + \frac{1 - e^2}{x^2}, \quad \dot{y} = \frac{-y}{r^3}, \quad \ddot{z} = \frac{-z}{r^3}.$$

Two integrals of (9) can be found at once. They are

$$(10)\quad \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{r} - \frac{1 - e^2}{x} + C, \quad \dot{z}y - z\dot{y} = h.$$

Also we notice that if a solution

$$x = X(e,t), \quad y = Y(e,t), \quad z = Z(e,t)$$

is known, then another solution is

$$x = aX\left(e, \frac{t - t_0}{a^{3/2}}\right), \quad y = aY\left(e, \frac{t - t_0}{a^{3/2}}\right), \quad z = aZ\left(e, \frac{t - t_0}{a^{3/2}}\right).$$

4. *Periodic Solutions of Period* 2π. We shall obtain solutions periodic with the period 2π. From the remark which we have just made we can infer the existence of solutions of any desired period. The existence of a solution developed in powers of e as a parameter follows from a theorem of MacMillan.* When $e=0$, a solution of the equations is

$$x = 1, \quad y = 0, \quad z = 0. \tag{11}$$

The solutions which we shall proceed to develop are of the form

$$\begin{cases} x = 1 + e^2x_2 + e^4x_4 + \cdots, \\ y = ey_1 + e^3y_3 + \cdots, \\ z = ez_1 + e^3z_3 + \cdots. \end{cases}$$

When developed in powers of the parameter, the equations of motion are

$$\begin{cases} \sum\limits_0 e^{2i}\dot{x}_{2i} = [\frac{3}{2}(y_1^2 + z_1^2) - 1]e^2 + \cdots, \\ \sum\limits_0 e^{2i+1}\ddot{y}_{2i+1} = -y_1e + [-y_3 \\ \qquad\qquad + \frac{3}{2}y_1(2x_2 + y_1^2 + z_1^2)]e^3 + \cdots, \\ \sum\limits_0 e^{2i+1}z_{2i+1} = -z_1e + [-z_3 \\ \qquad\qquad + \frac{3}{2}z_1(2x_2 + y_1^2 + z_1^2)]e^3 + \cdots. \end{cases} \tag{12}$$

Equating the coefficients of e in the last two equations we obtain

$$\dot{y}_1 + y_1 = 0, \qquad z_1 + z_1 = 0. \tag{13}$$

The solutions of these equations are

$$\begin{cases} y_1 = c_1 \cos t + c_2 \sin t, \\ z_1 = c_3 \cos t + c_4 \sin t, \end{cases} \tag{14}$$

where $c_1, \cdots, c_4$ are constants of integration. From the coefficients of e in the first of (12), we have

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* Transactions of this Society, vol. 13 (1912), p. 157.

$$(15)\quad \left\{\begin{aligned} x_2 &= \frac{3}{2}(y_1^2 + z_1^2) - 1 \\ &= \frac{3}{2}\left[\left(\frac{c_1^2 + c_2^2 + c_3^2 + c_4^2}{2}\right)\right. \\ &\quad + \left(\frac{c_1^2 + c_3^2 - c_2^2 - c_4^2}{2}\right)\cos 2t \\ &\quad \left. + (c_1c_2 + c_3c_4)\sin 2t\right] - 1. \end{aligned}\right.$$

Upon integration we obtain

$$x_2 = \left(\frac{3}{4}(c_1^2 + c_2^2 + c_3^2 + c_4^2) - 1\right)\frac{t^2}{2} + c_5t + c_6$$
$$- \frac{3}{16}(c_1^2 + c_3^2 - c_2^2 - c_4^2)\cos 2t - \frac{3}{8}(c_1c_2 + c_3c_4)\sin 2t,$$

c_5 and c_6 being arbitrary.

The periodicity conditions which we shall impose are that $x(2\pi) - x(0) = 0$ and $\dot{x}(2\pi) - \dot{x}(0) = 0$, with similar conditions for the other coordinates. Moreover, the coefficient of each power of the parameter must satisfy these conditions. When the periodicity conditions are applied to x_2 and $\dot{x}_2$ we see that

$$\left(\frac{3}{4}(c_1^2 + c_2^2 + c_3^2 + c_4^2) - 1\right)\frac{(2\pi)^2}{2} - 2\pi c_5 = 0,$$
$$\left(\frac{3}{4}(c_1^2 + c_2^2 + c_3^2 + c_4^2) - 1\right)(2\pi) = 0.$$

From these relations we now find that

$$(16)\qquad c_1^2 + c_2^2 + c_3^2 + c_4^2 = 4/3, \quad c_5 = 0.$$

Obviously the periodicity conditions imposed on y_1, $\dot{y}_1$, z_1 and $\dot{z}_1$ are satisfied identically.

When we proceed to the next approximation we have the equations

$$(17)\qquad \begin{cases} \ddot{y}_3 + y_3 = (3/2)(2x_2 + y_1^2 + z_1^2)y_1, \\ \ddot{z}_3 + z_3 = (3/2)(2x_2 + y_1^2 + z_1^2)z_1. \end{cases}$$

These equations become

$$\ddot{y}_3 + y_3 = \sum_i A_i^{(3)} \sin it + \sum B_i^{(3)} \cos it,$$

$$\ddot{z}_3 + z_3 = \sum_i C_i^{(3)} \sin it + \sum D_i^{(3)} \cos it, \quad (i = 1,3),$$

where the coefficients of the periodic terms represent known functions of the constants of integration. Upon integrating the first of these two equations, we have

$$y_3 = c_1^{(3)} \cos t + c_2^{(3)} \sin t - A_1^{(3)} \frac{t \cos t}{2} + B_1^{(3)} \frac{t \sin t}{2}$$
$$+ \text{ terms in } \cos 3t \text{ and } \sin 3t,$$

where $c_1^{(3)}$ and $c_2^{(3)}$ are the new constants of integration. When we impose the condition that y_3 and $\dot{y}_3$ shall be periodic we find that $A_1^{(3)} = B_1^{(3)} = 0$. In the same manner we can show that $C_1^{(3)} = D_1^{(3)} = 0$.

5. *Restrictions on the Constants.* These four equations impose new restrictions on the constants c_i. They become

$$(18)\qquad \begin{cases} B_1^{(3)} = 3(c_6 + \frac{1}{3})c_1 + c_1 M + c_2 N = 0, \\ A_1^{(3)} = 3(c_6 + \frac{1}{3})c_2 - c_2 M + c_1 N = 0, \\ D_1^{(3)} = 3(c_6 + \frac{1}{3})c_3 + c_3 M + c_4 N = 0, \\ C_1^{(3)} = 3(c_6 + \frac{1}{3})c_4 - c_4 M + c_3 N = 0, \end{cases}$$

where

$$M = \tfrac{3}{32}(c_1^2 + c_3^2 - c_2^2 - c_4^2), \quad N = \tfrac{3}{16}(c_1 c_2 + c_3 c_4).$$

When we multiply the four equations by c_4, $-c_3$, $-c_2$, c_1, respectively and add, the result is

$$(19)\qquad 3(c_6 + \tfrac{1}{3})(c_1 c_4 - c_2 c_3) = 0.$$

There are three cases according as one factor is zero or both factors are zero.

CASE I. $c_6+1/3=0$, $c_1c_4-c_2c_3\neq 0$.
The first of (16) shows that not all of the constants can be zero. It then follows from equations (18) that $M=N=0$. This gives us the two equations

$$c_1^2+c_3^2-c_2^2-c_4^2=0, \quad c_1c_2+c_3c_4=0.$$

If, for example, $c_2\neq 0$ we have from the second of these two equations

$$c_1^2=\frac{c_3^2c_4^2}{c_2^2}.$$

When this is substituted in the first it becomes

$$c_3^2c_4^2+c_2^2c_3^2-c_2^4-c_4^2c_2^2=0,$$

or

$$(c_2^2+c_4^2)(c_3^2-c_2^2)=0.$$

Since $c_2\neq 0$ we must have

$$c_3=c_2, \quad c_1=-c_4, \quad \text{or} \quad c_1=c_4, \quad c_3=-c_2.$$

Case I shows that as far as the approximation has been carried the motion is in a circle in a plane normal to the x axis. In fact we have

$$y_1^2+z_1^2=c_1^2+c_2^2=\tfrac{2}{3},$$
$$x_2=-\tfrac{1}{3}.$$

CASE II. $c_6+1/3=0$, $c_2c_3-c_1c_4=0$.
It is easy to show that this case is impossible. We would have as in Case I $M=N=0$, and would obtain the same relations between the constants. When these relations are substituted in $c_2c_3-c_1c_4=0$ that equation becomes $c_2^2+c_1^2=0$, which is impossible since not all of these constants can be zero.

CASE III. $c_6+1/3\neq 0$, $c_2c_3-c_1c_4=0$.
On multiplying the first of equations (14) by c_3, the second by c_1 and subtracting, since $c_2c_3-c_1c_4=0$, we obtain $c_3y_1-c_1z_1=0$, which shows that the motion lies in a plane passing through the x axis. (See Fig. 1.)

6. *The Orbits.* We shall next show that when the process is continued each approximation gives rise to terms of the same type as those already found, and, therefore, that the periodic orbits are either circles about the x axis, or else they lie entirely in a plane containing the x axis.

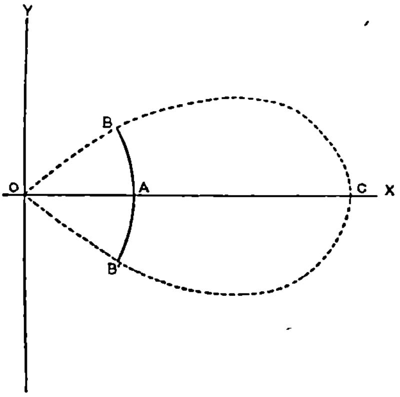

FIG. 1

The figure illustrates the path described by one of the particles for the case in which $e=1/2$. When $t=0$ it is at A. It reaches B when $t=\pi/2$, stops, and then returns along the same path to A, $t=\pi$. It then continues to B', $t=3\pi/2$, and finally completes the cycle by returning to A when $t=2\pi$. The position of the other particle is at all times symmetric to that of this one with respect to the y axis. The curve of zero velocity (conchoid) is indicated by $OBCB'O$.

For the discussion of Case I, let us suppose that when $t=0$, $y=0$. We then have

$$c_2 = c_3 = (\tfrac{2}{3})^{1/2}, \quad c_4 = -c_1 = 0.$$

The solution, as far as we have carried it, is

$$\begin{aligned} y &= e(c_2 \sin t) + e^3(c_2^{(3)} \sin t), \\ z &= e(c_2 \cos t) + e^3(c_3^{(3)} \cos t + c_4^{(3)} \sin t), \\ x &= 1 + e^2(-\tfrac{1}{3}). \end{aligned}$$

The equation for x_4 becomes

$$\begin{aligned}\dot{x}_4 &= 3(y_1y_3 + z_1z_3) + 2x_2 + \tfrac{9}{2}x_2^2\\ &= (\tfrac{3}{2}(c_2c_2^{(3)} + c_2c_3^{(3)}) - \tfrac{1}{6})\\ &\qquad + \tfrac{3}{2}\left\{c_2(c_3^{(3)} - c_2^{(3)})\right\} \cos 2t + \tfrac{3}{2}c_2c_4^{(3)} \sin 2t.\end{aligned}$$

Periodicity requires that

$$c_2(c_2^{(3)} + c_3^{(3)}) = 1/9,$$

and we have upon integration

$$x_4 = c_6^{(4)} - \tfrac{3}{8}c_2(c_3^{(3)} - c_2^{(3)}) \cos 2t - \tfrac{3}{8}8c_2c_4^{(3)} \sin 2t.$$

The equations for y_5 and z_5 now become

$$\begin{aligned}y_5 + y_5 &= \tfrac{3}{2}y_1(2x_4 + x_2^2 + 2(y_1y_3 + z_1z_3)),\\ \ddot{z}_5 + z_5 &= \tfrac{3}{2}z_1(2x_4 + x_2^2 + 2(y_1y_3 + z_1z_3)),\end{aligned}$$

which are when written out in full

$$\begin{aligned}\ddot{y}_5 + y_5 &= \tfrac{3}{2}(c_2 \sin t)\left\{(\tfrac{2}{9} + 2c_6^{(4)})\right.\\ &\qquad \left. + \tfrac{1}{4}c_2(c_3^{(3)} - c_2^{(3)}) \cos 2t + \tfrac{1}{4}c_2c_4^{(3)} \sin 2t\right\},\\ \dot{z}_5 + z_5 &= \tfrac{3}{2}(c_2 \cos t)\left\{(\tfrac{2}{9} + 2c_6^{(4)})\right.\\ &\qquad \left. + \tfrac{1}{4}c_2(c_3^{(3)} - c_2^{(3)}) \cos 2t + \tfrac{1}{4}c_2c_4^{(3)} \sin 2t\right\}.\end{aligned}$$

When the products of the trigonometric functions are expressed as functions of multiple angles it will be seen that for periodicity we must have

$$c_6^{(4)} = -1/9, \quad c_3^{(3)} = c_2^{(3)}, \quad c_4^{(3)} = 0.$$

The last of these conditions shows that z_3 contains no term in sin t. The complementary function for z_5 may contain a term in sin t but the next integration will show that the coefficient of this term is zero.

The process can be continued indefinitely and our solution is of the form

$$y = p_1 \sin t, \quad z = p_1 \cos t, \quad x = p_2,$$

where p_1 and p_2 are power series in e. The orbit is obviously a circle about the x axis.

We could have obtained this orbit by more elementary means. If we put

$$x = (1 - e^2)^{1/3},$$
$$y = (1 - (1 - e^2)^{2/3})^{1/2} \sin (t - t_0),$$
$$z = (1 - (1 - e^2)^{2/3})^{1/2} \cos (t - t_0),$$

it is easily seen that the equations of motion are satisfied.

For the discussion of the case where the motion, as far as the first approximation is concerned, lies in a plane containing the x axis, let us choose that plane $z=0$. We shall take as the origin of time the instant when $y=0$. With this choice we have $c_2=(4/3)^{1/2}$, $c_1=c_3=c_4=0$.

Under these restrictions the equation for z_3 is $\ddot{z}_3+z_3=0$, and its solution is

$$z_3 = c_3^{(3)} \cos t + c_4^{(3)} \sin t.$$

The constants of integration introduced in the complementary function of x_2 have already been determined by (16) and (18), and the equation for z_5 now becomes

$$\begin{aligned}\ddot{z}_5 + z_5 &= z_3(3x_2 + \tfrac{3}{2}y_1^2)\\ &= -\tfrac{1}{4}c_3^{(3)} \cos t - \tfrac{1}{8}c_3^{(3)} \cos 3t - \tfrac{1}{8}c_4^{(3)} \sin 3t.\end{aligned}$$

Now for a periodic solution it is obvious that $c_3^{(3)}=0$. Therefore z_3 contains only a term in sin t. If we continue this process imposing each time the condition that when $t=0$, $y=0$, we shall find that z as well as y contains only terms in $\sin (2k+1)t$. This means of course that the particle crosses the x axis when $t=n\pi$. The motion will always remain in the plane which contains the x axis and the tangent to the orbit at the instant the particle crosses that axis because all the forces are acting in this plane. This can be easily verified by a consideration of the differential equations.

7. *Indefinite Extension of the Process.* It remains to show that the process of determining the constants can be carried on indefinitely. For this purpose we may take $z=0$ as the

plane of motion and suppose that the following terms have been calculated:

$$(20)\qquad \begin{cases} x_{2j} = \sum_{0}^{j} a_{2i}^{(2j)} \cos 2it, & (j = 0,1,2,\cdots, n), \\ y_{2j+1} = d_{2j+1} \sin t + \sum_{1}^{j} a_{2i+1}^{(2j+1)} \sin (2i+1)t. \end{cases}$$

All the coefficients have been determined except d_{2n+1}. When we equate the coefficients of e^{2n+2} in the equation for x we find

$$\dot{x}_{2n+2} = x_{2n+2}(-1 + 3 - 2) + 3y_1y_{2n+1} + \text{known terms}.$$

This may be written in the form

$$(21)\qquad \dot{x}_{2n+2} = \tfrac{3}{2}c_2d_{2n+1}(1 - \cos 2t) + \sum_{0}^{n+1} A_{2i}^{(2n+2)} \cos 2it,$$

where all the coefficients are known except d_{2n+1}. The arguments of the periodic functions must be even multiples of the time for they all enter from x or y^2 which contain only terms with arguments of that form.

The integration of this equation will produce a secular term unless

$$\text{(i)}\qquad \tfrac{3}{2}c_2d_{2n+1} + A_0^{(2n+2)} = 0.$$

Having imposed this condition, we find that

$$(22)\qquad x_{2n+2} = b_{2n+2}t + d_{2n+2} + \sum_{1}^{n+1} a_{2i}^{(2n+2)} \cos 2it,$$

where b_{2n+2} and d_{2n+2} are constants of integration and

$$a_{2i}^{(2n+2)} = \frac{-A_{2i}^{(2n+2)}}{(2i)^2}.$$

For periodicity we must of course put

$$\text{(ii)}\qquad b_{2n+2} = 0.$$

We now determine y_{2n+3} by equating the coefficients of e^{2n+3} in the y equation. This gives

$$\ddot{y}_{2n+3} + y_{2n+3} = 3y_1 x_{2n+2} + \text{known terms},$$

which may be written in the form

$$(23) \quad \ddot{y}_{2n+3} + y_{2n+3} = 3c_2 d_{2n+2} \sin t + \sum_{0}^{n+1} A_{2i+1}^{(2n+3)} \sin (2i+1)t.$$

The arguments of the periodic functions are odd multiples of the time because they arise from terms such as y^{2k+1} and $y^{2k+1}x^{2h}$. If the coefficient of sin t in the right member is not zero a secular term will be introduced. This requires that

$$(\text{iii}) \qquad 3c_2 d_{2n+2} + A_1{}^{2n+3} = 0.$$

When the equation is integrated we have

$$(24) \quad y_{2n+3} = d_{2n+3} \sin t + b_{2n+3} \cos t + \sum_{1}^{n+1} a_{2i+1}^{(2n+3)} \sin (2i+1)t,$$

where

$$a_{2i+1}^{(2n+3)} = \frac{A_{2i+1}^{(2n+3)}}{1-(2i+1)^2}.$$

We may suppose that the origin of time has been fixed by the condition $y=0$ when $t=0$. This determines the remaining constant and we have

$$(\text{iv}) \qquad b_{2n+3} = 0.$$

We now have determined all the coefficients to and including x_{2n+2} and y_{2n+2}, except the coefficient of sin t in the latter. Equations (22) and (24) are of the same form as (20), with n replaced by $n+1$ and it is evident that the process can be carried on indefinitely.

8. *Development of x and y in Terms of the Parameter.* The development of x and y has been carried as far as terms involving the fifth power of the parameter. The expressions found are

$$(25)\quad \begin{cases} x = 1 + \left(-\dfrac{3}{8} + \dfrac{1}{4}\cos 2t \right) e^2 \\ \qquad + \left(-\dfrac{391}{4608} + \dfrac{29}{384}\cos 2t - \dfrac{5}{1536}\cos 4t \right) e^4, \\ y = (3)^{1/2} \left[\left(\dfrac{2}{3}\sin t \right) e + \left(\dfrac{1}{96}\sin 3t \right) e^3 \right. \\ \qquad + \left(-\dfrac{3307}{331776}\sin t + \dfrac{91}{36864}\sin 3t \right. \\ \qquad \left. \left. + \dfrac{37}{110592}\sin 5t \right) e^5 \right]. \end{cases}$$

For the case of the helium atom $k^2 = 1/2$ and the parameter e must be given the value $(7/8)^{1/2}$. For this value of the parameter the convergence is not rapid, and the terms which we have calculated do not give the same degree of accuracy as the numerical integration of Langmuir. However the two results are in sufficient agreement to verify each other.

The numerical results are easily checked by means of the energy integral

$$(10)\qquad \frac{1}{2}(\dot{x}^2 + \dot{y}^2) = \frac{1}{r} - \frac{1 - e^2}{x} + C.$$

The constant of integration is itself a power series in the parameter. As far as the sixth power of the parameter, it is

$$(26)\qquad C = -\frac{1}{3}e^2 + \frac{1}{16}e^4 + \frac{127}{10368}e^6.$$

YALE UNIVERSITY

RATIONAL QUINTIC SURFACES WITH TWO SKEW DOUBLE LINES*

BY A. R. WILLIAMS

The purpose of this paper is to obtain by different methods, and where possible to extend, the results given by Clebsch and Hill† for quintic surfaces having two distinct skew double lines. In a second paper, which will appear in an early issue of this Bulletin, I discuss a surface, due to Montesano, which has two consecutive skew double lines.

The equation of a quintic surface that has xy and zw for double lines, but not further specialized, contains twenty-four terms. If we take a point on each of them, say $(0:0:z_1:1)$ and $(x_1:1:0:0)$ we find that for a given surface there are thirteen pairs of values of x_1 and z_1 such that the line joining corresponding points lies in the surface. If we allow the coefficients to vary we find that eleven of these transversals may be assigned arbitrarily and the remaining two are then determined. That is, there is a single infinity of such surfaces all of which have the eleven transversals and the other two determined by them. The form of the equation shows that there are six pinch points on each of the double lines.

The surface is obviously rational, that is, its points may be put in one to one correspondence with the points of a plane. For the line drawn from any point P of the surface to intersect the two double lines meets a given plane, π_0, in a point P', which we may consider the image of P. A section of the quintic surface has of course two double points where its plane cuts the double lines. Therefore, establishing the correspondence in the way just indicated, as P moves on the plane section the surface generated by the line $P'P$ is of degree $2\times1\times1\times5-2\times1-2\times1$, or 6. And the two lines, a and b, double on the quintic, are triple on this sextic surface. Hence the images of the plane sections are sextic curves having in common two triple points where the lines a and b meet π_0, and thirteen simple points where the transversals meet it. If we did not know the number of the latter, we could infer it to be thirteen, since two sextics of the system can have only five variable intersections. As mentioned above, eleven of these transversals determine the other two, and there is a pencil of surfaces having the two double lines and the thirteen transversals. But the ruled sextic determined as above by any plane section of a quintic of the pencil meets the other quintics also in plane quintic curves. Thus we have ∞^3 and not ∞^4 such ruled sextics. Hence we have the following theorem.

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* Presented to the Society, San Francisco Section, April 2, 1927.

† Clebsch, Mathematische Annalen, vol. 1, p. 253.
Hill, Mathematical Review, vol. 1, p. 1.

THEOREM 1. *Of the ∞^4 plane sextics having in common two triple points and eleven simple points there are ∞^3 (instead of ∞^2) that pass also through two certain points determined by the given thirteen.*

We see how this determination is effected when we consider the images of the double lines. From each point of one double line we must draw the two tangents to the surface, one in each sheet, that meet the other. Hence the two images of a point of a double line lie on a line through the point in which the other double line meets π_0. Let the double lines a and b meet π_0 at B and A respectively. Since any plane through a double line is tangent to the surface where the residual cubic meets the double line, the image of a has a double point at B and a triple point at A, and is met by any line through B in three points and by any line through A in two (associated) points. Thus the image of a double line is a quintic which has a double point where the double line itself meets π_0, and a triple point where the other double line meets it, and which contains the points of intersection of the thirteen transversals with π_0. But a quintic that has a fixed triple point and a fixed double point is fully determined by 11 other simple points; and two such quintics have just two more intersections, which are the points we seek. This gives the following simple but striking result. In a plane, π_0, take two points A and B and 11 other points C. Let the two remaining intersections of the two quintics both of which contain the 11 points C, and which have respecitvely a triple point at A and a double point at B, and vice versa, be P and Q. Then if any two skew lines neither of which lies in π_0, be drawn through A and B, there will be a pencil of quintic surfaces having the two skew lines for double lines and containing the 13 transversals determined by the points C and P and Q.

The configuration in π_0 may be simplified by a quadratic transformation. Taking for fundamental triangle the points A and B in which the double lines meet the plane, and C, where any one of the transversals meets it, we see immediately that the sextics which are the images of the plane sections become quintics having double points at A and B, and passing through the twelve points that correspond to the points in which the other twelve transversals meet π_0. The images of the double lines become quartics, one having a double point at A and a simple point at B, and the other a node at B and a simple point at A, and both passing through the twelve new points. C goes into the line AB which is now the image of the line on the surface that originally corresponded to C. Evidently the point C can be chosen in thirteen ways. That is, the original plane system of sextics can be reduced to the system of quintics having in common two double points and twelve simple points in thirteen ways. This was noted by Clebsch. Referring now to the new system let l be the line on the surface whose image is AB. To sections of the surface by planes through l correspond quintics each composed of the line AB and a variable quartic through A and B and the other twelve base points. These quartics have only two variable intersections with AB, since l meets both double lines of the surface. It was seen above that the image of a double line is a quartic, having a double point at A (or B) and a simple point at B (or A), and passing through the twelve simple base points. Hence the images of the double lines are special curves of the pencil of quartics that correspond to the plane sections through l; and the fact that they have double points at A and B shows that any two curves of the pencil have two intersections at each of these points. This may be restated as follows.

THEOREM 2. *The fourteen fundamental points are the base points of a pencil of quartics that have a common tangent at two of them.*

On the image of a double line the points are paired, since to a point on a double line must correspond two points in the plane. Thus the quintic corresponding to any plane section meets the image of a double line in two (associated) points aside from the fundamental points. Two quartics of the pencil are tangent to AB. The corresponding plane quartics on the surface are tangent to l where it is tangent to the parabolic curve, and their planes are doubly stationary, that is, they are stationary planes in the developable of the stationary tangent planes. Clebsch remarks that there are twenty-five quartics of the plane pencil that have a node aside from A or B. The number would seem to be twenty-three, since the two special quartics of the pencil (the images of the double lines) count for two each. That is, $3(n-1)^2-4=23$ for $n=4$. Thus if we consider a pencil of cubics that have a common tangent there are only ten $(3\times 2^2-2)$ that have a node aside from the one whose node is at the point of tangency. Thus the plane sections through l (and hence any transversal) are in general doubly tangent to the surface, but twenty-three of them are triply tangent; and evidently two, that is, the planes determined by the transversal and the double lines, cut the surface in conics, and are quadruply tangent.

In the above the plane system has been derived by geometrical projection and quadratic transformation. If we start with the statement that the plane system consists of the quintics having in common two double points, A and B, and twelve simple points, we immediately see that there must be a relation among the base points, for otherwise they would determine only ∞^2 quintics. We may, however, easily find this relation by means of the theory of rational surfaces as developed by Caporali.* It tells us that the complete image of the double lines is a curve of degree eight which must consist of the two quartics described above. Considering them in connection with the pencil of quartics, the images of the plane sections through l, we find as before the relation connecting the base points.

Some other properties of the surface will now be developed. Let a be the double line whose image is the quartic with double point at A. To the quintic composed of this quartic and the line AB corresponds the section of the surface whose plane contains a and l. Therefore the point A must correspond to the conic cut from the surface by that plane. All directions at A correspond to points on this conic. The directions of the tangents of the image of a correspond to the points at which the conic meets a. To the direction AB corresponds the intersection of the conic with l which does not take place on the other double line, b. All sections of the surface by planes through l, including the conic in question, pass through the point where l meets b; and corresponding to that point we have at A the common direction of the pencil of quartics which are the images of those sections. The image of b has at A this same direction. Since b is a double line, its intersection with l corresponds to another point in the plane, which must lie on AB, and which is the fourth point in which the image of b meets AB, in addition to the double point at B and the simple intersection at A. Thus this pair of associated points on the image of b lie on a line, BA, through the double point B. In an exactly similar way all directions at B correspond to points on the conic whose plane contains l and the double line b.

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* *Sistemi Triplimente Infiniti*, Collectanea Mathematica, Hoepli, 1881.

It is a distinctive feature of this correspondence that any ray through the double point, A or B, of the quartic which is the image of a double line meets the curve in two associated points, that is, points which correspond to a single point of the surface. For the image of b and any line through A make up a quintic of the system, the corresponding section of the surface being a plane cubic whose plane contains b. This plane meets a in a point whose two images must lie on the line through A. Through any point of the surface there are two such plane cubics corresponding to rays through A and B in the plane. The points of a pair corresponding to a point of the double curve are such that the net of fundamental curves that pass through one pass also through the other. If two such points come together they are the image of a pinch point of the surface. Such points lie on the Jacobian of any net of fundamental curves. In our case, since we can draw six tangents to a uninodal quartic from its double point, there are six pinch points on each of the double lines. Again, from A, which is a simple point on the image of b, eight tangents can be drawn to that quartic. To these tangents correspond plane cubics on the surface which are tangent to b where it is tangent to the parabolic curve, and whose planes are stationary planes in the developable of the stationary tangent planes. There are of course eight such sections that contain a. To each of the twelve simple base points corresponds a line that meets both double lines. To a ray through A (or B) and one of these twelve points corresponds the conic whose plane contains b (or a) and the line corresponding to that base point. Thus there are twenty-six such conics, including the two whose planes pass through l. To the line joining any two simple base points, or to the conic through A and B and three simple base points, corresponds a skew cubic. There are thus $66+220$, or 286 of these. If two simple base points lie on a line through A the corresponding lines on the surface will meet on a, and their plane, which contains b, meets the surface in a residual line, whose image is the ray joining A to the base points. That is, two of the twenty-six conics just mentioned have become line pairs. Clebsch has pointed out that the ∞^3 hyperboloids which contain the double lines meet the surface in residual skew sextics whose images are the conics through A and B. The ∞^2 hyperboloids that contain also the line l intersect the surface in residual skew quintics that have for their images the line AB and the ∞^2 lines of the plane.

The genus of a curve of order n on the surface does not exceed the greatest integer in $(2n^2+2n+33)/20$. Let m be the order of the corresponding curve in the plane, and let it have points of multiplicity $\alpha_1, \alpha_2, \cdots, \alpha_{12}$ at the simple base points, and of multiplicity β_1 and β_2 at A and B. Then we have $5m-[\Sigma\alpha+2\Sigma\beta]=n$. Letting $\Sigma\alpha+2\Sigma\beta=k$, we have $m=(n+k)/5$. Then the genus of the plane curve, which is the same as that of the corresponding curve on the surface, is

$$\frac{(n+k-5)(n+k-10)}{50}-\frac{1}{2}\Big[\sum\alpha(\alpha-1)+\sum\beta(\beta-1)\Big].$$

Differentiating with respect to the twelve α's and two β's, we have fourteen equations from which it follows easily that for a maximum genus $k=4n+10$, the α's are all equal to $(n+3)/5$, and $\beta_1=\beta_2=(4n+7)/10$. For these values the expression for the genus becomes $(2n^2+2n+33)/20$.

The characteristic numbers of the tangent cone are obtained without difficulty by the method applicable to rational surfaces, or by the general formulas of Salmon and Plücker. For the sake of comparison with the forms of the surface to be treated later we may note that the order of the tangent cone is 16. Its class, or class of the surface, is 34. Its genus is 23; and it has 42 cuspidal edges, and 40 double edges.

In concluding this paper we may consider some special groupings of the base points. The general methods of producing nodes on unicursal surfaces are to allow two or more base points to coincide, or to place on a line a number of base points equal to the order of the fundamental curves. Thus, if two simple base points coincide we get an ordinary conic node. If three become consecutive, but not collinear, the result is a binode, B_3, that reduces the class by three, etc. In our case the fourteen fundamental points are the common intersections of a pencil of quartics that have a common tangent at two of them. Therefore, when we have chosen these two, say A and B, and the directions thereat, we have only nine at our disposal. The remaining three which are thus determined may be called residual points. We can not put five of the nine on a line, nor any of them on the line AB. But we can put four of them on a line, and we then get on the surface a fourteenth line, m, which meets l and the lines corresponding to the four base points, but which does not meet the double lines. To sections of the surface by planes through m correspond a pencil of quartics having double points at A and B, and passing through the five remaining points (of the nine) and the three residual points. An arbitrary quartic of the pencil meets the line containing the collinear base points four times, and thus the corresponding plane is quadruply tangent to the surface. But thirteen quartics of the pencil have a third node and their corresponding planes are quintuply tangent. If these ten points were arbitrary there would be but one quartic having double points at two of them and passing through the other eight. The section that contains m and l meets the double lines where l does. Its image in the plane, therefore, in addition to the line containing the four base points, is the line AB, and a cubic which has the common directions at A and B, and passes through the other eight points; thus satisfying twelve conditions. If we now put four of the five points still at our disposal on a line we get a fifteenth line, n, on the surface, which meets l and m and four others, but not the double lines. Clebsch remarked that no line can meet more than five of the thirteen transversals to the double lines. The residual section by the plane of l, m, n is a conic whose image in the plane is also a conic, which has the common directions at A and B and passes through the four other points, thus fulfilling eight conditions. Thus, if four points of such a set of fourteen are collinear, or if eight of them lie in sets of four on two lines, the remaining points are related in the manner just noted. If we arranged the nine points in a triangle, one at each vertex and two on each side, we should get three skew lines on the surface, each meeting l and four other lines, but not meeting the double lines.

THE UNIVERSITY OF CALIFORNIA

A NOTE ON THE RATIONAL PLANE QUARTIC CURVE WITH CUSPS OR UNDULATIONS*

BY J. H. NEELLEY

1. *Introduction.* In a recent paper† compound singularities of the rational plane quartic curve have been considered, but cusps and undulations were not incorporated in that article because of their widespread discussion by other writers. However, the errors or ambiguities in previous treatments of these two singularities are cleared up by this paper.

2. *A System of Cusp Invariants.* It is well known that all types of the rational plane quartic curve with simple singularities are given to within projection by plane sections of the Steiner Romische Surface S_3^4 of order three and class four. When referred to its tropes S_3^4 has the equation

$$(1) \qquad (x_0)^{1/2} \pm (x_1)^{1/2} \pm (x_2)^{1/2} \pm (x_3)^{1/2} = 0.$$

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*Presented to the Society, December 28, 1927.

†J. H. Neelley, *Compound singularities of the rational plane quartic curve*, American Journal of Mathematics, vol. 49 (1927), pp. 389-400.

If the intersection of the three double lines is taken as the unit point these lines have the equations

$$(2)\qquad \begin{cases} x_0 = x_1, & x_1 = x_2, \quad x_2 = x_0, \\ x_2 = x_3, & x_3 = x_0, \quad x_1 = x_3. \end{cases}$$

The ends of the double lines are pinch points of the surface with the coordinates

$$(1,1,0,0),\ (1,0,1,0),\ (1,0,0,1),\ (0,1,1,0),\ (0,1,0,1) \text{ and } (0,0,1,1),$$

and a plane on one of these points cuts out a curve with a cusp at the point. Let the coordinates of the cutting plane be $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ so if it is on one of the pinch points, say $(0, 1, 1, 0)$, we have $\alpha_1+\alpha_2=0$. As there are six pinch points there are six such forms each of which vanishes for some one of the pinch points. Therefore, one of the six factors of the product $\overset{6}{\Pi}(\alpha_i+\alpha_j)$ vanishes if the curve has a cusp. So this product may be used as an invariant which vanishes for a curve with a cusp. Letting σ_i represent the symmetric functions of the α's, this invariant may be written

$$(3)\qquad \sigma_1\sigma_2\sigma_3 - \sigma_3^2 - \sigma_1^2\sigma_4 = 0.^*$$

If we consider the curve with two cusps as cut out by a plane on two pinch points there are two forms of the type $\alpha_i+\alpha_j$ which vanish. So all possible products of five of the forms vanish and this gives as the two-cusp additional condition

$$\overset{6}{\sum}\left[\overset{5}{\Pi}(\alpha_i + \alpha_j)\right] = 0,$$

which in terms of the symmetric functions is

$$(4)\qquad 2\sigma_1\sigma_2^2 + \sigma_2\sigma_3 + \sigma_1^2\sigma_3 - \sigma_1\sigma_4 = 0.^*$$

But in so doing we neglect the fact that the plane cannot pass through both ends of one of the double lines. That is, a proper curve is cut out only if two forms such as $\alpha_0+\alpha_1$

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*This invariant was given by J. E. Rowe, Transactions of this Society, vol. 12 (1911), p. 306.

and $\alpha_2+\alpha_3$ do not both vanish. Imposing this condition on the α's the two-cusp invariant is found to be

$$\sigma_1\sigma_3 + \sigma_2^2 - 4\sigma_4 = 0. \tag{5}$$

Now let the plane $(\alpha x)=0$ be on three pinch points. Then three forms $\alpha_i+\alpha_j$ vanish simultaneously so that all products of four such forms vanish and we have

$$\sum^{15}\left[\prod^{4}(\alpha_i + \alpha_j)\right] = 0,$$

or

$$\sigma_1\sigma_3 + 32\sigma_4 + \sigma_2^3 + 2\sigma_1^3\sigma_2 = 0.^* \tag{6}$$

But again let us consider only proper curves and impose that no two forms such as $\alpha_i+\alpha_j$ and $\alpha_k+\alpha_l$ shall vanish simultaneously. This gives as the additional condition for a curve with three cusps the very simple invariant

$$\sigma_2 = 0. \tag{7}$$

3. *Degenerate Forms.* Now we consider the possible degenerate forms cut from this surface. Such are either a conic and the square of a line or the square of a conic or the squares of two lines. The conic and the square of a line is gotten if the cutting plane passes through only two pinch points and these must be the ends of one of the double lines of the surface. That is, if $\alpha_i+\alpha_j$ vanishes $\alpha_k+\alpha_l$ vanishes also, whence $\sigma_1=0$ for such. And this is the condition that the plane pass through the unit point which is necessary for degenerate forms including a line. As two forms $\alpha_i+\alpha_j$ vanish simultaneously the condition (4) of the previous section is satisfied. So for a conic and line we have

$$\sigma_1 = \sigma_3 = 0, \tag{8}$$

as $\sigma_2\neq 0$, since the plane does not pass through three pinch points.

The square of a conic is cut from the surface if the plane passes through three pinch points which lie in one of the

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[*] This invariant is given by J. E. Rowe, loc. cit.

reference planes. This means that condition (6) is satisfied and also $\sigma_2=\sigma_3=0$ due to condition (7) and the fact that the cutting plane is tangent to the surface. This gives $\sigma_4=0$ and so the square of a conic occurs if

$$(9) \qquad \sigma_2 = \sigma_3 = \sigma_4 = 0.$$

Now let the plane pass through two pinch points so that condition (5) is satisfied and also make $\sigma_1=0$ so that we have a pair of lines. This makes

$$(10) \qquad \sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0,$$

from consideration of relations (5) and (6).

So we have the following complete system:

A curve with one cusp has	$\sigma_1\sigma_2\sigma_3-\sigma_3^2-\sigma_1^2\sigma_4=0.$
Two cusps necessitates also that	$\sigma_1\sigma_3+\sigma_2^2-4\sigma_4=0.$
Three cusps adds to these	$\sigma_2=0.$
A conic and square of a line if	$\sigma_1=\sigma_3=0.$
The square of a conic if	$\sigma_2=\sigma_3=\sigma_4=0.$
The squares of two lines if	$\sigma_1=\sigma_2=\sigma_3=\sigma_4=0.$

4. *The Curve with Undulations.* There is another part of the literature of the rational plane quartic curve which should be revised. We refer to the theory of the curve with undulations as developed from consideration of the fundamental sextic which arises if the curve is represented parametrically. That is, if three forms

$$x_i = (\alpha_i t)^4, \qquad (i = 0,1,2),$$

represent the curve, the pencil of a polar quartics $(\beta t)^4+\lambda(\gamma t)^4$ fixes uniquely a sextic $(\rho t)^6$ which is apolar to the members of the pencil. The curve has been discussed by means of the invariants and covariants of this sextic.* Its line sections are the second polars of $(\rho t)^6$ and the pure second polars envelope the conic K† which is the locus of points in the plane from which a pencil of lines will cut out a pencil of

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*W. Stahl, *Über die rationale ebene Curve vierter Ordnung,* Journal fur Mathematik, vol. 104 (1889), pp. 302-320.

†W. Stahl, *Über die rationale ebene Curve vierter Ordnung,* Journal für Mathematik, vol. 101 (1886), pp. 300-325.

binary quartics having the invariant I_2 zero. This conic K breaks up into two lines when the quartic curve has an undulation; hence the statement of some writers that the sextic $(\rho t)^6$ is inadmissable when the curve has an undulation.* This statement is seemingly supported by the fact that the roots of $(\rho t)^6=0$ are the parameters of the points of the quartic curve where the lines of the curve and the conic K at its corresponding points are incident. But let us examine the curve with an undulation. Calling the parameter of the undulation $t=0$ and that of a point of inflection $t=\infty$, the join of these two points and the lines of the curve at them gives the following representation of the curve:

$$(11)\qquad \begin{cases} x_0 = a_0t^4, \\ x_1 = \qquad 4b_1t^3 + 6c_1t^2 + 4d_1t, \\ x_2 = \qquad\qquad\qquad\qquad 4d_2t + e_2, \end{cases}$$

whence the fundamental quartics are

$$(12)\quad (\beta t)^4 \equiv b_1t^2 + c_1t, \qquad (\gamma t)^4 \equiv b_1d_2t^4 + b_1e_2t^3 - d_1e_2t^2.$$

These quartics give

$$(13)\qquad (\rho t)^6 \equiv Kt^6$$

which is very definitely the sixth power of the undulation parameter. This sextic (13) permits the recovery of but one line section and that is the undulation line of the curve. But this line is also a line of the degenerate conic K whose equation is of the form

$$(14)\qquad x_0(K_0x_0 + K_1x_1 + K_2x_2) = 0.$$

So at $t=0$, the point fixed by the root of $(\rho t)^6=0$, we have the line of the curve incident with a line of the conic K even in this case. The unusual situation here is that the curve cannot be recovered from the sextic. This is generally possible as any three independent second polars of the sextic may be used as a reference scheme to recover the curve.

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* W. Stahl, loc. cit., vol. 104, p. 303 and p. 318; R. M. Winger, *Self-projective rational curves of the fourth and fifth orders*, American Journal of Mathematics, vol. 36 (1914), p. 64.

Consider a curve with undulations at $t=0$ and $t=\infty$. The join of the undulations and the undulation lines of the curve represent the curve as follows:

$$(15)\qquad \begin{cases} x_0 = a_0t^4, \\ x_1 = \qquad 4b_1t^3 + 6c_1t^2 + 4d_1t, \\ x_2 = \qquad\qquad\qquad\qquad e_2, \end{cases}$$

with

$$(16)\qquad (\beta t)^4 \equiv b_1t^2 + c_1t, \qquad (\gamma t)^4 \equiv b_1t^3 - d_1t,$$

which give

$$(17)\qquad (\rho t)^6 \equiv \lambda t^6 + \mu\tau^4,$$

or a pencil of sextics which includes the sixth powers of each undulation parameter. All second polars of (17) are lines of the pencil determined by the undulation lines of the curve and once more the curve cannot be recovered from the sextic as there are not three independent second polars. The conic K for this type curve has the equation

$$(18)\qquad cx_0x_2 = 0,$$

which is the product of the undulation lines, the two possible pure second polars of (17).

The curve with three undulations is well known but to make this article complete and to remove at least one erroneous statement concerning it we will give its theory. Let $t=0$, ∞ and c_1/b_1, where neither c_1 nor b_1 are zero, be the undulation parameters and use the undulation lines as a triangle of reference. Then the curve is given by

$$(19)\qquad \begin{cases} x_0 = a_0t^4, \\ x_1 = (b_1t - c_1)^4, \\ x_2 = e_2\,, \end{cases}$$

with

$$(20)\qquad (\beta t)^4 \equiv b_1^2t^3 - c_1^2t, \qquad (\gamma t)^4 \equiv b_1t^2 - c_1t$$

which have the net of apolar sextics

$$\lambda t^6 + \mu(b_1 t - c_1 \tau)^6 + \nu\tau^6. \tag{21}$$

This net includes the sixth powers of the three undulation parameters as members. So the curve may be recovered from any member of the net (21) which is a linear combination of each of these sixth powers. This is because there is a net of second polars of such a member and any three independent ones may be used to recover the curve. The equation of the conic K vanishes identically if the curve has three undulations. This is obviously necessary from the definition of K. That is, the pencil of lines determined by any two lines $(\lambda x)=0$, $(\mu x)=0$ cuts a pencil of quartics from the curve (19) for which the invariant I_2 vanishes. So the equation of K must be satisfied by any point $(x_0 x_1 x_2)$ of the plane. We state the results of this section as the following theorems.

THEOREM 1. *A curve with one undulation has as its fundamental sextic the sixth power of the undulation parameter and the curve cannot be recovered from the sextic.*

THEOREM 2. *A curve with two undulations has a pencil of fundamental sextics which includes the sixth powers of the undulation parameters and again the curve cannot be recovered from the sextics.*

THEOREM 3. *A curve with three undulations has a net of fundamental sextics which includes the sixth powers of the undulation parameters and the curve may be recovered from any member of the net which is a linear combination of all three sixth powers.*

CARNEGIE INSTITUTE OF TECHNOLOGY

THE BITANGENTS AT A POINT OF A DESMIC SURFACE

BY R. M. MATHEWS

A plane tangent to a surface of the fourth order cuts the surface in a curve of the fourth order with a singular point at the point of contact. Through this point there can then be drawn six lines tangent to the curve elsewhere. So through a point on a surface of the fourth order there are in general six bitangents.

The second points of contact of these bitangents lie on a conic which, for a desmic surface, degenerates into two lines. Thus we have two sets of three bitangents; numerous geometrical properties of one triple have been given by G. Humbert and others.* Concerning the second triple only some general properties that apply to both sets are known; no characterizing geometric properties have been published. This paper presents one.

For brevity in the equations we shall use the symbol $\overline{ij}$ to denote $(x_i^2 - x_j^2)$.

The pencil of desmic surfaces in variables (y) and parameter (x),

$$(1) \quad \begin{vmatrix} (y_0^2 - y_1^2)(y_2^2 - y_3^2), & (y_0^2 - y_2^2)(y_1^2 - y_3^2) \\ \overline{01}\cdot\overline{23} \quad , & \overline{02}\cdot\overline{13} \end{vmatrix} = 0,$$

determines a cubic complex, and the cubic cone of this complex of which an arbitrary point $P(x)$ is the vertex, cuts the plane $y_3 = 0$ in the cubic curve

$$(2) \quad \overline{03}y_1y_2(x_2y_1 - x_1y_2) + \overline{13}y_2y_0(x_0y_2 - x_2y_0) + \overline{23}y_0y_1(x_1y_0 - x_0y_1) = 0.$$

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* Humbert, G., *Sur la surface desmique du quatrième ordre,* Journal de Mathématiques, (4), vol. 7 (1891), pp. 353–398

Jessop, C. M., *Quartic Surfaces with Singular Points,* Cambridge, 1916.

Mathews, R. M., *Cubic curves and desmic surfaces,* Transactions of this Society, vol. 28 (1926), pp. 502–522, and ibid., vol. 30 (1928), pp. 19–23.

The tangent plane at $P(x)$ to the desmic surface through P is

$$(3)\quad \overline{13}\cdot\overline{23}\cdot\overline{12}x_0y_0+\overline{23}\cdot\overline{03}\cdot\overline{20}x_1y_1 \\ +\overline{03}\cdot\overline{13}\cdot\overline{01}x_2y_2+\overline{01}\cdot\overline{12}\cdot\overline{20}x_3y_3=0,$$

and cuts $y_3=0$ in a line ABC with

$$(4)\quad \begin{cases} A:\ \overline{03}x_1x_2:\overline{13}x_2x_0:\overline{23}x_0x_1, \\ B:\ \overline{03}(x_1x_2+x_0x_3):\overline{13}(x_0x_2+x_1x_3):\overline{23}(x_0x_1+x_2x_3), \\ C:\ \overline{03}(x_1x_2-x_0x_3):\overline{13}(x_0x_2-x_1x_3):\overline{23}(x_0x_1-x_2x_3). \end{cases}$$

The points A, B, C are on the cubic curve (2) and the lines PA, PB, PC are the known bitangents.*

The cubic curve (2) has for Hessian curve the cubic

$$(5)\quad \begin{vmatrix} \overline{23}x_1y_1-\overline{13}x_2y_2, & \overline{23}(x_1y_0-x_0y_1), & \overline{13}(x_0y_2-x_2y_0) \\ \overline{23}(x_1y_0-x_0y_1), & \overline{03}x_2y_2-\overline{23}x_0y_0, & \overline{03}(x_2y_1-x_1y_2) \\ \overline{13}(x_0y_2-x_2y_0), & \overline{03}(x_2y_1-x_1y_2), & \overline{13}x_0y_0-\overline{03}x_1y_1 \end{vmatrix}=0.$$

To write the coordinates of an arbitrary point Q on the line ABC, it is convenient to find $\overline{A}$ as the harmonic conjugate of A with respect to B and C, namely

$$(6)\qquad \overline{A}:\overline{03}x_0x_3:\overline{13}x_1x_3:\overline{23}x_2x_3$$

and then to write the coordinates for

$$Q\equiv\rho A+\sigma\overline{A}.$$

On solving $A\overline{A}$ simultaneously with the Hessian (5), we find that the values of the ρ/σ ratio for the points of intersection are roots of the cubic

$$(7)\quad 2\rho^3x_0x_1x_2x_3+\rho^2\sigma\sum x_0^2x_1^2+6\rho\sigma^2x_0x_1x_2x_3 \\ +\sigma^3\left(\sum x_0^2x_1^2-\sum x_0^4\right)=0.$$

Next, the line PQ is an arbitrary line through P on the tangent plane at P. We write the coordinates of an arbitrary point R on PQ in the form

$$R\equiv\lambda Q+\mu P$$

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* For these equations and results see Mathews, loc. cit.

and substitute in the equation of the desmic surface. The result is a fourth degree equation in the ratio λ/μ whose four roots give the intersections R of PQ with the surface. Since P is a double point on the intersection of the plane and surface this quartic equation reduces to the quadratic

$$(8)\quad \left\{\begin{aligned} &\lambda^2\{(z_0^2 - z_1^2)z_2^2 n - (z_0^2 - z_2^2)z_1^2 m)\} \\ &\quad + 2\lambda\mu\{[(z_0^2 - z_1^2)z_2x_2 + z_2^2(x_0z_0 - x_1z_1)]n \\ &\quad - [(z_0^2 - z_2^2)z_1x_1 + z_1^2(x_0z_0 - x_2z_2)m]\} \\ &\quad + \mu^2\{[(z_0^2 - z_1^2)\overline{23} + z_2^2\overline{01} + 4(x_0z_0 - x_1z_1)x_2z_2]n \\ &\quad - [(z_0^2 - z_2^2)\overline{13} + z_1^2\overline{02} \\ &\quad + 4(x_0z_0 - x_2z_2)x_1z_1]m\} = 0, \end{aligned}\right.$$

where $m=k\ \overline{02}\cdot\overline{13}$, $n=k\ \overline{01}\cdot\overline{23}$, ($k$ a factor of proportionality), and where the z's are the coordinates of Q.

Now, when PQ is a bitangent to the surface the two roots of this quadratic are equal. We set the discriminant equal to zero, substitute for the z's the coordinates of Q as

$$Q = \rho A + \sigma\bar{A}.$$

and obtain an equation of the sixth degree in the ratio ρ/σ. Three of the roots are $\sigma=0$ and $\rho/\sigma=\pm 1$ giving A, B, C, the points for the known bitangents, and leaving for the determination of the other three exactly equation (7) which is the condition that Q lie at the intersections of the line ABC with the Hessian cubic.

The cubic cone at P cuts an arbitrary plane, not through P, in a cubic curve which has a definite Hessian; when the points of the latter are joined to P we get a cubic cone which may be called the Hessian cone corresponding to the first. Thus we have the following theorem.

THEOREM. *The bitangents to a desmic surface at a point thereon lie in two triples; the lines of one triple lie on the cubic cone of the system proper to P, and of the other on the corresponding Hessian cone.*

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ON CONNECTED AND REGULAR POINT SETS*

BY R. L. WILDER

In a recent number of this Bulletin† G. T. Whyburn showed that if A and B are any two points of a connected and regular point set M, and K denotes the set of all points of M which separate‡ A and B in M, then $K+A+B$ is a closed and bounded point set. In the present paper I shall show that this theorem is susceptible of quite simple proof, and admits of two obvious generalizations which hold in space of n dimensions. The methods used in proving these generalizations are also employed to show that if N is a closed and bounded point set which lies in a connected subset of an open subset, F, of a connected and regular point set, M, then F contains a bounded, connected and regular set which contains N. These results are then applied to give certain theorems concerning continuous curves.

Use will be made of the notions *region of M* and *simple chain of regions* as introduced in my paper, *The non-existence of a certain type of regular point set*,§ as well as of Theorem 1 of that paper; these extend readily to n-dimensional space, if in the definition of *region of M* "circle" is replaced by "sphere." Furthermore, since the Borel property is employed I make note of the following lemma, the proof of which should be quite obvious.||

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* Presented to the Society, April 7, 1928.

† G. T. Whyburn, *Concerning connected and regular point sets*, this Bulletin, vol. 33 (1927), pp. 685–689.

‡ If A, B and X are points of a connected set M, then X is said to separate A and B in M if $M-X$ is the sum of two mutually separated point sets which contain A and B respectively.

§ This Bulletin, vol. 33 (1927), pp. 439–446. This paper will be referred to hereafter as N. E. R.

|| It will be understood hereafter without explicit statement that the results hold for sets imbedded in euclidean space of n dimensions.

LEMMA. *If M is a connected and regular point set, and N is a closed and bounded subset of M, and G is a collection of regions of M which cover N, then G contains a finite subset which covers N.*

(By virtue of the property of regularity, there exists, concentric with the sphere used in determining a region R of M, a sphere S such that *all* points of M interior to S are also points of R. When it is said, then, that a collection of regions, G, *covers* a subset N of M, it is meant that every point of N is within the sphere S which corresponds to some region of the collection G.)

DEFINITION. If N is a subset of a connected set M and P is a point of $M-N$, then P is said to *separate N in M* if $M-P$ is the sum of two mutually separated sets each of which contains at least one point of N.

THEOREM 1. *If N is a closed and bounded subset of a connected and regular point set M and K denotes the set of all points which separate N in M, then $K+N$ is a closed and bounded point set.*

PROOF. Suppose the set $K+N$ is not closed. Then there is a point P which is a limit point of this set and which does not belong to it. As N is closed, P is a limit point of K, but not of N. If x is any point of M, distinct from P in case P is a point of M, there exists a region of M which covers x and which neither contains P nor has P as a limit point. Let G denote the collection of all such regions. By the above lemma, G contains a finite set of regions, R_1, $R_2, \cdots, R_n$, such that every point of N is in at least one of these regions. For every i ($i=1, 2, \cdots, n$), let P_i be a point of N in R_i. For every two points P_i and P_{i+1} ($i=1, 2, \cdots, n-1$) there exists, by Theorem 1 of N. E. R., and by virtue of the fact that P does not separate these points in M,[*] a simple chain of regions of G from P_i to P_{i+1}. The regions $R_1, R_2, \ldots, R_n$ together with the regions which go to make up these simple chains form a finite set, g, of regions of G, and if H denotes the set of all points of M contained in regions of g, then H is a connected and bounded set. As P is not a limit point of H there exist in the vicinity of P points of K which are not in H. But clearly such points cannot separate N in M since N is a subset of H. Thus the supposition that $K+N$ is not closed leads to a contradiction.

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[*] Use is made in this connection and in the proof of Theorem 2 below, of Lemma 2 of my paper *A characterization of continuous curves by a property of their open subsets*, Fundamenta Mathematicae, vol. 11(1928), pp. 127–131.

To show that $K+N$ is bounded, let G be any collection of regions of M covering N and proceed as above to establish the existence of a bounded connected set, H, which is a subset of M and contains N. As every point of K must lie in H, it is clear that $K+N$ is bounded.

DEFINITION. If A and B are any two distinct subsets of a connected set M, and X is a point of $M-(A+B)$, then X is said to *separate A from B in M* if $M-X$ is the sum of two mutually separated sets which contain A and B, respectively.

THEOREM 2. *If A and B are any two distinct subsets of a connected and regular point set M, such that $A+B$ is closed and bounded, and K denotes the set of all points which separate A from B in M, then $K+A+B$ is a closed and bounded set.*

INDICATION OF PROOF. Select P and G as in Theorem 1. As P does not separate A from B in M, there is a connected subset of $M-P$ which contains a point P_1 of A and a point P_2 of B. Hence from G can be selected a simple chain from P_1 to P_2.

The corollarys of Theorems 1 and 2 applied to continuous curves* are obvious.

As a direct consequence of Theorems 1 and 2 and of the fact that the difference of two closed sets is both an F_σ and a G_δ,† we have the following theorem.

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* In this paper a continuous curve is considered as a closed, connected and regular point set not necessarily bounded.

† See F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig, 1914, p. 306. An F_σ is the sum of a denumerable set of closed sets and a G_δ is the set of points common to a denumerable set of open sets.

Theorem 3. *Let M be a connected and regular point set. Then if N is any closed and bounded subset of M, the set of all points which separate N in M is both an F_σ and a G_δ: and if A and B are any two distinct subsets of M such that $A+B$ is closed and bounded, the set of all points which separate A from B in M is both an F_σ and a G_δ.*[*]

By methods similar to those used in proving Theorem 1 and noting in addition the fact that a region in a connected and regular point set is itself a regular set (Theorem 2 of N. E. R.), we have the following result.

Theorem 4. *If K is a closed subset of a connected and regular point set M and N is a closed and bounded subset of a connected subset of $M-K$, then N lies in a bounded, connected and regular subset of $M-K$ which has no limit point in K.*

The analog of Theorem 4 for continuous curves may be stated as follows.

Theorem 5. *If N is a closed and bounded subset of an open subset, Q, of a continuous curve M, and N lies in some connected subset of Q, then N lies in a bounded continuous curve which is a subset of Q.*

Proof. Hahn has shown[†] that if P is any point of M and r is any positive number, there exists a continuous curve $M(P, r)$ which is a subset of M, contains every point of M less than a certain distance d (dependent on r) from P, and is such that all of its points are at a distance less than r from P. The set of all points $\{x\}$ of M such that x is joined to P by a connected subset of M every point of which is at a distance less than d from P constitutes a region of M and will be denoted by $M'(P, r)$. Clearly $M'(P, r)$ is a subset of $M(P, r)$.

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[*] In this connection it may be of interest to note that it has been shown that the set of all cut points of a continuous curve is an F_σ. See C. Zarankiewicz, *Sur les points de division dans les ensembles connexes*, Fundamenta Mathematicae, vol. 9 (1927), pp. 124–171, Theorem 17.

[†] H. Hahn, *Mengentheoretische Charakterisierung der stetigen Kurve*, Wiener Akademie Sitzungsberichte, vol. 123, Part IIa, pp. 2433–2489; see Theorem XXI, p. 2475. Although Hahn states his result for a bounded regular continuum, it is clear that it holds for any regular continuum.

If P is any point of Q, let r be a positive number less than the distance* from P to $M-Q$. Then the sets $M(P, r)$ and $M'(P, r)$ are subsets of Q. By a method similar to that used in proving Theorem 1 it can be shown that there exists a connected set H' which consists of a finite number of the regions of type $M'(P, r)$ and contains N. The set H composed of the sets $M(P, r)$ associated with those sets $M'(P, r)$ which constitute H' is a bounded continuous curve lying in Q and containing all points of N.

Theorem 5 is a generalization of a result obtained by R. L. Moore† to the effect that if Q is an open subset of a continuous curve and A and B are two points which lie in a connected subset of Q, then A and B are joined by a simple continuous arc which lies wholly in Q. As I have shown elsewhere‡ that this property is sufficient that a continuum be a continuous curve, it follows that the property stated in Theorem 5 also serves to characterize a continuous curve.

As I have indicated in a recent paper,§ subsets of a point set M may be separated in M in different senses. For our present purposes we employ the following definitions.

DEFINITION. If N is a subset of a connected set M and P is a point of $M-N$, then P is said to *separate N in M in the weak sense* if there exist two points of N which do not lie in a connected subset of $M-P$.

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* That is, the greatest lower bound of all distances Px, where x is a point of $M-Q$.

† *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), pp. 254–260, Theorem 1.

‡ *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), pp. 340–377, Theorem 18.

§ *A characterization of continuous curves by a property of their open subsets*, Fundamenta Mathematicae, vol. 11 (1928), pp. 127–131. The terminology "*separates* in the strong (or weak) sense" should not be confused with "*disconnects* in the strong (or weak) sense" as introduced by R. L. Moore in *Concerning the cut-points of continuous curves and of other closed and connected point-sets*, Proceedings of the National Academy of Sciences, vol. 9 (1923), pp. 101–106.

DEFINITION. If A and B are any two distinct subsets of a connected set M and X is a point of $M-(A+B)$, then X is said to *separate A from B in M in the weak sense* if $M-X$ contains no connected subset which contains points of both A and B.

As examples of cases where these two definitions of "separate" are satisfied, but where the preceding definitions* are not satisfied, consider the following examples.

Examples. Let M_n ($n=1, 2, 3, \cdots$) denote the straight line interval joining the points (0, 0) and (1, $1/n$). Also, denote the points (1/2, 0), (1, 0) and (0, 0) by A, B, and X, respectively, and let $M=A+B+\sum_{n=1}^{\infty}M_n$.

The set M is connected, but the set $M-X$ contains no connected set containing A and B. Hence X separates A from B in the weak sense. However, there is no separation of $M-X$ into two mutually separated subsets containing A and B, respectively, and thus X does not separate A from B in M in the strong sense, that is, in the sense of the definition which immediately precedes Theorem 2.

If we let N denote the set of all points with rational coordinates in the interval [0, 1] of the X axis, except (0, 0), and define M_n and X as above, and let M now denote the set $N+\sum_{n=1}^{\infty}M_n$, it is easy to see that X separates N in M in the weak sense, but not in the strong sense, i.e., in the sense of the definition preceding Theorem 1.

On the basis of Lemma 2 of my paper *A characterization of continuous curves by a property of their open subsets*† we have the following extension of Theorems 1–5.

THEOREM 6. (1) *Theorems* 1, 2 *and* 3 *still hold true if "separate" be interpreted to mean "separate in the weak sense"*; (2) *Theorem* 4 *still holds true if the words "N is a closed and bounded subset of a connected subset of $M-K$" be replaced by "N is a closed and bounded subset of $M-K$ such that there is no separation of $M-K$ into two mutually separated sets each*

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* That is, the definitions preceding Theorems 1 and 2, respectively.

† Loc. cit.

of which contains points of N;" and (3). *Theorem* 5 *still holds true if the words "N lies in some connected subset of Q" be replaced by "there is no separation of Q into two mutually separated sets each of which contains points of N."*

In conclusion we may note the following application of part (1) of Theorem 6 to the theory of irreducibly connected sets.*

THEOREM 7. *If the connected and regular point set M is irreducibly connected about a closed and bounded set N, then M is a bounded continuous curve.*

PROOF. Let P be any point of $M-N$. Then $M-P$ contains no connected subset which contains N, since M is irreducibly connected about N. That is, $M-N$ is the set of points of M which separate N in M in the weak sense, and accordingly by Theorem 6, part (1), the set $(M-N)+N=M$ is bounded and closed. Hence M is a bounded continuous curve.

It may be pointed out that Whyburn's Theorem 2† to the effect that if a connected and regular point set M is irreducibly connected between two of its points A and B, then M is a simple continuous arc from A to B, is a corollary of Theorem 7 above. For since by Theorem 7 such a set, M, is a continuous curve, A and B are the end points of a simple continuous arc, t, of M. It is clear, then, that $M \equiv t$.

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* A connected set M is said to be *irreducibly connected* about one of its subsets, N, if it has no proper connected subset which contains N. See H. M. Gehman, *Concerning irreducibly connected sets and irreducible continua*, Proceedings of the National Academy of Sciences, vol. 12 (1926), pp. 544–547.

† Loc. cit. I might say here that in establishing the first of those results concerning simple closed curves to which Professor Whyburn kindly calls attention in this connection, I found it necessary to prove as a lemma the definition of arc stated in his Theorem 2. I did not mention this in my abstract (this Bulletin, vol. 32 (1926), p. 123, abstract No. 15) and have not yet published the paper. However, the proof which I developed in that connection is quite different from that given by Professor Whyburn as well as from the proof indicated in the present paper.

A PROPERTY OF THE LEVEL LINES OF A REGION WITH A RECTIFIABLE BOUNDARY*

BY G. A. PFEIFFER

1. *Introduction.* Before stating the result of this paper let me recall that a *level line of a region*† in a plane is the locus of the equation $g(x, y; a, b)=c$, where $g(x, y; a, b)$ is the Green's function of the region which has the point (a, b) as its pole, and c is a positive constant. The set of all level lines of the region, with the point (a, b) fixed and c any positive constant, is called a *pencil of level lines* of the region, and the fixed point (a, b) is called the *pole* of that pencil. Any pencil Σ of level lines of a region which is in a plane and which has a connected boundary containing more than one point, is the image of the set of circles concentric with and interior to any circle K under any transformation Π which maps in a one-to-one and conformal way the interior of K on Σ, such that the pole of the pencil of level lines corresponds to the center of K; and, conversely, the image of the set of circles concentric with and interior to a circle K under any transformation Π which maps in a one-to-one and conformal way the interior of K on a planar region Σ is a pencil of level lines of Σ, which has the image under Π of the center of K as its pole. Thus, a pencil of level lines of Σ is a one-parameter set of simple closed curves, and the value of the parameter t of a level line G of such a pencil of level lines may be taken as the length of the radius of the circle to which G corresponds under Π. Taking K as the unit circle, then t varies between 0 and 1. The symbol $[G_t]$ denotes a pencil of level lines of the region

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* Presented to the Society, May 7, 1927, as part of a paper of the title *Certain sequences of curves which approach a rectifiable boundary from within.*

† A region in a plane is a set of points in a plane such that there exists a planar neighborhood of each point of the set which contains only points of the set.

Σ such that the level line G_t corresponds under Π to the circle which is concentric with the unit circle and which has a radius of length t $(0<t<1)$.

Now, if $\Lambda(t)$ denotes the function of t defined for $0<t<1$ and such that $\Lambda(t_1)$ is the length of the level line G_{t_1} of the pencil of level lines $[G_t]$, $0<t<1$, then it is a result of a theorem of Hardy, referred to below, that $\Lambda(t)$ is an increasing continuous function of t for $0<t<1$; that is, if $0<t_1<t_2<1$, then $\Lambda(t_1)<\Lambda(t_2)$. It is a simple consequence of the formula for the length of an analytic transform of a rectifiable curve, which is derived in §2 below, that $\lim_{t\to 0}\Lambda(t)=0$ and nothing further is said about that; but much of the following proof is devoted to showing that if the boundary of Σ is a rectifiable simple closed curve, then $\lim_{t\to 1}\Lambda(t)$ is the length of the boundary of Σ.

The result of this paper is contained in the following two theorems. The theorems are closely connected and their proofs are combined in the demonstration which follows.

THEOREM. *The function* $\Lambda(t)$ *of* t, *which is defined in the interval* $0\leqq t\leqq 1$, *and which is such that* $\Lambda(t_1)$, *if* $0<t_1<1$, *is the length of the level line* G_{t_1} *of the pencil* $[G_t]$, $0<t<1$, *of level lines of the planar region* Σ *whose boundary is a rectifiable simple closed curve and such that* $\Lambda(0)=0$ *and* $\Lambda(1)=$ *the length of the boundary of* Σ, *is an increasing* continuous function of* t *in the (closed) interval* $0\leqq t\leqq 1$.

DEFINITION. An *approximating sequence of regions* of the region Σ is a sequence of regions $\{\Sigma_n\}$, $n=1, 2, 3, \cdots$, such that every limit point of each Σ_n is a point of Σ and every point of Σ belongs to all but a finite number of the regions Σ_n.†

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* That is, if $0\leqq t_1<t_2\leqq 1$, then $\Lambda(t_1)<\Lambda(t_2)$.

† It follows readily from this definition that if the region Σ is bounded then an approximating sequence of regions $\{\Sigma_n\}$, $n=1, 2, 3, \cdots$, of the region Σ contains a subsequence of regions $\{\Sigma_{n_i}\}$, $i=1, 2, 3, \cdots$, which is such that (a) every limit point of any region Σ_{n_i} belongs to Σ and to the succeeding region $\Sigma_{n_{i+1}}$ and (b) every point of Σ is in all but a finite number of the regions Σ_{n_i}.

DEFINITION. An *approximating sequence of curves* of the region Σ is a sequence of curves $\{C_n\}$, $n=1, 2, 3, \cdots$, such that each curve C_n is the boundary of a region Σ_n of an approximating sequence of regions $\{\Sigma_n\}$, $n=1, 2, 3, \cdots$, of the region Σ.

THEOREM. *If the boundary of a planar region Σ is a simple closed curve which is rectifiable, then there exist approximating sequences of curves $\{C_n\}$, $n=1, 2, 3, \cdots$, of the region Σ such that the curves C_n are level lines of any given pencil of level lines of Σ and, if l_n denotes the length of the curve C_n, $l_n < l_{n+1}$ and* $\lim_{n\to\infty} l_n$ *is the length of the boundary of Σ.*

2. *The Length of an Analytic Transform of a Rectifiable Curve.* Let the function $w=f(z)$ be analytic in the interior, $i(K)$, of the unit circle, K, and map in a one-to-one way $i(K)$ on the region Σ of the theorem. Let C be a rectifiable curve in $i(K)$ and C' its image under the transformation $w=f(z)$. Then the length, l', of C' is

$$\lim_{n\to\infty} (|\Delta w_{n_1}| + |\Delta w_{n_2}| + \cdots + |\Delta w_{n_r}|),$$

where $\Delta w_{n_i}=f(z_{n_i})-f(z_{n_{i-1}})$, where $z_{n_0}, z_{n_1}, z_{n_2}, \cdots, z_{n_r}, z_{n_{r+1}}=z_0$ are points on C such that $|z_{n_i}-z_{n_{i-1}}|<\delta_n>0$, and where $\lim_{n\to\infty} \delta_n=0$.

If Δs_{n_i} is the length of the arc of C whose end points are z_{n_i} and $z_{n_{i-1}}$ and which does not contain as an inner point the point $z=z_{n_0}$, then

$$l' = \lim_{n\to\infty} \sum_{i=1}^{r} \left|\frac{\Delta w_{n_i}}{\Delta z_{n_i}}\right| \Delta s_{n_i}, \qquad \Delta z_{n_i} = z_{n_i} - z_{n_{i-1}};$$

and, because of the uniformity of the approach of $\Delta w/\Delta z$ to dw/dz along C, dw/dz being continuous on C, it follows that

$$l' = \lim_{n\to\infty} \sum_{i=1}^{r} \left(\lim_{z\to z_{n_i}} \left|\frac{\Delta w}{\Delta z}\right| \right) \Delta s_{n_i}.$$

Hence

$$l' = \int_C \left|\frac{dw}{dz}\right| ds.$$

3. *The Level Lines of a Polygonal Region.* Let $i(J)$ denote a region whose boundary is a simple polygon J, and let $w=f(z)$ be a function which is analytic in the interior $i(K)$ of the unit circle K and which maps in a one-to-one way $i(K)$ on $i(J)$. Then $w=f(z)$ is analytic at any point of the circle K whose image is not a vertex of J and if $w=f(a)$ is a vertex of J, then at any point of $i(K)$ different from $z=a$ in some neighborhood of $z=a$ the derivative of $w=f(z)$ is $(z-a)^{\mu}\,\lambda(z)$, where $\lambda(z)$ is analytic at $z=a$ and $-1<\mu<1$.* In fact, $\mu=\alpha/\pi-1$, where α is the measure in radians of the interior angle of the polygon J whose vertex is the point $w=f(a)$.

Let the point $w=f(a_i)$ be a vertex of the polygon J and U_{a_i} a neighborhood of $z=a_i$ such that at every point of $i(K)$ which is in U_{a_i} and different from $z=a_i$, $f'(z)=(z-a_i)^{\mu_i}\,\lambda_i(z)$, where $\lambda_i(z)$ is analytic at $z=a_i$ and $-1<\mu_i<1$. Further, let Γ_i denote a circular arc concentric with K, and contained in U_{a_i}, such that its mid-point $z=b_i$ is on the radius of K through $z=a_i$. Let $z=c_i$ be an end point of this arc. Then, if $-1<\mu_i<0$ and M_i is a bound of $|\lambda_i(z)|$ in U_{a_i},

$$\int_{\Gamma} |f'(z)|\,ds \leqq M_i \int_{\Gamma_i} (|z-a_i|)^{\mu_i} ds \leqq M_i \int_{\Gamma_i} (|z-b_i|)^{\mu_i} ds.$$

If η is an arbitrary positive number, then there exists a positive number ξ such that $|z-b_i|/(\widehat{zb_i}) \geqq 1-\eta$ if $\widehat{zb_i}<\xi$, where $\widehat{zb_i}$ denotes the length of the sub-arc of Γ_i whose end points are $z=z$ and $z=b_i$. Then

$$\int_{\Gamma_i}(|z-b_i|)^{\mu_i} ds \leqq (1-\eta)^{\mu_i} \int_{\Gamma_i} (\widehat{zb_i})^{\mu_i} ds$$

$$= 2(1-\eta)^{\mu_i} \int_0^{l_i/2} s^{\mu_i} ds \doteq 2(1-\eta)^{\mu_i} \frac{1}{\mu_i+1} \cdot \frac{l^{\mu_i+1}}{2^{\mu_i+1}},$$

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* Bieberbach, *Lehrbuch der Funktionentheorie*, vol. II, p. 34 and p. 37. Also Study, *Vorlesungen über ausgewählte Gegenstande der Geometrie*, Part II, p. 85.

where l_ι is the length of Γ_ι and if $\eta < 1/2$, then

$$\int_{\Gamma_\iota} (\,|\,z - b_\iota\,|\,)^{\mu_\iota} ds < \frac{1}{2^{2\mu_\iota}(\mu_\iota + 1)} l_\iota^{\mu_\iota+1}.$$

Again, if $0 \leqq \mu_\iota < 1$ and M_ι is a bound of $|\lambda_\iota(z)|$ in U_{a_ι}, we have

$$\int_{\Gamma_\iota} |\,f'(z)\,|\, ds \leqq M_\iota \int_{\Gamma_\iota} (\,|\,z - a_\iota\,|\,)^{\mu_\iota} ds \leqq M_\iota \int_{\Gamma_\iota} |\,c_\iota - a_\iota\,|\, ds$$

$$\leqq M_i\left(\,|\,a_\iota - b_\iota\,|\, + \frac{l_\iota}{2}\right) l_\iota,$$

where l_ι is the length of the arc Γ_ι. Hence if ϵ is any positive number there exists a positive number δ_ι such that if Γ_ι is any circular arc which is concentric with and either interior to or on the given circle K and contained in U_{a_ι} and which has a length $l_\iota < \delta_\iota$, then

$$\int_{\Gamma_\iota} |\,f'(z)\,|\, ds < \epsilon.$$

Now, let σ_ι denote an arc on the circle K which has $z = a_\iota$ as its mid-point and a length l_ι which is less than δ_ι and, further, such that no two arcs σ_ι have a point in common and let R denote the set of all points which are interior to K and which do not belong to any sector bounded by the arc σ_ι and the radii of K through its end points; also let $\overline{R}$ denote the set of points consisting of the points of R and of the boundary of R. Then $w = f'(z)$ is analytic in $\overline{R}$ and hence there exists a positive number δ such that $|f'(z_1) - f'(z_2)| < \epsilon$ if $z = z_1$ and $z = z_2$ are any two points in $\overline{R}$ such that $|z_1 - z_2| < \delta$. Now, if d is a positive number less than the radius of each U_{a_ι}, let K' denote a circle interior to and concentric with K and having a radius which differs from that of K by less than d and also less than δ. Then let τ_ι denote any arc of K which contains no arc σ_ι and whose end points are also end points of arcs σ_ι and let σ_ι' and τ_ι' denote the arcs of K' which are composed of the points of intersection of the circle K'

and the radii of K through all the points of σ_i and all the points of τ_i respectively. It follows that

$$\int_{\tau_i'} | f'(z) | \, ds = \int_{\tau_i} \frac{r-d}{r} (| f'(z) | + \eta(z)) ds,$$

where r is the radius of K and $|\eta(z)| < \epsilon$. If h denotes the length of the polygon J and h' the length of the transform of K' under the transformation $w = f(z)$ and n the number of vertices of J, then

$$\begin{aligned} h' &= \sum_{i=1}^{n} \int_{\sigma_i'} | f'(z) | \, ds + \sum_{i=1}^{n} \int_{\tau_i'} | f'(z) | \, ds \\ &< n\epsilon + \sum_{i=1}^{n} \int_{\tau_i} | f'(z) | \, ds + \frac{r-d}{r} \epsilon h, \end{aligned}$$

and

$$h' > \sum_{i=1}^{n} \int_{\tau_i} | f'(z) | \, ds - \frac{d}{r} h - \frac{r-d}{r} \epsilon h.$$

Since

$$h - n\epsilon < \sum_{i=1}^{n} \int_{\tau_i} |f'(z) | \, ds < h,$$

it follows that

$$h - n\epsilon - \frac{d}{r} h - \frac{r-d}{r} \epsilon h < h' < n\epsilon + h + \frac{r-d}{r} \epsilon h.$$

Now if $d < \epsilon$, then

$$h - \left(n + \frac{h}{r} + h \right) \epsilon < h' < h + (n + h)\epsilon,$$

or

$$| h - \; | h' < \left(n + \frac{h}{r} + h \right) \epsilon.$$

Hence if ρ is any positive number and d is sufficiently small then $|h - h'| < \rho$.

From this result follows immediately, as far as it concerns the $\lim_{n\to\infty} l_n$, the special case of the second theorem of the paper in which the region Σ is the interior of a simple polygon in a plane.

4. *The Region Σ in General.* Let Σ be a region in the w-plane and let the boundary of Σ be a rectifiable simple closed curve, C. Then no level line of Σ has a length greater than the length of the boundary of Σ. For let there exist a level line of Σ, say G, which has a length, g, which is greater than the length, l, of C. It is assumed that $w=0$ is an interior point of G; no loss of generality follows from this assumption. Then there exists a function $w=f(z)$ which is analytic in the unit circle and which maps in a one-to-one and conformal way the interior of the unit circle on Σ such that $f(0)=0$ and $f'(0)=1$ and such that G is the image under the transformation $w=f(z)$ of a circle, H, concentric with the unit circle, K. Further, let $\{P_n\}$, $n=1, 2, 3, \cdots$, be a sequence of simple polygons inscribed in C which are such that $\lim_{n\to\infty} d_n=0$, where d_n is the length of a side of P_n which is not shorter than any other side of P_n, and such that $w=0$ is an interior point of each P_n. Then there exists a function $w=f_n(z)$ which maps the interior of the circle $|z|=1$ on the interior of P_n in a one-to-one and conformal way such that the point $w=0$ corresponds to the point $z=0$ and the derivative of the function $w=f_n(z)$ at $z=0$ is unity. By a theorem* of Carathéodory, and the fact that there is only one function which maps the interior of the unit circle on Σ in a one-to-one and conformal way such that $w=0$ corresponds to $z=0$ and the derivative of the mapping function at $z=0$ is unity, it follows that the sequence of functions $\{w=f_n(z)\}$, $n=1, 2, 3, \cdots$, approaches the function $w=f(z)$, $|z|<1$, uniformly on any closed set of points which is contained in the interior of the unit circle. Consequently the sequence of derivatives of the functions $w=f_n(z)$, $\{w=f_n'(z)\}$, $n=1, 2, 3, \cdots$, converges uniformly to the derivative of $w=f(z)$ on any closed set of points which is in the interior of the unit circle.

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* See Mathematische Annalen, vol. 72 (1912), pp. 120–126. Also Bieberbach, *Lehrbuch der Funktionentheorie*, vol. II, pp. 12–15.

Now, let ν be a positive number less than $g-l$ and p a positive integer such that

$$|f'(z) - f_p'(z)| < \frac{\nu}{2\pi},$$

for z on H. Then

$$|f_p'(z)| = |f'(z)| + \eta(z),$$

where $|\eta(z)| < \nu/2\pi$ for z on H, and

$$\int_H |f_p'(z)|\, ds = \int_H |f'(z)|\, ds + \int_H \eta(z) ds.$$

But

$$\int_H |f'(z)|\, ds = g \text{ and } \int_H |f_p'(z)|\, ds$$

is the length of the image of H under the transformation $w=f_p(z)$, $|z|<1$. The latter image is a level line of the pencil of level lines of the polygonal region bounded by P_p, which has the point $w=0$ as its pole. If the length of this level line is denoted by g_p, then $g_p>g-\nu$ and hence $g_p>l$.

According to the result for polygonal regions which was obtained above, there exists a circle, H', with center $z=0$ and a radius of length less than unity but greater than the length of a radius of H such that the length, g_p', of the image of H' under the transformation $w=f_p(z)$ differs from the length, l_p, of P_p by an amount less than g_p-l. Since $l_p<l$ it follows that $g_p'<g_p$. But this result contradicts the fact that by a theorem* of Hardy in connection with the formula for the length of an analytic transform of a rectifiable curve, which is given above. It follows that $g_p'>g_p$.

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* See Proceedings of the London Mathematical Society, (2), vol. 14 (1915), pp. 269–277. Also Landau, *Ergebnisse der Funktionentheorie*, p. 85. The theorem is that if $w=f(z)$ is analytic and not constant for $|z|<R$, then $\int_0^{2\pi}|f(re^{i\theta})|\,d\theta$, $z=re^{i\theta}$, $0\leqq\theta\leqq 2\pi$ and $0<r<R$, is a continuous increasing function of r for $0<r<R$.

Only a special case of this theorem is used above. The functions concerned are only those which map in a one-to-one and conformal way the interior of the unit circle on the interior of a simple polygon.

Thus the length of any level line of Σ is not greater than the length of the boundary of Σ. The theorem of Hardy then implies that the length of any level line of $\sum$ is less than the length of the boundary of Σ.

That any sequence of level lines $\{G_{t_n}\}$, $t_n < t_{n+1}$, $n=1, 2, 3, \cdots$, and $\lim_{n\to\infty} t_n = 1$, which belong to any given pencil of level lines $[G_t]$, $0<t<1$, of level lines of any planar region Σ whose boundary is connected and contains more than one point is an approximating sequence of curves of Σ follows easily from the one-to-one conformal mapping of the interior of the unit circle on Σ, which determines the sequence of level lines $\{G_{t_n}\}$, $t_n<t_{n+1}$, $n=1, 2, 3, \cdots$, as the image of a sequence of circles, $\{H_n\}$, $n=1, 2, 3, \cdots$, which are concentric with and interior to the unit circle and such that $\lim_{n\to\infty} r_n = 1$, where r_n is the length of the radius of the circle H_n. Evidently, the level line G_{t_n} is in the interior of the level line $G_{t_{n+1}}$.

Now, if the boundary of Σ is a rectifiable simple closed curve of length l, it follows readily from certain known results* that if l_n is the length of the rectifiable curve C_n of the approximating sequence of curves $\{C_n\}$, $n=1, 2, 3, \cdots$, of the region Σ and if $\lim_{n\to\infty} l_n$ exists, then we have $\lim_{n\to\infty} l_n \geqq l$. Hence, with what has preceded, if l_{t_n} is the length of the level line G_{t_n} of the sequence of level lines $\{G_{t_n}\}$, $t_n<t_{n+1}$, $n=1, 2, 3, \cdots$, $\lim_{n\to\infty} t_n=1$, then $l_{t_n}<l$ and, since by the theorem of Hardy $l_{t_n}<l_{t_{n+1}}$, $n=1, 2, 3, \cdots$, it follows that $\lim_{n\to\infty} l_{t_n}=l$. Thus $\lim_{t\to 1} \Lambda(t)=l$ and the sequence of level lines $\{G_{t_n}\}$, $t_n<t_{n+1}$, $n=1, 2, 3, \cdots$, $\lim_{n\to\infty} t_n=1$, is an approximating sequence of curves of Σ as specified in the second theorem.

COLUMBIA UNIVERSITY

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* In particular, Theorem V, p. 519 of Hahn, *Theorie der reellen Funktionen*, vol. I.

CORRESPONDENCE OF GAUSS AND GERLING

Briefwechsel zwischen Carl Friedrich Gauss und Christian Ludwig Gerling. By Dr. Clemens Schaeffer. *Herausgegeben im Auftrage der Gesellschaft zur Beforderung der gesamten Naturwissenschaften zu Marburg.* Berlin, Otto Elsner, 1927. xx+820 pp. Price 36 marks; bound, 40 marks.

It is not too much to say that Germany sets a standard not reached by any other country in the recognition, through publications of this kind, of the work of a nation's great scientists. More than a half a century ago there appeared six volumes of the correspondence between Gauss and Schumacher; a quarter of a century ago there were added two volumes of letters exchanged between him and others; and beside these there has been published a considerable amount of correspondence between him and both Bessel and Humboldt. There now appears the present work of more than eight hundred pages, giving 388 letters dating from 1810 to 1854, of which 163 are those written by Gauss and 225 by Gerling.

Such extensive publications as these relating to the friendly intercourse between the great Göttingen scientist and his fellow workers are not often possible. Such a body of material is not available, even in the case of men of highest standing,—men whose letters would naturally be preserved. Some, like Cayley, were not prolific letter writers; while others, like Sylvester, wrote apparently for the love of writing. It was not, however, because Gauss was accustomed to correspond so freely with his friends that his letters have been so freely made known to the world, nor was it merely because Germany or a German learned society desired to honor his memory. It is in a large measure due to the fact that German readers are numerous enough to purchase volumes of this kind and thereby render their publication possible. Such source material is invaluable to the historian, but there are few historians to use it; the support of the publication must have come from the large body of cultured scientists who find in the past a stimulus to their work for the future. In our own country, with all its wealth, such publications seem to be almost impossible. Perhaps it is because letter writing is with us a lost art; more probably it is because reading is such.

Gerling was professor of mathematics, physics, and astronomy in the the University of Marburg from 1817 to 1864. He was repeatedly offered positions elsewhere, but for nearly a half a century he continued to fill the chair for which he had been chosen in his early years. He was eleven years younger than Gauss and an acquaintance formed in Göttingen had been maintained until the latter's death in 1855.

Professor Schaeffer, the editor, has shown himself admirably adapted to the work. His tastes are those of both Gauss and Gerling. He is professor of physics at Breslau, and hence his field of major interests was at least one of the fields which each writer cultivated with a success that the world recognizes. As editor he has been painstaking in giving the letters as they would have been written today, with modern spelling and punctuation, and in supplying a large number of footnotes of a biographical and bibliographical nature. While many readers may wish that the text had been given precisely as the original letters were written, it should be said that all alterations from those mentioned are made in brackets so that the reader is sure of the text as Gauss or Gerling would have prepared it at the present time.

It is, of course, impossible to call attention to all points of historical interest in such a collection, but some idea of the scientific range may be obtained from a few references. Gerling told Gauss as late as Nov. 22, 1818 (see p. 182) that the construction of the 17-gon as given in the *Disquisitiones Arithmeticae* was not clear; indeed, that it was merely suggestive; and he asked for a complete proof. This Gauss gave (letter of Jan. 6, 1819, p. 185) in the form which is now familiar, together with a historical note (p. 188) concerning his discovery of the method. The work of Gauss and Weber on the electric telegraph is well known, and in his letter of Aug. 26, 1835, he goes into the subject of his later improvements (p. 447). Gerling's reply (p. 451) shows how much interest was being aroused at that time in the new invention.

We usually think of Gauss as a great mathematical genius and as a great astronomer, forgetting that he ranked relatively as high as a geodecist. Those who are interested in this phase of his work will find a large amount of material in this correspondence, not merely with respect to triangulations and accuracy of observation, but as to the advanced phases of the subject and as to the improvement of instruments and the application of the method of least squares to geodetic work.

As to his pure mathematical interests, there are numerous references to his early work on critical points of a triangle (p. 340), the question of the angle (p. 195), and the various theories of parallels, although these are generally in the nature of obiter dicta rather than of proof. Speaking of Legendre's treatment of parallels, for example, he remarks "das für mich gar keine Beweiskraft in seinem Schluss liegt," and he proceeds succinctly to give his reasons, and similarly in various other matters of a similar kind

Naturally the work should and will find place in all mathematical and reference libraries of importance.

David Eugene Smith

EXTERIOR BALLISTICS

In the March-April number of this Bulletin, (vol. 34, pp. 229–232) appears a review of Moulton's *Exterior Ballistics*. This has certain features not closely connected with the book under review which appear in need of comment, and in one case correction.

The first of these is the statement that "the student and the professor of mathematics have at their disposal no book from which a trajectory can be actually computed" the reason being* that the necessary "tables are regarded as confidential information by the U. S. Army."

There are, however, at least three places in which these tables are accessible to any one interested. These are *The Method of Numerical Integration in Exterior Ballistics* by Dunham Jackson, the *Exterior Ballistic Tables* of 1924, and Herrmann's *Exterior Ballistics*. The first two of these are War Department documents, but are not confidential. The last is published by the Naval Institute.

The second feature concerns the value of the method of numerical integration. It has become clear from a correspondence with the reviewer that he regards this method as a useless refinement in the present stage of experimental ballistics. He agrees, however, it has the merit of not adding further errors to those of experimental origin. Its use in Moulton's book to the exclusion of other methods accords with the avowed object of that book of remaining applicable over a considerable period of experimental advance. As regards actual computations, that advance is even now taking place. Many trajectories are now computed at Aberdeen using more recent data on the resistance function than those of the G table. The advantage of numerical integration here becomes evident. The computer is merely given a new table of the resistance function and he proceeds as before. The physicist does not have to ask the mathematician "can you integrate these new differential equations that my experiments give?"

The fact that numerical integration is extremely laborious is not brought out in the review, although of course well known to the reviewer. The individual steps are easy to perform and easy to understand but are very numerous. It would appear that this fact should be stressed in any general estimate of the value of the method, as it constitutes the outstanding objection to its use.

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* Professor Rowe desires to add that, in his opinion, the theory and the tables should be given in the same book. THE EDITORS.

SHORTER NOTICES

The Geometry of René Descartes, translated from the French and Latin by David Eugene Smith and Marcia L. Latham. With a facsimile of the first edition, 1637. Chicago and London, The Open Court Publishing Company, 1925. xiii+246 pp., with portrait. Price, $4.

La Géométrie de René Descartes. Nouvelle Edition. (Avec portrait de Descartes d'après Frans Hals.) Paris, J. Hermann, 1927. Price, 21 francs.

It is indeed a healthy sign of the popular interest in the history of mathematics that so many publishers in European countries, and in America, are occupied with the reproduction of the classics of science. One can find few works more worthy of reproduction than this fundamental achievement of Descartes. In a very real sense modern mathematics begins with his analytic geometry. Also in a very real sense Descartes is the discoverer rather than the inventor of the work. We now can see that this work is the product of the varied developments of mathematics made in Europe before Descartes, particularly in Italy, in France, and in England, development of ideas which were based upon the work of the Greeks, the Hindus, and the Arabs. The contemporary and even more modern work by Fermat illustrates the fact that the ideas were "in the air." Fortunately both were Frenchmen.

The French edition of Descartes' Geometry is the second publication of this work in this form, although this fact is not stated in the publication. The first edition appeared in 1886, issued by the same publisher. The French edition unfortunately modernizes the terminology.

Two German editions have also appeared, translated by Ludwig Schlesinger. The first appeared in 1894 and sold for three marks sixty pfennigs which is again the price for the new edition of 1923, a noteworthy achievement. The French edition sells at 21 francs.

The American edition is far superior to the others mentioned in that it gives an exact facsimile of the text, with translation and notes. The price, four dollars, is just about five times that of a French or German edition.

The notes are, in general, adequate. The vital point that Descartes studiously avoids negative abscissas and ordinates does not receive treatment. Some of the notes, for example, note 49 on page 33, are actually misleading on this point. In recent publications Wieleitner has stressed this late complete recognition of the negative. Tropfke (*Geschichte der Elementar Mathematik*, vol. 6, Leipzig, 1924, p. 109) asserts that only with Newton in 1704 do the four quadrants receive equal recognition, and that most texts of the eighteenth century did not grasp it.

The English version can be highly commended to all students of mathematics, a fine rendition of a classic which will even today be a source of inspiration to lovers of mathematical science.

L. C. KARPINSKI

Mathematische Formelsammlung. By O. Th. Bürklen, revised by F. Ringleb. Berlin and Leipzig, Walter de Gruyter, 1927. 241 pp.

This little volume of the Sammlung Göschen contains a very complete list of the important formulas and theorems of what the mathematician would call elementary mathematics. The subjects included are arithmetic, algebra, elementary theory of numbers, plane and solid geometry, plane and spherical trigonometry with applications to geography and astronomy, plane and solid analytic geometry, differential and integral calculus, differential geometry, and differential equations.

The table of contents and a brief index make it very easy to locate a formula or theorem and the typography is excellent. The book may be recommended as a convenient source for those of us who hesitate to trust our memories for formulas and theorems which we seldom use.

Under the heading "Literatur" there occurs only one citation, namely, the new edition of Pascal's Repertorium by Salkowski and Timerding, which is likely to form the basis of any other collection of mathematical formules.

W. R. Longley

Les Bases de la Géométrie et de la Physique. L'Invariance de L'Espace Euclidien. Par Clément Laurès. Paris, Blanchard, 1928. 125 pp.

According to a statement on page 1, the physicist will say that the remarkable usefulness of euclidean geometry in physics is due to the fact that it investigates the relations of invariable solid bodies, and that measurements made in accordance with it reveal veritable constants of nature; while the mathematician will affirm that there is an infinitude of systems of geometry all equally possible and that the basis of choice among them rests on considerations of convenience, and not on logical necessity. To M. Laurès, the latter answer is unsatisfactory, and that of the relativists is still more so. If the parallel postulate of Euclid is indemonstrable, then (according to the author, p. 4) geometry has no solid foundation. To show how to demonstrate this postulate is the avowed purpose of the book—a purpose which the author (p. 5) "presents with confidence to the enlightened public." He says (p. 5): "The indemonstrability of this postulate is the first article of the contemporary *Credo* of science. The reader will see how this Credo is false on this point." "The non-euclidean geometries were one of the most stupid inventions of the nineteenth century." "The human reason is sick." And here the author undertakes to administer that intellectual remedy which shall restore it to health. With this restoration he would also remove the "stupidity" of relativity from modern science. "Whatever contains truth," he says on page 5, "can be very simply explained by means of invariant euclidean space." It is improbable that any mathematical reader will now be seriously interested in an argument whose avowed purpose is to demonstrate the parallel postulate of Euclid.

R. D. Carmichael

Fonctions Entières et Fonctions Méromorphes d'une Variable. By G. Valiron. Mémorial des Sciences Mathématiques, Fascicule II. Paris, Gauthier-Villars, 1925. 56 pp.

Sur les Fonctions Hypergéométriques de Plusieurs Variables. Les Polynomes d'Hermite et autres Fonctions Sphériques dans l'Hyperspace. By Paul Appell. Mémorial des Sciences Mathématiques, Fascicule III. Paris, Gauthier-Villars, 1925. 75 pp.

So brief a time has elapsed since the appearance of Valiron's former work, *Lectures on the General Theory of Integral Functions*, that one naturally expects the present small volume to follow essentially the development given in the former book. The condensation of material in order to confine the subject to fifty pages is most noticeable. On this account the beginner in this field will probably prefer the older book, where the pace is more leisurely and the exposition more detailed. The bibliography is more complete in the later treatment. The author's own fertility almost caused him embarrassment, since he just exhausted the alphabet in listing his own titles.

In the work on the hypergeometric functions of several variables Appell has brought together a brief account of the work, extending over a number of years, of a number of mathematicians, of whom he himself is very conspicuous. The work consists of generalizations in rather natural directions. Guided by the celebrated series of Gauss, four different series in two variables and four to five parameters are constructed. For the functions so defined properties analogous to those of the hypergeometric function are developed. One expects to find nothing especially novel or interesting, and his expectations are confirmed. It seems to the reviewer that a number of celebrated mathematicians have deprived lesser ones of much fine material for theses. It is, of course, useful to have an account of what has been done in the field, even though it has limited significance, and it gives gratification to have it done so ably.

K. P. WILLIAMS

Exercices de Calcul Différentiel et Intégral. By Abbé Potron. Volume I. Paris, Hermann, 1926. xviii+332 pp.

This book is a typical representative of the set of French "Exercises" or "Collections of problems." The final goal here is to train the students for the technically difficult examination for the "Certificate in Differential and Integral Calculus."

Accordingly, the problems which have been given at such examinations for a number of years in Paris and in other cities, are "analysed" here, or decomposed into their simplest elements, where the solution is reduced to a mere question of routine computation. The "synthesis," namely the problems as they have been actually given and their solutions, are promised in the second volume.

This method perhaps is well fitted for the particular purpose as mentioned above, but it is at least questionable as to its general educational value. However, American teachers will find many a good example to be assigned to students for their home-work, among the total of 1080 problems gathered in this book.

The problems are classified into six parts: I. Derivatives, differentials, geometric applications. II. Integral calculus. III. Differential equations (ordinary). IV. Partial differential equations. V. Functions of a complex variable. VI. Supplements. Each part is subdivided into smaller chapters, and the latter in turn into paragraphs. Each paragraph is supplied with short outlines of the corresponding parts of the theory, which not always are above reproach, particularly when the attempt is made to be rigorous.

Most completely treated are the applications of calculus to geometry, especially those which require formal computations. Some other topics, however, are treated quite inadequately: they might have been omitted altogether without any loss; as for instance, ruled surfaces (2 problems), contact of plane curves (1), improper integrals (3), elliptic and hyperelliptic integrals (6), transformations of surfaces (3), trigonometric series (4), calculus of variations (6) and so on.

The order of the problems is not always the best possible; some problems are repeated at different places. Some problems are not very clearly or carefully stated; see for instance, p. 8 (20, 21), p. 124 (43).

The book is very neatly printed, although misprints are not infrequent.

J. D. TAMARKIN

A Treatise on the Analytical Dynamics of Particles and Rigid Bodies; with an Introduction to the Problem of Three Bodies. By E. T. Whittaker. Third Edition. Cambridge, University Press, 1927. xiv+456 pp.

The first edition of this book, which was published in 1904, received an extensive review by E. B. Wilson in this Bulletin (vol. 12 (1906), pp. 451–458). The second edition, which appeared in 1917, (see this Bulletin, vol. 26 (1920), p. 183) differed from the first mainly in the matter of references to or brief outlines of more recent researches.

In the third edition the first fourteen chapters, with some corrections and additional references, have been reproduced photographically from the second edition. Chapter XV on the general theory of orbits and Chapter XVI on integration by series have been completely rewritten in order to present the subject as it has been developed by the researches of the last eleven years. As illustrations of periodic and asymptotic orbits the author has treated the paths which small particles describe in the gravitational field due to a single attracting mass when the newtonian law of gravitation is replaced by the laws belonging to the general relativity theory. There is also a brief account of Synge's geometry of dynamics, in which dynamical problems are treated by aid of the tensor analysis. References only are given to the work of Birkhoff on the classification of the various types of motion of a dynamical system with two degrees of freedom.

Wilson's excellent account of the characteristics of the first edition makes it unnecessary to comment in more detail upon the present edition.

W. R. LONGLEY

Höhere Algebra. By Helmut Hasse. Volume 2: *Gleichungen hoheren Grades.* (Sammlung Göschen.) Berlin, de Gruyter. 1927, 160 pp.

This second part of Hasse's algebra is still more interesting than the first. The contents are apparently the same as in most textbooks on higher algebra: the theory of general algebraic equations, Galois group theory, leading up to the condition for a solution by radicals.

This algebra is however no textbook in the common sense of the word, but a modern scientific review of the fundamentals of algebra, in particular the Galois fields, wherein the last traces of the theory of functions in algebra are eliminated.

The definition of the abstract rings and fields and the fundamental properties of them were already given in vol. I. A feature of the present volume is the introduction of the ideas of Steinitz developed in his famous paper *Algebraische Theorie der Körper* (Journal für Mathematik, vol. 137). Steinitz's classification of the abstract fields, fields of characteristic 0 and fields of characteristic p, is introduced: to the last class belong for instance all finite fields, and all fields having a finite subfield. All fields containing the rational field have the characteristic 0. By construction of successive adjunction fields a Galois field is obtained, and the main theorems on the connection between groups and equations are proved for all perfect (vollkommen) fields, that is, all fields with characteristic 0 and fieds with characteristic p having certain properties.

It is difficult to give a satisfactory account of this excellent little book in a few lines, but it can be warmly recommended to *all* mathematicians interested in algebra.

OYSTEIN ORE

Compléments de Géométrie Moderne. By Charles Michel. Paris, Vuibert, 1926. 317 pp. Price (unbound) 35 frs.

The name *Modern Geometry* is used by different writers to cover a vast and varied body of doctrine, from, say, the Simson line and up to the Lie theory, or anything else. What may be meant by "compléments" is still less certain. The preface of the book might be expected to throw some light upon the vague and unassuming title, but the book has no preface. Thus reduced to use the publisher's announcement as a substitute, one soon becomes convinced that a more ambitious caption would come much nearer to be descriptive of the contents of the work. For the author presupposes on the part of the reader a familiarity with the more elementary projective properties of conics, quadrics, and algebraic curves and surfaces in general, with the use of imaginaries in geometry, etc. In other words, he expects his reader to have mastered the advanced parts of a French *Traité de Géométrie*, like Hadamard's, or Rouché et Comberousse's, and in addition a book like E. Duporcq's *Premiers Principes de Géométrie Moderne.*

With this much as a background the author treats pencils and nets of conics and quadrics, the cubic curve both in the plane and in space, Steiner's surfaces. The method of presentation throughout the book is almost exclusively synthetic, except for the first chapter, where the author accords "au calcul sa part inévitable." That the author thus expedites matters is quite certain, but to say that the analytical attack and the implied principle of continuity are unavoidable is to forget the brilliant and fruitful labors of von Staudt and his successors.

The author has made a very judicious choice in the vast material available and has produced a book which duplicates no other single book in the existing literature. The book is written in the good traditional style of French clarity and is free of excessive condensation. The reader is further helped by the figures which appear when and where they are most needed. Bibliographical references are few in number, there is no index, and the table of contents is brief and summary.

We have quite a few books which may be used as a basis for a first course in synthetic projective geometry, particularly in English. The situation changes radically when it comes to a second course in this subject. M. Michel's book may be used to great advantage for such a purpose in our American schools, and the set of nearly one hundred exercises at the end of the book may be quite helpful in this connection.

NATHAN ALTSHILLER-COURT

Linienspektren und periodisches System der Elemente. By Friedrich Hund. Volume IV of the series Struktur der Materie, edited by M. Born and J. Franck. Berlin, Springer, 1927. vi+221 pp.

In this book a very successful attempt is made to give a recapitulation and a uniform, systematic presentation of the results obtained in the unravelling and classification of line spectra from the pioneer work of Bohr through the investigations by Catalán, Russell, Saunders, Pauli, Heisenberg, Schrödinger, and others. In order to present only one kind of problem to the reader at a time the author has intentionally made very little explicit use of the new quantum mechanics. Nevertheless, he makes a special point of calling attention to the difficulties and limitations inherent in the correspondence principle. Hence, from different points of view the book contains material which should appeal to the astronomer and the chemist as well as to the theoretical physicist and the spectroscopist.

Not only is the presentation relatively non-mathematical, in the sense that advanced analysis has been replaced by a vectorial algorithm devised by the author, but it is also unusually clear and logical. By commencing with the rough and then proceeding to the finer properties of spectra, the atomic model so-called is refined step by step until it becomes possible to give at least a qualitative account of all the details of the most complex spectra. By no means the least valuable feature of the book consists in the numerous, excellent tables which constitute the best and most extensive classification of spectral terms and relationships extant. It is thus evident that the text is an unusually valuable contribution to the field of line spectra.

H. S. UHLER

Interpolation. By J. F. Steffensen. Baltimore, Williams and Wilkins, 1927. 248 pp.

Numerische Infinitesimalrechnung. By Martin Lindow. Berlin and Bonn, Ferd. Dummlers, 1928. 176 pp.

Professor Steffensen's book is the outgrowth of the lectures which the author has given to actuarial students at the University of Copenhagen and is, with a few additions and simplifications, a translation of the Danish edition published in 1925. The book is intended as a text for students in American colleges and requires as mathematical equipment only an elementary knowledge of the differential and integral calculus. In a few places where the gamma function has been used the paragraphs have been printed in smaller type and may be omitted without breaking the continuity of the text.

The topics covered are (1) the general theory of interpolation and extrapolation including the standard formulas usually associated with the names of Newton, Gauss, Stirling, Bessel, and others; (2) numerical differentiation; (3) numerical integration; (4) numerical solution of differential equations.

Professor Steffensen's treatment is more rigorous than is usual in books on interpolation. This is important not merely from the point of view of the pure mathematician but also because of the increased number of formulas with workable remainder terms. It should not be supposed, however, that this adds to the difficulty of reading the text. The style is clear and, after the meaning of the symbols has been mastered, the book should prove very valuable to the increasing number of Americans who require some knowledge of this field of mathematics. The formulas and methods are illustrated by simple numerical examples, but the value of the book for class room use would be increased if it contained some problems to be solved by the student.

Dr. Lindow's book covers essentially the same ground as the preceding one. In the section on the numerical integration of differential equations he has made applications to the problem of the simple pendulum and the restricted problem of three bodies. There are no problems for the student but this book is not intended for class use in American colleges. A useful feature for the computer is the inclusion of seventeen tables giving the numerical coefficients for the expansions required in the various standard formulas.

W. R. LONGLEY

NOTES

The July, 1928, number of the Transactions of this Society (volume 30, No. 3) contains the following papers: *Transversality in space of three dimensions*, by E. Kasner; *On integral equations with discontinuous kernels*, by J. D. Tamarkin and R. E. Langer; *On approximation to an arbitrary function of a complex variable by polynomials*, by J. L. Walsh; *Transformations of nets*, by V. G. Grove; *Allgemeine Eigenschaften der Cantorschen Kohärenzen*, by M. Zarycki; *On the irregular cases of the linear ordinary difference equation*, by C. R. Adams; *On the convergence of quadrature formulas related to an infinite interval*, by J. V. Uspensky; *Second-order linear systems with summable coefficients*, by J. H. Sturdivant; *Concerning the arc curves and basic sets of a continuous curve*, by W. L. Ayres; *A solution of the matric equation $P(X)=A$*, by W. E. Roth; *Concerning the cut points of continua*, by G. T. Whyburn; *Some theorems on the connection between ideals and group of a Galois field*, by O. Ore.

The July, 1928, number of the American Journal of Mathematics (volume 50, No. 3) contains: *On hyperelliptic θ-functions with rational characteristics*, by O. Zariski; *Certain perfect groups generated by two operators of orders two and three*, by H. R. Brahana; *On triadic Cremona nets of of plane curves*, by F. Farnum; *Number relations between types of extremals joining a pair of points*, by D. E. Richmond; *An intrinsic treatment of Poisson's integral*, by F. W. Perkins; *On the invariant combinants of two binary quintics*, by T. W. Moore; *A boundary value problem of ordinary self-adjoint differential equations with singularities*, by M. C. Gray; *Nets of conics in the real domain*, by A. D. Campbell; *Rational tacnodal and oscnodal quartic curves considered as plane-sections of quartic surfaces*, by L. T. Moore and J. H. Neelley.

The April, 1928, number of the Annals of Mathematics (series 2, volume 29, No. 2) contains: *The canonical form of a one-parameter group*, by P. Franklin; *A class of real quadratic forms in infinitely many variables*, by F. H. Murray; *The general geometry of paths*, by J. Douglas; *On a geometrical theory of continuous groups. II. Euclidean and hyperbolic groups of three-dimensional space*, by B. de Kerékjártó; *A method of numerical solution of the problem of Plateau*, by J. Douglas; *Minimizing two types of definite integral*, by P. R. Rider; *On two types of plane rational curve*, by H. Hilton; *A correspondence between matrices and quadratic ideals*, by C. C. MacDuffee; *A class of functional equations*, by E. Hille; *Definitions of abstract groups*, by G. A. Miller; *Concerning a set of metrical hypotheses for geometry*, by J. L. Dorroh; *Closed point sets on a manifold*, by S. Lefschetz.

A new Bulletin of the National Research Council series has recently appeared, entitled *Selected topics in algebraic geometry*. It contains chapters by A. B. Coble, A. Emch, S. Lefschetz, F. R. Sharpe, C. H. Sisam, and V. Snyder.

The Open Court Publishing Company has just issued, through the University of Chicago Press, the first volume, *Notations in Elementary Mathematics*, of a work entitled *A History of Mathematical Notations*, by Professor F. Cajori, of the University of California.

By the will of Lieutenant Colonel A. J. C. Cunningham, who died February 8, 1928, the London Mathematical Society will receive his library, and also £1,000 for the improvement of methods of factorization of large numbers, and £2,000 for the publication of Colonel Cunningham's works. One twelfth of the residuary estate also will go to the London Mathematical Society, and one twelfth to the mathematical subsection of the British Association, for preparing new mathematical tables in the theory of numbers.

Professor J. W. Lasley, of the University of North Carolina, has been elected chairman of the mathematics section of the North Carolina Academy of Science, and Professor W. W. Elliott, of Duke University, secretary of that section.

The Albert medal of the Royal Society of Arts has been awarded to Sir Ernest Rutherford, for his pioneer researches in the structure of matter.

Dr. Henri Deslandres, director of the observatory of Paris, has received the honorary degree of doctor of mathematics and physics from the University of Leyden.

Professor Emile Picard, of the University of Paris, celebrated, on May 6, 1928, his fiftieth scientific anniversary.

Cambridge University has conferred an honorary doctorate on Professor A. Einstein.

Professors David Hilbert, Paul Langevin, and Ludwig Prandtl have been elected foreign members of the Royal Society of London.

Dr. J. H. Jeans, secretary of the Royal Society, has been knighted.

Dr. Cristóbal de Losada y Puga, of Lima, Peru, a member of this Society, has been elected a corresponding member of the Royal Academy of Sciences of Madrid.

Sir Ernest Rutherford has been elected an associate of the Royal Academy of Belgium.

Williams College has conferred the honorary degree of doctor of science on Professor W. A. Bratton, of Whitman College.

Columbia University has conferred an honorary doctorate on President W. W. Campbell, of the University of California, director of the Lick Observatory.

Professor Joaquin Gallo, director of the National Astronomical Observatory of Mexico, has received the honorary degree of doctor of science from Northwestern University.

The University of Indiana has conferred an honorary degree on Professor J. A. Miller, of Swarthmore College.

Princeton University has conferred the honorary degree of doctor of science on Professor R. A. Millikan.

The chair of physics of Bryn Mawr College has been named in honor of Marion Reilly; for partial support of it, a fund to be known as the Marion Reilly Memorial is being created by a committee of her friends and classmates, under the chairmanship of Mrs. Monroe Buckley.

Professor Karl Pearson has been elected a foreign honorary member of the American Academy of Arts and Sciences.

Professor U. Amaldi, of the Higher School of Architecture at Rome, has been elected a corresponding member of the Reale Accademia dei Lincei.

The R. Accademia delle Scienze of Turin has elected the following corresponding members: E. Fermi, of the University of Rome; A. Signorini, of the University of Naples; L. Tonelli, of the University of Bologna; G. Vitali, of the University of Padua. Professor Vitali has been named corresponding member of the R. Istituto Veneto; he has also been awarded the prize for mathematics for the year 1927 by the Italian Society of Sciences.

Proferror W. B. Ford, of the University of Michigan, has been granted leave of absence for the current academic year and will travel in Europe. Representing the Carnegie Endowment for International Peace, he will lecture at the Universities of Leiden and Utrecht in Holland, the University of Brussels in Belgium, the Universities of Lille and Grenoble in France, and the University of Pisa in Italy.

Professor M. I. Pupin has received the honorary degree of doctor of laws from Middlebury College. He has also recently been elected an honorary member of the American Institute of Electrical Engineers.

Professor D. E. Smith, retired, of Columbia University, has returned from Europe, where he has served as a visiting Carnegie professor of international relations. Representing the Carnegie Endowment for International Peace, he lectured at University College, London, and at the Universities of Montpellier and Toulouse.

Associate Professor Ludwig Berwald, of the German University of Prague, has been appointed to a professorship at the German Technical School at Prague. Dr. Paul Funk has been promoted to a professorship of mathematics at the same technical school.

At the University of Göttingen, Professor H. Bohr, of the Copenhagen Technical School, has been appointed to a professorship of mathematics; Dr. Hans Lipps and Dr. Max Schuler have been promoted to associate professorships.

Dr. Alwin Walther has been appointed professor of mathematics at the Darmstadt Technical School.

Assistant Professor C. R. Adams, of Brown University, has been promoted to an associate professorship of mathematics.

Dr. Constance R. Ballantine has been appointed associate in mathematics at the University of Washington.

Assistant Professor I. A. Barnett, of the University of Cincinnati, has been promoted to an associate professorship of mathematics.

Mr. C. F. Bowles has been promoted to an assistant professorship of mathematics at the South Dakota State School of Mines.

Associate Professor J. A. Bullard, of the United States Naval Academy, has been appointed to a professorship of mathematics at the University of Vermont.

Assistant Professor W. B. Campbell, of Colgate University, has been appointed by the Baptist Mission Board lecturer in mathematics at Judson College, Rangoon, Burma.

Dr. Elizabeth Carlson has been promoted to an assistant professorship at the University of Minnesota.

Dr. M. G. Carman has been appointed head of the department of mathematics at Murray State Teachers College, Murray, Ky.

Associate Professor A. F. Carpenter, of the University of Washington, has been promoted to a professorship of mathematics.

Dr. W. F. Cheney has been promoted to an assistant professorship of mathematics at Tufts College.

Assistant Professor Lennie P. Copeland has been promoted to an associate professorship at Wellesley College.

Assistant Professor C. H. Currier has been promoted to an associate professorship of mathematics at Brown University.

Mr. L. J. Deck has been promoted to a professorship of mathematics at Muhlenberg College.

Assistant Professor R. F. Deimel, of Stevens Institute of Technology, has been promoted to an associate professorship of mechanics.

Dr. R. D. Doner has been appointed associate professor of mathematics at the Alabama Polytechnic Institute.

Assistant Professor W. E. Edington has been promoted to an associate professorship of mathematics at Purdue University.

Assistant Professor H. A. Fales, of Columbia University, has been promoted to an associate professorship of chemistry.

Assistant Professor I. H. Fenn, of the Brooklyn Polytechnic Institute, has been promoted to an associate professorship of mathematics.

Assistant Professor G. H. Graves has been promoted to an associate professorship of mathematics at Purdue University.

Dr. D. C. Harkin has been appointed associate professor at the Alabama Polytechnic Institute.

Assistant Professor D. A. Hatch, of Lafayette College, has been promoted to an associate professorship of mathematics.

Dr. C. E. Horne has been appointed professor of mathematics at the University of Porto Rico.

Assistant Professor Daniel Hull has been promoted to be full professor and head of the department of physics at the University of Notre Dame.

Dr. Vern James has been appointed head of the department of mathematics at Menlo Junior College.

Mr. F. C. Jonah has been appointed assistant professor of mathematics at Adelbert College.

Dr. E. G. Keller has been appointed assistant professor of mathematics at the University of Texas.

Assistant Professor Claribel Kendall, of the University of Colorado, has been promoted to an associate professorship of mathematics.

Associate Professor F. C. Kent has been promoted to a professorship of mathematics at the Oregon State Agricultural College.

Associate Professor E. P. Lane has been promoted to a professorship of mathematics at the University of Chicago.

Mr. H. I. Lane has been appointed assistant professor of mathematics at the University of South Dakota.

Dr. C. G. Latimer has been appointed professor of mathematics at the University of Kentucky.

Dr. E. E. Libman, of the University of Illinois, has been promoted to an assistant professorship of physics.

Dr. R. B. Lindsay has been promoted to an assistant professorship of physics at Yale University.

Dr. R. G. Lubben has been appointed adjunct professor of mathematics at the University of Texas.

Assistant Professor G. F. McEwen has been promoted to a full professorship at the Scripps Institution of Oceanography.

Dr. Elsie J. McFarland has been promoted to an assistant professorship of mathematics at the University of Oklahoma.

Dr. A. J. Maria has been appointed assistant professor of mathematics at the University of Illinois.

Assistant Professor R. W. Marriott has been promoted to a professorship of mathematics and astronomy at Swarthmore College.

Dr. Florence M. Mears has been appointed acting assistant professor of mathematics at Pennsylvania State College.

Professor J. M. Melchiors has been appointed professor of physics at Loyola University, Matoon, Wis.

Dr. G. M. Merriman has been appointed assistant professor of mathematics at Grinnel College.

Associate Professor J. C. Morehead has been promoted to a professorship of graphics in the department of architecture of the Carnegie Institute of Technology.

Associate Professor F. R. Morris has been promoted to be professor and head of the department of mathematics and engineering at Fresno State College.

Assistant Professor D. S. Morse has been promoted to an associate professorship of mathematics at Union College.

Assistant Professor Marston Morse has been promoted to an associate professorship of mathematics at Harvard University.

Associate Professor F. D. Murnaghan, of Johns Hopkins University, has been promoted to a professorship of applied mathematics.

Associate Professor J. R. Musselman, of Johns Hopkins, has been appointed professor of mathematics in the Woman's College of Western Reserve University.

Assistant Professor J. A. Northcott has been promoted to an associate professorship of mathematics at Columbia University.

Dr. H. P. Pettit has been appointed professor of mathematics at Marquette University.

Mr. H. S. Pollard has been appointed assistant professor of mathematics at Miami University.

Assistant Professor R. G. Putnam, of New York University, has been promoted to an associate professorship of mathematics.

Associate Professor L. J. Reed, of Johns Hopkins, has been promoted to a professorship of biometry and vital statistics.

Professor P. R. Rider, of Washington University, has been appointed Sterling Research Fellow at Yale for the year 1928–29.

Associate Professor H. M. Robert has been promoted to a professorship at the United States Naval Academy.

Assistant Professor H. P. Robertson, of the California Institute of Technology, has been appointed visiting assistant professor of mathematical physics at Princeton University for the year 1928–29.

Dr. C. F. Roos has been appointed assistant professor of mathematics at Cornell University.

Dr. W. E. Roth has been appointed assistant professor at the University of Wisconsin, in the university extension division at Milwaukee.

Dr. C. A. Rupp has been appointed assistant professor of mathematics at Pennsylvania State College.

Dr. Hazel E. Schoonmaker has been appointed assistant professor of mathematics at the New Jersey College for Women.

Assistant Professor H. A. Simmons has been promoted to an associate professorship at Northwestern University.

Assistant Professor Herman L. Smith has been promoted to an associate professorship of mathematics at Louisiana State University.

Mr. A. F. C. Stevenson has been promoted to an assistant professorship of mathematics at the University of Toronto.

Dr. M. H. Stone has been promoted to an assistant professorship of mathematics at Harvard.

Dr. Marian M. Torrey has been promoted to an assistant professorship of mathematics at Goucher College.

Dr. P. S. Wagner has been appointed professor of mathematics at Lebanon Valley College, Annville, Pa.

Dr. L. E. Ward has been promoted to an assistant professorship of mathematics at the University of Iowa.

Assistant Professor Warren Weaver, of the University of Wisconsin, has been promoted to a professorship of mathematics.

Dr. G. T. Whyburn has been promoted to an adjunct professorship of mathematics at the University of Texas.

Dr. Frederick Wood has been appointed professor of mathematics at Hamline University.

Miss Frances M. Wright has been appointed to an assistant professorship at Elmira College.

The following appointments to instructorships in mathematics are announced:

University of Arkansas, Mr. D. P. Richardson;

Brooklyn Polytechnic Institute, Mr. D. E. Whitford;
Bryn Mawr College, Mr. H. H. Pixley;
Cornell University, Mr. L. J. Paradiso;
Hunter College, Miss R. L. Anderson;
University of Illinois, Dr. Bessie I. Miller;
Johns Hopkins University, Dr. L. M. Blumenthal;
Lafayette College, Mr. J. H. Fithian;
Lehigh University, Dr. H. B. Hammatt, Messrs. Keeler and Hugh Stanley, Dr. W. J. Trjitzinsky, Mr. A. N. Van Arnam;
University of Michigan, Mr. F. S. Beale;
Oregon State Agricultural College, Miss Melva K. Thompson;
University of Rochester, Miss Rose A. Whelan;
Sweet Briar College, Miss Julia W. Bower;
Williams College, Mr. H. L. Dorwart;
Winthrop College, Miss Alice A. Grant;
University of Wisconsin, Mr. J. I. Vass.

Professor Luigi Bianchi, director of the Scuola Normale Superiore at Pisa, died June 6, 1928, at the age of seventy-two.

Professor Arthur Schoenflies, of the University of Frankfurt on Main, died May 27, 1928, at the age of seventy-five.

Professor Otto Staude, of the University of Rostock, is dead.

Professor Heinrich Weber, formerly professor of physics at the Braunschweig Technical School, is dead.

Professor R. E. Allardice, of Stanford University, died May 29, 1928, at the age of sixty-six. Professor Allardice had been a member of the American Mathematical Society since 1901.

Professor L. W. Dowling, of the University of Wisconsin, died on September 16, 1928, at the age of sixty-one. He had been a member of this Society since 1897.

Mr. J. H. Woodward, consulting actuary, of New York City, died May 15, 1928. Mr. Woodward was a member of the American Mathematical Society.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

APPELL (P.). Le problème géométrique des déblais et remblais. (Mémorial des Sciences Mathématiques, No. 27.) Paris, Gauthier-Villars, 1928.

BAUER (G.). See BIEBERBACH (O.).

BIEBERBACH (O.). Vorlesungen über Algebra. Unter Benutzung der dritten Auflage des gleichnamigen Werkes von Gustav Bauer in vierter vermehrter Auflage dargestellt. Leipzig, Teubner, 1928. 10+334 pp.

BIENIAS (M.). Untersuchung über die kollinearen Erzeugungen der Kummerschen Fläche. Hildesheim, Verlag Borgmeyer, 1928. 39 pp.

BORGER (R. L.). Analytic geometry. New York, McGraw-Hill, 1928. 334 pp.

BRASCH (F. E.), editor. Sir Isaac Newton, 1727–1927, a bicentenary evaluation of his work. Prepared under the auspices of the History of Science Society and cooperating societies. Introduction by D. E. Smith. Baltimore, Williams and Wilkins, 1928. 240 pp.

BURNSIDE (W.). Theory of probability. Edited, with a memoir, by A. R. Forsyth. Cambridge, University Press, 1928. 30+106 pp.

CASTELNUOVO (G.). Calcolo delle probabilità. Volume II. Bologna, 1928. 8+243 pp.

COTTON (E.). Approximations successives et équations différentielles. (Mémorial des Sciences Mathématiques, No. 28.) Paris, Gauthier-Villars, 1928.

FLADT (K.), herausgegeben von. Quellenhefte zur Elementarmathematik. Band 1: Elementargeometrie, von K. Fladt. Leipzig, Teubner, 1928. 73 pp.

FORSYTH (A. R.). See BURNSIDE (W.).

GAMBIER (B.). Déformation des surfaces étudiées du point de vue infinitésimal. (Mémorial des Sciences Mathématiques, No. 26.) Paris, Gauthier-Villars, 1928.

GUICHARD (C.). Les courbes de l'espace à n dimensions. (Mémorial des Sciences Mathématiques, No. 29.) Paris, Gauthier-Villars, 1928.

HILB (E.). See LAGALLY (M.).

KLIEM (F.). Apollonius. Berlin, Salle, 1927. 75 pp.

KOPEČNÝ (J.). Über die Gleichung $x^n+y^n=z^n$. Teil 1. Pressburg, Wigand, 1928. 16 pp.

LAGALLY (M.). Vorlesungen uber Vektor-Rechnung. (Mathematik und ihre Anwendungen, herausgegeben von E. Hilb, Band 2.) Leipzig, Akademische Verlagsgesellschaft, 1928. 18+358 pp.

LEVI-CIVITA (T.). Der absolute Differentialkalkül und seine Anwendungen in Geometrie und Physik. (Die Grundlehren der mathematischen Wissenschaften, Band 28.) Berlin, Springer, 1928. 11+310 pp.

LOBATSCHEVSKY CELEBRATION. Ad annum MCMXXVI centesimum a geometra kazaniensi N. J. Lobacewski noneuklideae geometriae systematis inventi concelebrandum. Kazan, Physico-Mathematical Society, 1927. 112 pp.

——In memorian N. I. Lobatschevskii, volume II: Collections des mémoires presentés par les savants de divers päys à la Société Physico-Mathématique de Kazan à l'occasion de la célébration du centenaire de la découverte de la géométrie non-euclidienne. Kazan, 1927.

MALET (H.). Exposé élémentaire du calcul vectoriel et de quelques applications. Avec une préface de Maurice d'Ocagne. Paris, Gauthier-Villars, 1928. 72 pp.

MALSCH (F.). Zahl und Raum. Band I-VIII. Leipzig, Quelle und Meyer, 1927.

NEWTON BICENTENARY. See BRASCH (F.E.).

D'OCAGNE (M.). See MALET (H.).

POPP (K.). Die Bedeutung der Mathematik und Astronomie für die Gegenwartskultur und ihre Entwicklung im Unterricht der höheren Schule. München, Oldenbourg, 1927. 61 pp.

SMITH (D. E.). See BRASCH (F. E.).

SYMON (A.). The new geometry. Part 3: Circles. Glasgow, Robert Gibson, 1928.

WEIDA (F. M.). The logarithmic slide-rule. A pamphlet to simplify instruction. New York, Macmillan, 1927. 7 pp.

WENZEL (A.). Galilei. Berlin, Salle, 1927. 72 pp.

PART II. APPLIED MATHEMATICS

ABBOTT (C. G.). See MITCHELL (S. A.).

VON ANGERER (E.). See WIEN (W.).

BAKKER (G.). See WIEN (W.).

BAXTER (H. E.). See YOUNG (G.).

BENNY (L. B.). Mathematics for students of technology. A senior course. London, Oxford University Press, 1927. 28+451 pp.

BERLAGE (H. P.). Versuch einer Entwicklungsgeschichte der Planeten. Leipzig, Akademische Verlagsgesellschaft, 1927. 68 pp.

BESSON (H.). See HOPPE (E.).

BOEGEHOLD (H.). Geometrische Optik. Berlin, Gebrüder Borntraeger, 1927. 375 pp.

BOLLNOW (O. F.). See EDDINGTON (A. S.).

BRAGG (W. H.). The structure of an organic crystal. London, Longmans, 1928. 32 pp.

DE BROGLIE (L.). See DE BROGLIE (M.).

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CASPARI (W. A.). The structure and properties of matter. London, Benn, 1928. 78 pp.

CHAZY (J.). La théorie de la relativité et de la mécanique celeste. Tome I. Paris, Gauthier-Villars, 1928.

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CODD (L. W.). See HESS (V. F.).

CREW (H.). The rise of modern physics. Baltimore, Williams and Wilkins, 1928. 356 pp.

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DUVAL (A. B.) et HEBRARD (L.). Traité pratique de navigation aérienne. 2e édition. Paris, Gauthier-Villars, 1928. 196 pp.

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FISHER (R. A.). Statistical methods for research workers. 2d edition, revised and enlarged. (Biological Methods and Manuals, No. 5.) Edinburgh, Oliver and Boyd, 1928. 12+269 pp.

FÖPPL (A.) und FÖPPL (L.). Drang und Zwang. München, Oldenbourg, 1928.

FÖPPL (L.). See FÖPPL (A.).

FORSTMANN (A.) und SCHRAMM (E.). Die Elektronenröhre. Ihre Theorie und ihre praktische Anwendung in Empfangs- und Verstärkerschaltungen. Berlin, Schmidt, 1927. 239 pp.

FORTRAT (R.). Introduction à l'étude de la physique théorique. 6e et 7e fascicules. Paris, Hermann, 1927. 100+71 pp.

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GRAETZ (L.). Handbuch der Elektrizität und des Magnetismus. Band V, Lieferung 2. Leipzig, Barth, 1928. 357 pp.

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GUNTHER (J.). (WALTER DE HAAS.) Was ist Magnetismus? Stuttgart, Franck'sche Verlagsbuchhandlung, 1927. 78 pp.

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DE HAAS (W.). See GUNTHER (H.).

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HARMS (F.). See WIEN (W.).

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HAY (A.). An introductory course of continuous current engineering. 3d edition. London, Constable, 1928. 13+383 pp.

HEBRARD (L.) See DUVAL (A. B.).

HESS (V. F.). The electrical conductivity of the atmosphere and its causes. Translated from the German by L. W. Codd. London, Constable, 1928. 18+204 pp.

HÖHN (E.). Über die Festigkeit der gewölbten Böden und der Zylinderschalen. Berlin, Springer, 1927. 223 pp.

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JOLLEY (L. B. W.). Alternate current rectification and allied problems. 3d edition, revised and enlarged. London, Chapman and Hall, 1928. 545 pp.

JOOS (G.). See WIEN (W.).

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LINKER (P. B. A.). Grundlagen der Wechelstromtheorie. Berlin, Stilke, 1928.

LUNNON (R. G.). New worlds for old. The realm of modern physics. London, Methuen, 1928. 5+106 pp.

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MELLANBY (A. L.). See UNWIN (W. C.).

METZ (A.). Temps, espace, relativité. Paris, Beauchesne, 1928. 212 pp.

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THE THIRTY-FOURTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY

The thirty-fourth summer meeting and twelfth colloquium of the Society were held at Amherst College, Amherst, Massachusetts, from Tuesday to Friday, September 4–7, 1928, preceded by the summer meeting of the Mathematical Association of America.

The colloquium lectures by Professor A. B. Coble were delivered on Tuesday and Wednesday mornings and Thursday afternoon with seventy-seven members in attendance; a report of these lectures, by Professor H. S. White, will appear in an early issue of this Bulletin. On Thursday morning the Society held sectional sessions for the reading of papers, a Section of Geometry and a Section of Algebra, Analysis, and Foundations. General sessions were held on Thursday afternoon and Friday morning. The joint dinner of the Society and the Association, with Ex-President Olds of Amherst, as toastmaster, was held at the Lord Jeffery Inn on Wednesday evening, with about 150 persons present.

During the latter part of Tuesday afternoon, visiting mathematicians and their friends enjoyed an excursion to Mount Holyoke College where they were entertained at tea; and Wednesday afternoon was devoted to an automobile excursion up the beautiful valley of the Connecticut River. President and Mrs. Pease tendered a reception to the guests on Tuesday evening at the Lord Jeffery Inn. The college dormitories were opened for the visitors, and meals were furnished at the Lord Jeffery Inn near the campus, all at very modest rates. The college and town placed their recreational facilities at the disposal of the mathematicians, a feature which was much appreciated.

Resolutions were passed expressing profound appreciation of the cordial hospitality of the College, and especially of its officers, President Pease, Ex-President Olds, and Dean

Esty, whose kindness in making the remarkably efficient arrangements for the comfort and convenience of the visitors met with praise on every hand. The beautiful surroundings of the venerable college was another of the features which made this meeting one of the most pleasant in the Society's history.

Greetings were cabled from the Society to the International Mathematical Congress meeting simultaneously at Bologna and a reply was received reciprocating the felicitations.

The attendance included the following one hundred twenty-eight members of the Society:

C. R. Adams, R. B. Adams, Allison, Anderton, Bacon, A. A. Bennett, Theodore Bennett, Blichfeldt, G. A. Bliss, Blumberg, Bradshaw, Brinkmann, H. S. Brown, Bullitt, Cairns, B. H. Camp, C. C. Camp, G. A. Campbell, Carlen, F. E. Carr, Coble, Abraham Cohen, Coolidge, A. H. Copeland, L. P. Copeland, Crathorne, Crawley, Currier, Darkow, H. T. Davis, Decker, Dederick, Dines, Doak, DoBell, Dorwart, Douglas, Dresden, Esty, Femler, W. B. Ford, Fort, Fry, Gehman, D. C. Gillespie, J. W. Glover, V. G. Grove, M. G. Haseman, E. R. Hedrick, Hickson, Huber, Hurwitz, J. I. Hutchinson, Louis Ingold, Ingraham, Dunham Jackson, Jeffery, M. I. Johnson, R. A. Johnson, Karnow, Kempner, Langer, Langman, Latimer, Leib, Lotka, MacColl, MacDuffee, MacMillan, Maizlish, H. P. Manning, Martin, Michie, W. E. Milne, Mode, C. L. E. Moore, T. W. Moore, Moriarty, Richard Morris, F. H. Murray, Neelley, Oakley, Olds, F. W. Perkins, L. R. Perkins, Phalen, Pierpont, C. S. Porter, Ransom, C. J. Rees, R. G. D. Richardson, D. E. Richmond, Rider, H. L. Rietz, Ritt, Robison, E. D. Roe, Schelkunoff, Schoonmaker, Seely, Shaub, Sheffer, Simons, Slobin, Slotnick, A. H. Smith, C. E. Smith, S. E. Smith, W. M. Smith, Sprague, Stetson, Struik, Tamarkin, Eugene Taylor, Thurston, Torrey, Tracey, L. E. Ward, J. H. Weaver, Warren Weaver, M. E. Wells, A. H. Wheeler, R. A. Whelan, H. S. White, Wiener, E. W. Wilson, W. A. Wilson, Yeaton.

The Secretary announced the election of the following forty-four persons to membership in the Society:

Mr. William Vinton Bond, University of Gottingen;
Mr. Moffatt Grier Boyce, Western Reserve University;
Professor Marguerite D. Darkow, Pennsylvania State College;
Professor Prentice Dearing Edwards, Ball Teachers College, Muncie, Indiana;
Mr. Carl M. Erikson, Michigan State Normal College;
Mr. William Welch Flexner, Princeton University;
Professor Robert Jeffrey Hannelly, Phoenix Junior College, Phoenix, Arizona;

Dr. Harry Gordonson Kaplan, Stuyvesant High School, New York City;
Mr. Herman Karnow, University of Colorado;
Mr. Akitsugu Kawaguchi, Hokkaido Imperial University, Sapporo, Japan;
Miss Dorothea Agatha Kern, Swarthmore College;
Miss Edna E. Kramer, Wadleigh High School, New York City;
Dr. Horace C. Levinson, Chicago, Ill.
Assistant Professor Edward James McShane, Wichita, Kansas;
Professor Tatsuo Mori, Naval Academy, Yedajima, Hiroshima-Ken, Japan;
Mr. David Clarence Morrow, University of Chicago;
Mr. Cletus Odia Oakley, University of Illinois;
Professor Kurt Edward Rosinger, Pennsylvania State College;
Mr. Robert Clarence Shook, University of Chicago;
Dr. Francis Raymond Stark, Western Union Telegraph Company, New York City;
Dr. Marie Johanna Weiss, Stockton, California;
Mr. Yue Kei Wong, University of Chicago;

Nominees of the Mathematical Association of America:
Dr. H. W. Bailey, University of Illinois;
Mrs. Mary Hegeler Carus, LaSalle, Illinois;
Dr. C. C. Craig, University of Michigan;
Mr. W. C. Risselman, University of Minnesota;
Professor C. C. Wylie, University of Iowa.

Nominees of the General Electric Company:
Dr. E. J. Berg, Mr. D. H. Harms, Mr. M. L. Henderson, Mr. A. Howard, Mr. W. C. Johnson, Jr., Dr. I. Langmuir, Mr. J. K. Leibing, Mr. R. G. Lorraine, Mr. R. H. Park, Mr. V. Petrovsky, Mr. D. C. Prince, Mr. W. F. Skeats, Mr. J. C. Smith, Mr. J. J. Smith, Mr. I. A. Terry, Mr. L. Wetherill;

Nominee of the Prudential Insurance Company:
Mr. Valentine Howell.

Five applications for membership were received.

It was announced that Professor G. H. Hardy, of the University of Oxford, has accepted the invitation of the Society to give the sixth Josiah Willard Gibbs Lecture in connection with the Annual Meeting in New York City, his subject being *An introduction to the theory of numbers.*

The following appointments were announced: to represent the Society at the inauguration of President F. B. Robinson, of the College of the City of New York, Professor T. S. Fiske; as Committee on Arrangements for the sixth Gibbs Lecture, Dean H. E. Hawkes and Professor G. W. Mullins; to represent the Society at the International Congress in Bologna, Professors R. C. Archibald, G. D. Birkhoff, H. F. Blichfeldt, Edward Kasner, Oswald Veblen, and Virgil Snyder; as Committee on Arrangements for the summer meeting of 1929, Professors A. J. Kempner (chairman), E. L. Dodd, C. A. Hutchinson, R. G. D. Richardson, and C. H. Sisam.

The invitations from Brown University for the summer meeting of 1930 and from the University of Minnesota for the summer meeting of 1931 were accepted.

It was announced that for 1928–29 the National Academy of Sciences had allotted to the Society from funds donated by the General Education Board the sum of $3500, to be distributed as follows: Bulletin, $500; Transactions, $1500; American Journal of Mathematics, $1500.

A list of nominations for trustees, and officers and other members of the Council was adopted and ordered printed on the ballot.

The Secretary announced that, in response to the request for copies of No. 6 of volume 33 of the Bulletin, members had volunteered to contribute more than enough to make up the shortage.

To meet the deficit for the current and succeeding years, Professor R. C. Archibald has procured a list of more than one hundred subscribers to a special fund, the amount per year being over thirteen hundred dollars. It was voted to extend to him the hearty thanks of the Council for his timely and untiring efforts in behalf of the finances.

The Council voted to thank Professor Clara E. Smith for her able efforts in the campaign of the Membership Committee.

At a joint meeting of the Trustees of the Mathematical Association and the Council of the Society, there was an informal discussion of various international projects which have been suggested for consideration.

Titles and abstracts of the papers read at this meeting follow below. The papers numbered 1 to 16 were read before the Section of Geometry, Professor Coolidge presiding; those numbered 17–36 before the Section of Algebra, Analysis, and Foundations, Professor Jackson presiding; those numbered

37 to 45 at the general session on Thursday afternoon, Professor Glover presiding; and those numbered 46 to 70 at the general session on Friday morning, Vice-President Rietz presiding. Professor Errera was introduced by Associate Secretary Dresden, Mr. Feenberg by Professor Ettlinger, and Professor Hufford by Professor Davis. The papers of Altschiller-Court, Ayres, Craig, Dorroh, Edmonson, Ettlinger, Feenberg, Graustein, Gronwall, Hightower, Hollcroft, Kasner, Ketchum, Knebelman, Lane, Lubben, Michal, R. L. Moore, Morse, Putnam, Sharpe, Sheffer (third paper), Simmons, Thomas, Walsh, Whyburn, and N. R. Wilson were read by title.

1. Professor C. L. E. Moore: *Linear complexes of curves in Riemannian space.*

The complex of curves defined by the equation $\sum \lambda_i dx^i = 0$ has many of the properties of hypersurfaces. In this paper the author investigates some of these similarities, including parallelism, geodesics, normal curvature, mean curvature, geodesic curvature, and torsion. There is a first and a second fundamental form, and consequently Gauss-Codazzi relations, together with the results that depend upon them.

2. Dr. Jesse Douglas (National Research Fellow): *Determination of all descriptive and voluminar differential invariants of a space of K-spreads.*

This paper refers to the author's previous paper, *The geometry of systems of K-spreads*, abstract in this Bulletin, Jan., 1928. The problem is to find all functions of the components of affine connection Γ^i_{jk} and their partial derivatives which are unchanged (1) under an arbitrary transformation of parameters on the K-spreads (descriptive), (2) under a transformation of parameters with constant jacobian (voluminar). It is found that the most general descriptive invariant is a function of a certain fundamental one, Π^i_{jk}, and its derivatives, and the most general voluminar invariant is a function of two fundamental ones, Γ^a_{ak}, V^i_{jk}, and their derivatives.

3. Professor S. A. Schelkunoff: *On rotations in ordinary and null spaces.*

The author gives a new proof that the characteristic roots of the ordinary group of rotations are exponential functions of the angles of component simple rotations. He then studies the quasi-orthogonal group. The characteristic roots are still unit complex numbers, but do not, in general, occur in conjugate pairs. Invariant "axes" of quasi-rotations form a quasi-orthogonal set of lines whose direction components are readily determined; the same is true of ordinary rotations as a special case. After the problem of decomposition of general rotations and quasi-rotations into simple component rotations is solved, the author passes to the converse problem: to determine the coefficients of a quasi-orthogonal transformation if the direction components of the invariant axes and the corresponding angles of rotations are given. Ordinary rotations are included as a special case. Necessary and sufficient conditions for a set of lines to be a set of invariant axes are then formulated.

4. Professor J. H. Neelley: *Concerning covariant forms of the rational plane quartic curve with compound singularities.* Second paper.

This paper considers the effects of the tacnode and the oscnode upon covariant forms of the rational plane quartic curve. The curve is treated parametrically, and binary forms which give covariant sets of parameter values are observed. Many cases of multiple roots occur and the corresponding special relations of curve and covariant are discussed. Some curves degenerate and some forms vanish identically. Special attention is given the complete systems of the fundamental pencils of binary quartics, as well as combinants of pencils of line sections for each type of curve.

5. Professor Edward Kasner: *Note on the derivative circular congruence of a polygenic function.*

This paper appeared in full in the September-October issue of this Bulletin.

6. Professor W. C. Graustein: *On the trisectors of the angles of a triangle.*

A projective method of proof of the theorem in elementary geometry to the effect that the triangle formed by the points of intersection of adjacent trisectors of the internal angles of any triangle is equilateral uncovers 27 equilateral triangles each of which has as vertices points of intersection of trisectors, "internal," "intermediate," or "external," of the given triangle. The 27 equilateral triangles fall into three sets of nine each, and each set of nine into three triads. The vertices of the triangles of a set are, taken collectively, the same for every set, and constitute the 27 finite intersections of three triads of parallel lines. Every two triangles of a triad are perspective, the three centers of perspective of the pairs of triangles of a triad lie on a line, and the three lines of centers of perspective of the three triads of a set go through a point. The three lines are equally spaced about the point in the case of each of two of the three sets, and in the case of the odd set, they are perpendicular to the sides of the given triangle. Finally, the three points of intersection of the lines of centers of perspective for the three sets are collinear.

7. Professor W. C. Graustein: *A geometric problem in which order of choice is important.*

The answer to the question as to whether, when a collineation between two distinct planes in space is given, it is always possible to move the planes rigidly into new positions in which the lines joining corresponding points go through a point, depends on an order of choice. If a metric space is assumed from the start, and a collineation between two finite planes in it is then chosen, the question must be answered in the negative. If the point of departure is a projective space in which there is given a projective correspondence between the points of two planes, it is always possible to introduce the metric (which is necessary to make rigid motions possible) so that the question can be answered affirmatively.

8. Professor Nathan Altshiller-Court: *On five mutually orthogonal spheres.*

The following are some of the results obtained in this paper: (1) If five spheres are mutually orthogonal, each of them is the conjugate sphere of the orthocentric tetrahedron determined by the centers of the remaining four. (2) Conversely: the five conjugate spheres of the five tetrahedra determined by an orthocentric group of five points in space are mutually orthogonal. (3) The sum of the squares of the radii of the five spheres determined by the centers of five given mutually orthogonal spheres is equal to twice the sum of the squares of the radii of the given spheres. (4) The ratio of the square of the radius of one of five mutually orthogonal spheres to the reciprocal of the volume of the tetrahedron determined by the centers of the remaining four is constant in absolute value. (5) The sum of the reciprocals of the squares of the radii of the ten circles along which five such spheres cut is zero. (6) The sum of the squares of the distances of the centers of five such spheres from the center of gravity of these five points is equal to four fifths of the sum of the squares of the radii of the spheres. (7) The tetrahedral pole of a plane of similitude of four of the spheres with respect to the tetrahedron determined by their centers coincides with the pole of the same plane with respect to the fifth sphere.

9. Dr. M. S. Knebelman (National Research Fellow): *Groups of motion in related Riemann spaces.*

By considering the existence of r linearly independent solutions of Killing's equations, it is shown that if a space admits an r-parameter group of motions whose covariant components are $\xi_i^{(1)}, \cdots, \xi_i^{(r)}$, then every space in geodesic correspondence with it admits an r-parameter group of motions whose components are $e^{2\phi}\xi_i^{(1)}, \cdots, e^{2\phi}\xi_i^{(r)}$, ϕ being the function whose gradient determines the geodesic correspondence. From this theorem that of Beltrami on spaces of constant curvature follows as a simple corollary. By considering the same equations for two conformal spaces, it is shown that if one of them admits an r-parameter group of motions, the other admits the same group of motions.

10. Professor F. R. Sharpe: *The problem of plane involutions of order $t>2$.*

In this paper it is shown that there are three ways of defining a plane involution I_t of order t: (1) by the curves of a net of genus p, (2) by the curves of two projective nets, (3) by a pencil of curves of genus p, and the algebraically corresponding curves of a linear system of dimension r. The method (3) gives, for $r>2$, families of involutions which exist for each $p\leqq\frac{1}{2}(t-1)(t-2)$. The method (2) gives single types of I_t which exist if, when they are reduced to the form (1), $p\leqq\frac{1}{2}t(t-1)$. The method (1) gives single types with $p\leqq 2t-2$. A fairly complete study is made for $t=3$ and a less complete discussion is given for $t=4$. The mapping of I_t on S_r, $r\geqq 3$, is also considered. The paper will appear in the American Journal of Mathematics.

11. Professor T. R. Hollcroft: *Multiple lines with fixed coincident tangent planes on algebraic surfaces.*

Noether derived a formula for the postulation of a multiple line of any given multiplicity on an algebraic surface of given order, but did not treat the cases in which some of the tangent planes to the surface through the line become coincident. The plane section of an algebraic surface containing an i-fold line is a curve with an i-fold point. An algebraic surface can have as many varieties of i-fold lines as an algebraic plane curve has varieties of i-fold points. In this paper, there is found the postulation of a line of given multiplicity on an algebraic surface of given order such that the plane section of the surface at the line is any algebraic singularity whose coincident tangents are fixed lines.

12. Professor E. P. Lane: *Canonical configurations associated with a surface.*

This paper appears in full in the present issue of this Bulletin.

13. Professor V. G. Grove: *Contributions to the general theory of transformations of nets.*

In this paper the author considers further generalizations of the theory of transformations F, K, and Ω of conjugate nets to the general net. In particular he discusses the part that the projective invariant C of Graustein plays in the general theory. The products of transformations C of general nets are considered. The concept of triads of nets is capable of generalization to general nets. Certain of the theorems derived are true generalizations of Slotnick's theorems relative to products of transformations F of conjugate nets. Others are true only for non-conjugate nets.

14. Dr. M. M. Slotnick: *On the projective differential geometry of conjugate nets.*

The analytical work connected with the study of conjugate nets in a projective 3-space can be made symmetrical by introducing a point on the axis congruence associated with the net. The point chosen is the harmonic conjugate of the point of the net with respect to the two focal points of the axis. The complete duality between the axis and ray congruences is analytically evident in a simple way. The results of Wilczynski and Green are obtained very readily. Relations between the point and tangential invariants of a net are found by means of which special types of nets are identified in simple analytical manner.

15. Dr. M. M. Slotnick: *Semi-parallel maps of lines of curvature.*

If the lines of curvature of two surfaces are mapped by a semi-parallel map as defined by Graustein (Annals of Mathematics, (2), vol. 27, pp. 271–278), the surfaces are either surfaces of Monge or molding surfaces whose director-developables are either in a Combescurian relation or identical.

16. Professor H. A. Simmons: *Affine differential geometry of curves and surfaces.* Preliminary communication.

Two results have been obtained in this paper. (i) We have found a complete system of invariants and covariants of plane curves under affine transformations, and have made geometrical interpretations of these invariants and covariants. (ii) We have found a complete system of seminvariants of ruled surfaces under affine transformations. In obtaining result (ii), an incomplete system of partial differential equations was encountered, the completion of which was laborious; in the corresponding projective theory one meets complete systems only. We hope to continue the affine treatment of both curves and surfaces.

17. Professor W. A. Wilson: *On irreducible cross-cuts of plane simply, connected regions.*

If R is a simply connected plane region whose frontier K is bounded, H is a closed subset of $\bar{R}$, m and n are points of $R-R\cdot H$, and H separates m from n in R but no closed proper part of H has that property, H is called an irreducible cross-cut of R between m and n. If also H does not separate m from n in the plane, it is called regular. If H is an irreducible cross-cut of R between m and n, then it is a part of the frontiers of the components of $R-R\cdot H$ containing m and n and $H=\overline{R\cdot H}$; and conversely. The principal results are as follows. If H is a bounded regular irreducible cross-cut of R between each pair of n points, $H\cdot K$ is the sum of at least n closed sets and H is a continuum irreducible between each pair of them. If $H\cdot K$ is the sum of n closed sets between each pair of which H is an irreducible continuum, there are at least n components of the complement of $H+K$ whose frontiers contain H as a proper part.

18. Professor R. G. Lubben: *Concerning limiting sets in abstract spaces.*

The author discusses theorems concerning limiting sets, some of which are extensions of results presented at an earlier meeting of the Society (see this Bulletin, vol. 32 (1926), p. 14). He considers, in particular, necessary and sufficient conditions, both with reference to the collections of point sets and to the spaces in which they are contained, that the limiting sets of such collections should have certain prescribed properties. A property of especial interest is the *distributive property*: A space is said to have the *distributive property* provided that if in that space K is a closed point set and G is a collection of point sets and each point of K belongs to some subset of K which is the limiting set of some sub-collection of G, then K itself is the limiting set of a sub-collection G_1 of G. If it be specified in addition that G_1 be countable, the space is said to have the *countably distributive property*. (1) Every regular Hausdorff space having the distributive property and satisfying the first countability axiom is locally compact. (2) In order that a metric space have the countably distributive property it is necessary and sufficient that the space be locally compact and separable.

19. Professor C. G. Latimer: *A generalization of Eisenstein's canonical cubic and associated forms.*

In this paper we shall consider a certain canonical cubic and associated forms. These forms are defined in exactly the same way as Eisenstein's like-named forms, except that instead of employing a prime q, in the form $3n+1$, we shall use an integer D, which contains no square factor, every prime factor of which is in the form $3n+1$. We shall find that if a is prime to a certain integer, then a necessary and sufficient condition that there exist an associated form belonging to a is that every rational prime factor of a be the norm of a prime ideal in a certain cubic field with the discriminant D^2. When D is a prime, our results are equivalent to Eisenstein's.

20. Dr. I. M. Sheffer (National Research Fellow): *A proof of the fundamental theorem of algebra.*

The proof here given, which is believed to be new, is not "elementary," since it makes use of the implicit function theorem and analytic continuation. Yet it may be termed simple, and may possess features of interest.

21. Mr. F. J. Feinler: *A reduced Bernoulli polynomial and its properties.*

From the initial relation $(B+1)^n-B^n=0$, $n\geqq 2$, is derived a set of shortened polynomials. The coefficients show remarkable characteristics, connecting them with Luca's échiquier of V_n.

22. Dr. W. L. Ayres (National Research Fellow): *Concerning the avoidable points of continua.*

Urysohn (Mathematische Annalen, vol. 98(1927), p. 301) and Zarankiewicz (Bulletin de l'Académie Polonaise, 1927, p. 194) have defined avoidable points of sets. From each of these *strong* properties we may obtain a *weak* property by substituting connected sets for continua. It is shown in this paper that the Urysohn strong and Zarankiewicz weak properties are independent; and both are weaker than the Zarankiewicz strong property and stronger than the Urysohn weak. A point P of a continuum M has the Urysohn weak [strong] property if and only if (1) M is connected im kleinen at P, (2) P is not a strong [weak] cut point im kleinen of M.

If every point of a plane continuum M has the Urysohn weak property, then (1) M is a cyclically connected continuous curve such that no two complementary domains of M have a boundary point in common (and conversely), and (2) no countable subset of M cuts M. If the point P of a continuum M has the weak Urysohn property with respect to every subcontinuum of M, then P has the strong Urysohn property with respect to every subcontinuum of M.

23. Dr. W. L. Ayres: *On the end points of continua.*

It is proved that a point P of a continuous curve M is an end point of M if and only if P has the Urysohn weak (or strong) property with respect to every subcontinuum of M. An example is given to show that this is not true for either the strong or weak Zarankiewicz properties. This new definition of an end point is examined for its relations in a general continuum to the definitions studied by Gehman (Transactions of this Society, vol. 30(1928), pp. 63–84). It is shown that the set of points of a bounded continuum having Gehman's property 6 is totally disconnected, and an example is given to show that in space of more than two dimensions Gehman's properties 3 and 6 are independent.

24. Dr. W. L. Ayres: *Conditions under which every arc of a continuous curve is a subset of a maximal arc of the curve.*

Let M denote a bounded plane continuous curve. In order that every arc of M should be a subset of a maximal arc of M either of the following conditions is necessary and sufficient: (1) the set of non-cut points of M of Menger order two should contain no continuum; (2) if α is any arc-segment of M (that is, an arc minus its end points) which is an open subset of M, then every point of α should be a cut point of M. In order that every arc of M which can be extended in M in both directions should be a subset of an arc of M joining two end points of M, either of the following is necessary and sufficient: (1) every point of M which is on the boundary of some complementary domain of M should be a limit point of the cut points of M, (2) every point of M which belongs to some simple closed curve that lies in the boundary of some complementary domain of M should be a limit point of the end points of M. We shall say that an arc β of M can be *extended in both directions in* M if there exists an arc γ of M such that the arc β is a subset of the arc-segment of γ.

25. Professor R. G. Putnam: *On the separability of classes* (E).

R. L. Moore has obtained (Fundamenta Mathematicae, vol. 8, p. 189) theorems concerning the relation between the separability of a class (D) of Fréchet and the proposition that every uncountable point set has a limit point. In the present paper classes (E) are considered, and similar theorems are obtained.

26. Mr. J. L. Dorroh: *Concerning a set of axioms for the semi-quadratic geometry of a three-space.*

In his paper *Sets of metrical hypotheses for geometry* (Transactions of this Society, vol. 9(1908), pp. 487–512), R. L. Moore raised the question whether the semi-quadratic geometry of a three-space is a consequence of the set O of order axioms and the set C of congruence axioms employed therein, together with M, the proposition that every segment has a midpoint, and P_2, a form of the parallel axiom. He stated that this question can be answered in the affirmative if it can be shown that in every space satisfying O, C and M, all right angles are congruent to each other. In the present paper it will be shown that all right angles are congruent to each other in every such space. In view of a recent result of the author's, it follows that three-dimensional semi-quadratic geometry follows from O, C, and P_2.

27. Professor R. L. Moore: *On the structure of a continuum.*

For each point P of the bounded plane continuum M, let M_{P_1} denote the set of all points X of M such that X is not separated from P in M by uncountably many distinct points of M, let M_{P_2} denote the set of all points X such that X is not separated from P in M by c mutually exclusive point sets each consisting of finitely many points of M, let M_{P_3} denote the set of all points that can be joined to P by a connected subset of M not containing uncountably many points every two of which are separated in M by a finite point set, and let M_{P_4} denote the component containing P of the product of all point sets g such that g contains P and is an element of some upper semi-continuous collection of mutually exclusive continua which fills up M and which is a continuous curve with respect to its elements. For each i let G_i denote the collection of all M_{P_i}'s for all points P of M. Every G_i is a continuous curve of elements, G_1 being dendritic and G_3 regular. The elements of G_4 will be called graphatomic subsets of M.

28. Professor G. T. Whyburn: *Concerning the cyclic elements of continuous curves.*

The cyclic elements of a continuous curve M of classes (a), (b), and (c) are, respectively, the maximal cyclic curves, cut points, and end points of M. In this paper it is shown that for each continuous curve M in n-space, a plane continuous curve K exists such that there is a continuous (1,1) correspondence, preserving class, between the cyclic elements of M and the cyclic elements of K. There does not exist a general continuous curve K such that, for every continuous curve M, a continuous (1,1) correspondence, preserving class, exists between the cyclic elements of M and the cyclic elements of some "subcontinuum of the elements of K." Let M be any continuous curve in n-space. Then (1) if no two maximal cyclic curves of M have a common point, every two points of M not belonging to the same maximal cyclic curve are separated in M by uncountably many points of M; (2) a subset C of M is a connected collection of cyclic elements of M if and only if for each connected subset H of M, $C\cdot H$ is vacuous or connected; and (3) if C is such a collection, and N is any component of $M-C$, then $\overline{C}\cdot\overline{N}\equiv$one point.

29. Professor R. U. Hightower: *On the classification of the elements of a ring.*

A method is presented in this paper for classifying the elements of an abstract ring R. The set C of the classes $C_0, C_1, \cdots$ forms a ring. C contains a subring $\overline{C}$, whose elements, taken together, form a subring $\overline{R}$ of the ring R. The class C_0 is an ideal of R that is in $\overline{R}$. The necessary and sufficient condition that C_0 be the conductor of $\overline{R}$ relative to R is that $\overline{C}$, which is the ring of classes of $\overline{R}$ modulo C_0, contain no proper subring C' that is invariant in C. To each ideal in R prime to C_0 there corresponds a regular ideal in $\overline{R}$, and conversely. Hence the conditions established by Masazo Sono for unique factorization of ideals in R hold for ideals in $\overline{R}$. Elements are assumed to be abstract. But if R, considered as an infinite abelian group with respect to addition, has a finite base, and if $\overline{R}$ is a subgroup of R of finite index, $\overline{R}$ is a ring defined by an algebraic equation.

30. Professor N. R. Wilson: *On finding ideals.*

This paper is a continuation of a paper in volume 29 of the Transactions of this Society, on finding the basis of an algebraic number field. A similar canonical basis is set up and operations with ideals so expressed are discussed. Congruences for finding the factors of $[p]$, given the field, are then developed, illustrated by the cubic.

31. Professor H. A. Simmons: *On a cyclo-symmetric diophantine equation.*

In this paper we first show that the cyclo-symmetric equation, (1): $1/(x_1x_2 \cdots x_r)+1/(x_2x_3 \cdots x_{r+1})+ \cdots +1/(x_nx_1 \cdots x_{r-1})=1/a$, where a is an integer $\geqq 1$, has the solution $(x_1, x_2, \cdots, x_n)\equiv(w_1, w_2, \cdots, w_n)$, where $w_i=1$, $(i=1, 2, \cdots, r-1)$, $w_r=a+1$, $w_i=aw_1 \cdots w_{i-r}+1$, $(i=r+1, r+2, \cdots, n-1)$, $w_n=aw_1 \cdots w_{n-r} (1+w_{n-r+1}+w_{n-r+1}w_{n-r+2} + \cdots +w_{n-r+1}w_{n-r+2} \cdots w_{n-1})$. Next we show if $n>r$, the equation obtained from (1) by merely changing its right member to b/a, where a, b are relatively prime positive integers, and $a=(m+1)b-1$, m being a positive integer, has a solution very much like the one written above. Then we prove special properties of our solutions which cause us to anticipate that they contain the maximum numbers x that can appear in any solutions of the two equations which we consider.

32. Professor L. L. Dines: *The resultant of two power series in two variables.*

The resultant of two homogeneous polynomials $f(x,y)$, $g(x,y)$ is a certain rational integral function of their coefficients, whose properties are well known. In the present paper the notion of the resultant is extended to two formal power series $f=f_m+f_{m+1}+\cdots$, $g=g_n+g_{n+1}+\cdots$, where f_i and g_i are homogeneous polynomials of degree i in two variables. Many of the properties of the resultant in the algebraic case have analogues in the transcendental case, and these are developed. There are applications in the problem of the analytic implicit functions defined by two equations $F(x, y, z_1, z_2, \cdots, z_s)=0$, $G(x, y, z_1, z_2, \cdots, z_s)=0$.

33. Professor L. L. Dines: *Linear integral inequalities of the first kind.*

In earlier papers (Transactions of this Society, vol. 30, p. 425, and this Bulletin, vol. 33, p. 695) the author has considered the linear integral inequality $\phi(x)+\int_a^b k(x,s)\phi(s)ds>0$, and its generalization. The inequality $\int_a^b k(x,s)\phi(s)ds>0$ presents greater difficulty, as is to be expected in view of the known theory of integral equations of the first kind. It does not possess a relationship to its adjoint integral equation $\int_a^b \phi(s)k(s,x)ds=0$ similar to that possessed by the inequality previously studied to its adjoint equation. If, however, the inequality symbol $>$ be replaced by the modified symbol $>'$ (meaning "is somewhere greater than and nowhere less than"), the resulting inequality of the first kind admits, at least for restricted classes of kernels, a relationship to its adjoint equation quite similar to that possessed by the inequality previously studied. The theory is extended in the sense of general analysis to the inequality $Jk\phi>'0$.

34. Professor A. H. Copeland: *Admissible numbers in the theory of geometrical probability.*

Geometrical probability is concerned with a set of related events having the power of the continuum. Thus in order to prove the consistency of the fundamental assumptions of geometrical probability, it is necessary to show the existence of a corresponding set of related admissible numbers. Let $\pi(E)$ be the probability that a point, P, of an n-dimensional continuum, belong to a given set, E, where $\pi(E)$ is an absolutely additive function. Then corresponding to every set E it is required to find an admissible number, $x(E)$, belonging to the set $A[\pi(E)]$. The events, $x(E)$, are related as follows. The function $x(E)$ must be absolutely additive and such that $x(E_1)\cdot x(E_2)=0$ whenever $E_1\cdot E_2=0$. Finally there must exist a fundamental domain, Δ, such that $x(\Delta)=1$. It is proved in this paper that the above restrictions upon the function $x(E)$ cannot all be satisfied for every set E, but that these conditions can be satisfied for a very general type of sets E. Thus the fundamental assumptions of geometrical probability are consistent provided we apply them to this type of sets E.

35. Professor A. H. Copeland: *Independent event histories.*

The condition that a set of numbers (or event histories) be independent is expressed by a certain set of equations which must be satisfied by these numbers. Thus, in a sense, independence is the exception and not the rule. On this account we might expect that, whereas the set of admissible numbers has the power of the continuum, a set of independent numbers must be finite or at most denumerable. However, this is not the case. It is proved in this paper that there exists a set, E, of independent admissible numbers such that each of the sets $E\cdot A(p)$, where $0<p<1$, has the power of the continuum, $A(p)$ being the set of admissible events associated with the probability p, and $E\cdot A(p)$ being the logical product of the sets E and $A(p)$. From this point of view independence is not exceptional.

36. Dr. F. W. Perkins: *A note on certain continuous non-differentiable functions.*

This note contains a discussion of some properties of a class of functions of which an example of a special type has already been given by the author under the title *An elementary example of a continuous non-differentiable function* (American Mathematical Monthly, vol. 34(1927), pp. 476 ff.).

37. Professor A. Errera: *A theorem about graphs and an application to the theory of numbers.*

Certain "incomplete" graphs of the third degree (that is, line graphs whose 0-cells are incident with one or three 1-cells) can be factored into a graph of first degree (0-cells incident with one 1-cell) and an incomplete graph of the second degree (0-cells incident with one or two 1-cells). The number of factorizations obeys certain laws, and this method can be used to prove a known theorem on Fibonacci numbers $1, 3, 4, 7, 11, \cdots$.

38. Professor E. R. Hedrick: *Geometric representations of fundamental quantities for non-analytic functions.*

In this paper, the author shows how to represent geometrically the values of the usual fundamental quantities E, F, G, and the value of the jacobian J, for a non-analytic function of a complex variable. These representations lead to geometric proofs of theorems regarding the fundamental quantities, including some whose proofs are ordinarily given otherwise, and some which are new. Thus, the well known theorem regarding the meaning of the equations $E=G$, $F=0$ appears from the geometric figures.

39. Professor D. J. Struik: *Correlation and coherency from the group standpoint.*

Correlation theory in statistics is essentially a study of a second degree matrix under certain rotational or affine transformations. These conceptions can be generalized to so-called coherency matrices, of which a very general type has been shown to be useful in quantum mechanics.

40. Dr. T. H. Gronwall: *On the wave equation of the hydrogen atom.*

This note gives a quite elementary derivation of the non-relativistic set of energy levels and characteristic functions of the hydrogen atom, and it is also proved that the set obtained is complete.

41. Dr. G. A. Campbell: *The practical application of the Fourier integral.*

The growing practical importance of transients and other non-periodic phenomena makes it desirable to simplify the application of the Fourier integral and to extend the range of problems which the practical man can solve in closed form by this method. To this end the known evaluations of Fourier integrals should be brought together in a table. A beginning has been made in the compilation of such a table, the results being presented, however, not as definite integrals but as paired functions, one function being the coefficient for the cisoidal oscillation, $\exp(i2\pi ft)$, and the other function the reciprocally related coefficient for the unit impulse. This arrangement simplifies practical applications, since it is the coefficients of which immediate use is made. Many pairs of coefficients of great importance in practical applications exist only as the limits of pairs which are regular in the sense of involving only convergent Fourier integrals. In the compilation a sharp distinction is drawn between these two types of pairs. A natural introduction to pairs seems to be supplied by the parabolic cylinder functions and the Gram-Charlier expansion.

42. Professors H. T. Davis and M. E. Hufford: *Note on diffraction and the wave theory of light.*

At the April meeting of the Society in Chicago, calculations based on the Lommel theory of diffraction through a circular orifice were presented. A comparison was made between the calculated and experimental values of the radii for a pattern of sixty-six rings. While the mathematical theory predicted the broadening and darkening of the outer rings of the plate, and theory and experiment were in general agreement, individual radii showed as much as 15 per cent error. A re-check of the experiment reveals an error in the wave length used in the first plate, and the agreement between experiment and calculation is now reduced to a negligible per cent. The mathematical interest is in the asymptotic formulas found for the Lommel functions.

43. Professor Marston Morse: *Singular points of vector fields under general boundary conditions.*

Earlier results of Brouwer, Hopf, Lefschetz, Alexander, Birkhoff, and others, in so far as they dealt with vector fields under boundary conditions assume that the vectors are either all directed exteriorly or all interiorly at the boundary. The author shows how to obtain the index sum of the field under the most general boundary conditions. This sum is evaluated in terms of the Euler characteristic of the given manifold and of the Euler characteristics of certain complexes on the boundary determined by the given field of vectors.

44. Professor H. T. Davis: *Integral equations of infinite order.*

T. Lalesco has studied the problem presented by the solution of the functional equation formed by equating to a known function a linear operator in negative powers of z, where z is the differential operator d/dx. The present paper shows that the Pincherle-Bourlet generatrix equation used in solving functional equations in positive powers of z also holds for functional equations in negative powers. This fact is used to derive a general solution of integral equations of infinite order from which certain expansions in the Heaviside calculus of electrical circuit theory come as special cases.

45. Professor H. W. Brinkmann: *On quadratic fields in which every genus of ideals contains but one class.* Preliminary report.

This preliminary paper deals with the determination of those quadratic imaginary fields which have two genera of ideals, each genus containing one class. Such fields are next in order of simplicity to those in which there is but one class of ideals and in which, therefore, the usual unique factorization theorem holds for integers. Interesting applications to elementary number theory appear as special results of the work.

46. Professors Einar Hille and J. D. Tamarkin: *On a general problem of summability of Fourier series.* Preliminary communication.

This paper deals with two general classes of regular transformations, the one introduced by Nörlund and the second discussed first by Hurwitz and Silverman and subsequently by Hausdorff. An attempt is made to obtain a classification of these transformations according as the Lebesgue-Hardy theorem (the summability of the Fourier series of an integrable function $f(x)$ to the value $f(x)$ almost everywhere) does or does not hold true.

47. Professor C. C. Camp: *Expansions associated with partial differential equations and auxiliary conditions on interior as well as bounding loci.* Preliminary communication.

In 1916, C. E. Wilder presented problems in the theory of ordinary differential equations with auxiliary conditions at more than two points. The present paper extends this theory to the case of partial differential equations in p variables with auxiliary conditions which may be made to depend for their solution on p ordinary differential systems each of the Wilder type. In addition the author is extending the treatment to partial differential equations of the type $\sum_{l=1}^{p}[\partial u/\partial x_l+\lambda a_l(x_l)u]=0$, where none of the $a_l(x_l)$ vanishes. It is believed that the theory is extensible to the case in which the $a_l(x_l)$ actually change sign in the intervals considered.

48. Dr. A. J. Lotka: *A numerical example in integral equations.*

The problem of the distribution-in-time of the progeny of a population element, which has been treated on a former occasion (this Bulletin, 1927, p. 403) by demonstrating the additive property of the Thiele seminvariants for successive generations, can also be solved by means of an integral equation. A complete numerical solution is given. This presents itself as an aperiodic component with logarithmic decrement, and a series of superposed damped oscillatory components with incommensurable periods. The constants appearing in the solution are functions of the Thiele seminvariants of the first generation. Problems of the same general character also occur in the computation of the current renewals of manufacturing plant, commercial equipment, etc.

49. Miss Rose Whelan: *Approximate solutions of certain general types of boundary value problems from the standpoint of integral equations.* Preliminary communication.

In this paper, the method of approximating the solutions of a general type of boundary value problem in the theory of linear differential and integro-differential equations by means of associated difference systems is justified. In each case both the boundary value problem and the corresponding difference system are reduced to equivalent Fredholm integral equations and the order of difference of the kernels is found. A treatment is given of the problem of determining the order of the difference between solutions, characteristic numbers, and characteristic solutions of two integral equations when the order of difference between kernels is known.

50. Professor J. I. Hutchinson: *Properties of functions represented by the Dirichlet series $\sum(av+b)^{-s}$ or by linear combinations of such series.*

In this paper, the function, $Z(s)$ defined by the series $\sum(av+b)^{-s}$, is studied with reference to its zeros and its behavior in certain parts of the complex plane. A formula for the real zeros is given. A critical strip is determined for the imaginary zeros, and some clue as to their distribution is indicated by particular examples. A complete survey is made of the problem of determining linear combinations $f(s)$ of the functions $Z(s)$ for different integral values of b, a being a fixed integer, so as to satisfy functional equations of the type considered by Cahen for a a prime. Various theorems are given concerning the imaginary zeros of the functions $f(s)$, the number of which is proved to be infinite. The functional equations are generalized so as to include the Dirichlet L-functions with imaginary characters as special cases.

51. Professor C. R. Adams: *On the linear partial q-difference equation of general type.*

The equation studied in this paper is (1): $\sum_{i,j=0}^{n} a_{ij}(x,y)f(q^i x, r^j y) = 0$, in which the $a_{ij}(x,y)$ are known functions of the complex variables x and y, analytic at $(0,0)$; q and r are constants, not zero and of moduli $\neq 1$; and $f(x,y)$ is to be determined. Series which formally satisfy (1) are found, and direct proof of their convergence is given. In this manner it is shown that in general there exists a one-parameter family of sets of solutions of the equation (1), each set comprising n solutions analytic in the vicinity of $(0,0)$. The situation at (∞,∞), or at $(0,\infty)$, or at $(\infty,0)$, is entirely parallel if the coefficient functions $a_{ij}(x,y)$ are analytic there. In addition to the regular case, in which the roots of the characteristic equation are restricted by the usual sort of hypothesis from being multiple, infinite, zero, etc., a large class of irregular cases is treated. The non-homogeneous equation obtained by adding to the left member of (1) a known function $b(x,y)$ analytic at (0,0) is also discussed. The methods employed are applicable to equations like (1) in more than two independent variables.

52. Dr. I. M. Sheffer (National Research Fellow): *Expansions in generalized Appell polynomials, and related questions.*

By generalized Appell polynomials we mean the polynomials $\{G_n(x)\}$ defined by $e^{tx}\{A_0(t)+x^kA_1(t)+\cdots+x^kA_k(t)\}=\sum G_n(x)t^n$. They make their appearance in the study of certain integral equations in the complex plane. We study expansions in these polynomials of (a suitable class of) analytic functions, and exhibit the relation of these expansions to the solutions of the integral equations mentioned. The theory is intimately tied up with certain linear differential equations of infinite order. The connection between such differential equations and integral equations, which is here brought out, is then utilized to solve two other classes of equations: (1) partial differential equations of infinite order, with constant coefficients; (2) "Laurent" differential equations of infinite order, with constant coefficients.

53. Dr. I. M. Sheffer: *Systems, finite and infinite in number, of linear differential equations of infinite order, with constant coefficients.*

The author has given existence theorems governing a finite system of linear differential equations of infinite order, with constant coefficients. There the method of attack was one of operators, and by means of suitable differential operators, the system was first reduced to a single equation of infinite order, after which certain solutions of this one equation were shown to represent solutions of the original system. In the present paper we consider the same system of equations, and the extension to systems of infinitely many equations, and show how such systems can be thrown into the form of systems of contour integral equations in the complex plane. These integral equations are then inverted: we obtain solutions which are themselves contour integrals.

54. Professor R. L. Jeffery: *Sequences of functions which define a definite integral containing a parameter.*

This paper is concerned with determining conditions on the bounded function $f(x,y)$ which insure the uniform convergence of the Riemann sum $\sum_{i=1}^{n} f(\xi_i,y)\ (x_i-x_{i-1})$ to its limit $F(y)$, when this limit exists independent of the choice of ξ_i on (x_{i-1}, x_i). Sufficient conditions are obtained for the uniformity of this convergence, and these conditions are shown to be both necessary and sufficient for a certain class of functions. The paper concludes with a consideration of the analogous problem for a Riemann-Stieltjes sum.

55. Professor R. L. Jeffery: *The existence of the Lebesgue-Stieltjes integral.*

A definition of a Lebesgue-Stieltjes integral of a function $f(x)$ with respect to a non-decreasing function $\phi(x)$ has been given by Hildebrandt (this Bulletin, vol. 24, pp. 188–190) involving the idea of the measurability of a set of points with respect to $\phi(x)$. Making use of a theorem due to Carathéodory (*Vorlesungen uber Reelle Funktionen*, §153) we set $\phi(x) = \nu(x) + \chi(x)$, where ν depends only on the discontinuities of ϕ, and where χ is continuous. It is then shown that the necessary and sufficient condition that every measurable set E be measurable with respect to ϕ is that χ be absolutely continuous. It then follows that if χ be absolutely continuous, every bounded and measurable function $f(x)$ has a Lebesgue-Stieltjes integral with respect to ϕ.

56. Professor D. E. Richmond: *A new proof of certain relations of Morse in the calculus of variations in the large.*

In a recent paper (Transactions of this Society, April, 1928) on the calculus of variations in the large, Morse characterizes the type of an extremal arc whose end points are not conjugate, by the number of mutual conjugate points within the arc. Among other applications, he establishes a set of relations between the numbers M_i of extremal arcs of type i, which join two given points in an extremal-convex region. The hypotheses include the existence of a field of extremals covering the region of operation. The present paper derives these relations from Morse's hypotheses using only well known continuity properties of the solutions of differential equations. Our methods enable us to state these relations in a somewhat more restrictive form than that given by Morse, and also to gain an insight into the relative position of extremal arcs of different type. It should be stated, however, that Morse has since announced results without the field hypothesis and also for the n-dimensional case. Our methods do not appear to apply to these more general cases.

57. Professor H. J. Ettlinger: *A countable infinite system of linear differential equations with summable coefficients and the associated difference system.*

The system considered in this paper is of the form $y(x) = \int_0^x A(t)y(t)dt + a$, where a is a vector in a countable infinity of dimensions having length, and A is a matrix with a countable infinity of rows and columns, each element of which is summable (Lebesgue) on $0<x<1$. If $y(x)$ is a vector each of whose components is absolutely continuous on $(0,1)$ and such that the length of y, $|y|$, exists, and $|Ay|$ and $|yA|$ are each not greater than $\phi(x)|y|$, where $\phi(x)$ is positive and summable on $(0,1)$, then the given system has a unique solution. The associated difference system is defined and solved, and it is shown that the limit of the solution of the difference system is the solution of the system above. The inequality $|\int_0^x y(t)dt| \leqq \int_0^x y|(t)|\,dt$ is established, and used to show that the method of successive approximations is available in both the algebraic and the differential systems.

58. Mr. E. Feenberg: *A countable infinity of ordinary differential equations in abstract space.*

A class of scalar functions H of vectors ω, in a countable number of dimensions, including the length function, is defined by the following properties of $H(\omega)$: $H(\omega) \geqq 0$, $H(0)=0$, $H(\omega_1-\omega_2) \leqq H(\omega_1)+H(\omega_2)$, and $H(0, 0, \cdots, \omega_j, 0, 0) = H(0, 0, \cdots, \omega_i, \cdots, 0, 0) \equiv H(\omega_j)$ when $\omega_i = \omega_j$. In the preceding paper, Professor Ettlinger has given an existence theorem for a linear differential system and an associated difference system in a countable number of variables where the coefficients are summable. The present paper extends the theorem to the following non-linear system: $y(x) = \int_0^x f(t, y)dt + a$, where a is a constant sector in a countable number of dimensions, y and f are vectors in a countable number of dimensions, and $H(a)$ exists, where $H(\omega)$ is a particular function of the class defined above and on $0 \leqq x_1 \leqq x_2 \leqq 1$, we have the following inequalities:
(1) $H(\int_{x_1}^{x_2} f(t,\omega)dt) \leqq \int_{x_1}^{x_2} \phi(t)dt$;
(2) $H(\int_{x_1}^{x_2} [f(t,\omega_1) - f(t,\omega_2)]dt) \leqq \int_{x_1}^{x_2} \psi(t)H(\omega_1-\omega_2)dt$;
(3) $H(\int_{x_1}^{x_2} [f_j(t,\omega_1) - f_j(t,\omega_2)]dt) \leqq \int_{x_1}^{x_2} \psi(t)H(\omega_1-\omega_2)dt$,
where $\phi(x)$ and $\psi(x)$ are positive and summable on (0, 1).

59. Mr. H. V. Craig: *On the reduction of the Euler equations in the calculus of variations to normal form.*

J. H. Taylor has given two methods of solving the Euler equations associated with $\int_{t_1}^{t_2} F(x,\dot{x})dt$ for the highest derivatives. The present paper modifies these methods so as to apply to the problem containing derivatives of any order.

60. Professor A. D. Michal: *The group manifold of finite continuous point and functional transformation groups.*

The differential geometry of the manifold of the r parameters of an r-parameter continuous group of transformations in n variables $x^1, x^2, \cdots, x^n$ has recently been studied by Cartan with the aid of ingenious geometric methods peculiar to his way of looking at an affinely connected manifold. From the analytic and invariant-theoretic standpoint, however, Cartan's geometric methods are by no means the last word on the subject. Furthermore, the proofs of many of his theorems bring into play the variables $x^1, \cdots, x^n$. In the present paper some of the essential results of Cartan's theory are developed analytically in such a way as to make the theory applicable to finite continuous groups of functional transformations in function space as well as to point transformations in n dimensions. Various fundamental tensor invariants of the group manifold are, for the first time, exhibited in an arbitrary coordinate system.

61. Professor J. M. Thomas: *Incomplete systems of partial differential equations.*

The theory of a system of total differential equations which is completely integrable is known to correspond exactly to the theory of a jacobian system of linear homogeneous partial differential equations, but the usual treatment of the incompletely integrable case for the two types of systems does not preserve this correspondence. The present paper gives a treatment displaying the equivalence of the two sorts of systems in the incompletely integrable case.

62. Professor J. M. Thomas: *Riquier's existence theorems.*

This paper contains an exposition of the principal results obtained by Riquier in his researches on systems of partial differential equations. It has been offered to the Annals of Mathematics.

63. Professor J. L. Walsh: *Note on the expansion of analytic functions in series of polynomials and in series of other analytic functions.*

This note continues the study contained in several papers by the author in the Transactions of this Society. The present note gives particular attention to expansions in terms of polynomials associated with an arbitrary Jordan region.

64. Professor J. L. Walsh: *On the degree of approximation to an analytic function by means of rational functions.*

Let C be an arbitrary rectifiable Jordan curve in the z-plane, and let the function $f(z)$ be defined on C. A necessary and sufficient condition that $f(z)$ should be regular-analytic on C is that there should exist rational functions $r_n(z)$ of degree $2n$, $n=0, 1, \cdots$, with no poles or limit points of poles on C, such that we have $|f(z)-r_n(z)| \leqq M/R^n$, $R>1$, for all z on C.

65. Mr. Nat Edmonson: *Poisson's integral and plurisegments on the hypersphere.*

In a recent paper (American Journal, vol. 49, p. 153) Evans and Bray obtained necessary and sufficient conditions for the existence on the surface of the unit sphere S of a bounded additive function $F(s)$ of segments, such that $u(M)$, a function harmonic in the interior of S, is given by the Stieltjes integral $u(M)=(1/(4\pi))\int S\{(1-v^2).(MP^3)\}dF(S_P)$. The present paper extends their results to the hypersphere, in n-dimensional space, defined by the parametric equations $x_1=\cos\theta_1$, $x_2=\sin\theta_1\cos\theta_2, \cdots$. $x_{n-1}=\sin\theta_1\cdots\sin\theta_{n-2}\cos\theta_{n-1}$, $x_n=\sin\theta_1\cdots\sin\theta_{n-2}\sin\theta_{n-1}$. The essential feature is the definition of a system of polar caps and proper segments on S in such a way that an orthogonal system is obtained; the results of Evans and Bray are then readily generalized to n dimensions. The condition (a) of the main theorem of the present paper is eliminated by a method of sequence extraction which is based on the fact that the function $F(r_i,s)=\int_S u(M)dM$ is of uniformly limited variation; Evans and Bray applied the theory of convergence in the mean to accomplish this end. In other papers now being prepared, the limit approached by $u(M)$ as the point M approaches the boundary of the hypersphere, and functions of configurations analogous to regular curves on the three-dimensional sphere are being studied.

66. Dr. P. W. Ketchum: *Analytic functions of hypercomplex variables.*

All analytic functions of a hypervariable are monogenic only in case the hypervariable is both associative and commutative. For such hypervariables, all analytic functions are monogenic, and, conversely, all monogenic functions are analytic. Furthermore, the analogues of the two Cauchy integral theorems, Morera's theorem, Taylor's and Laurent's expansions, Mittag-Leffler's theorem, and Weierstrass's factor theorem hold for analytic functions. In this paper, a restricted variable is defined as one whose analytic functions range over a larger class of hypernumbers than the variable itself. The conditions under which all analytic functions of a restricted hypervariable satisfy given differential equations are studied. In particular, variables are obtained all of whose analytic functions satisfy Laplace's equation. The functions of two such harmonic hypervariables are studied in considerable detail. The totality of harmonic functions cannot be obtained from a single hypervariable.

67. Dr. P. W. Ketchum: *A complete solution of Laplace's equation by an infinite hypervariable.*

In this paper, a study is made of the restricted hypervariable $w=xe_0+ye_1+ze_{-1}$, where e_0, e_1, and e_{-1} are three of an infinite set of units defined by the relations $e_n=i^n \cos n\alpha$, $e_{-n}=i^n \sin n\alpha$, where $\cos n\alpha$ and $\sin n\alpha$ are to be regarded as hypernumbers and not as a collection of function values. Every analytic function of w is monogenic, and conversely every monogenic function of w is analytic. The line and surface integrals of analytic functions vanish over closed lines and surfaces, respectively, provided they include no singularities. Every analytic function of w satisfies Laplace's equation. With every analytic solution, $f_0(x,y,z)$, of Laplace's equation there can be associated an infinite set of functions f_k such that $\sum f_k e_k$ is an analytic function of w. The real part of a power series in w is equivalent to an expansion in solid spherical harmonics.

68. Professor H. A. Simmons: *On the ratio of an arc to its chord.*

In this paper it is shown that if $y=f(x)$ represents an arc with a continuously turning tangent, and if $f(x)$, $f'(x)$ come under one of the cases (i), (ii), (iii), (iv) below, then $\lim_{c\to\infty}(s/c)=1$, s standing for length of arc and c standing for length of corresponding chord: (i) $\lim_{x\to\infty}f(x)=\infty$, $\lim_{x\to\infty}f'(x)=\infty$; (ii) $\lim_{x\to a}f(x)=\infty$, $\lim_{x\to a}f'(x)=\infty$; (iii) $\lim_{x\to\infty}f(x)=\infty$, $\lim_{x\to\infty}f'(x)=\text{const.}$; (iv) $\lim\ x_{x\to\infty}f(x)=\text{const.}$, $\lim_{x\to\infty}f'(x)=0$.

69. Professor R. E. Langer: *On the simultaneous expansion of arbitrary functions.*

This paper is devoted to a discussion of the ordinary differential system $u''+p_1(x,\rho)u'+p_2(x,\rho)u=0$, $\alpha_{i1}(\rho)u'(a)+\alpha_{i2}(\rho)u(a)+\alpha_{i3}(\rho)u'(b)+\alpha_{i4}(\rho)u(b)=0$, $i=1, 2$, the coefficients $p_i(x,\rho)$ being polynomials of degree i in ρ and the $\alpha_{ij}(\rho)$ also being polynomials. The existing theory for this system asserts the expansibility of an "arbitrary" function in a series of characteristic solutions. It is here shown how two arbitrary functions $f_1(x)$, $f_2(x)$ may be expanded simultaneously with one determination of coefficients. Of these expansions that for $f_1(x)$ corresponds roughly to the expansion previously known, and the discussion of its convergence follows on familiar lines. The associated expansion for $f_2(x)$ on the other hand is shown to be convergent to the value $f_2(x)$ for an arbitrary choice of the functions $f_1(x)$, $f_2(x)$, only provided the boundary conditions are peculiarly related to the differential equation. For general boundary conditions of the type above it is shown that the expansion is summable to the value $f_2(x)$ by a specified type of Riesz typical means.

70. Professor R. E. Langer: *The boundary problem associated with a differential system rational in the parameter.*

The asymptotic forms for the solutions of the differential equation $u''+r_1(x,\rho)u'+r_2(x,\rho)u=0$ are known provided the coefficients are of the form $r_i(x,\rho)=\sum_{j=-i}^{\infty} r_{ij}(x)\rho^{-j}$, and certain other hypotheses are met. The boundary problem associated with this equation and the conditions $\alpha_{i1}\, u'(a)+\alpha_{i2}u(a)+\alpha_{i3}\, u'(b)+\alpha_{i4}\, u(b)=0$, $i=1, 2$, the $\alpha_{ij}(\rho)$ being polynomials, has, however, been discussed only for the cases in which the $r_i(x,\rho)$ are polynomials of degree i in ρ. This paper considers the case in which these cofficients are rational of the form $r_i(x,\rho)=\sum_{j=-i}^{i} r_{ij}(x)\rho^{-j}$. It is shown that there exists then a double set of characteristic values clustering both at $\rho=\infty$ and at $\rho=0$, and the simultaneous expansions for a double set (four in number) of arbitrary functions with one determination of coefficients is obtained. These expansions are shown to be equiconvergent in pairs with related expansions obtained from associated differential systems which are polynomial in ρ.

ARNOLD DRESDEN,
Associate Secretary

SCHMIDT'S ORTHOGONAL ENNUPLE AND THE FRENET FORMULAS FOR A CURVE

BY A. J. MC CONNELL

This note deals with §32 (pp. 103–107) of Eisenhart's *Riemannian Geometry.*

The object of the above section is to define an orthogonal ennuple from a given set of n linearly independent vectors and hence to deduce the Frenet formulas of a curve in a space of n dimensions. One criticism of Eisenhart's ennuple is that when the original set of vectors forms an orthogonal ennuple the defined ennuple does not reduce to it, and in this note I have modified the definition so that this disadvantage is removed. Also, to obtain equations (32.18) in his book, Eisenhart has assumed that $b_{p-1} = (b_{p-1}^2)^{1/2}$, and this is incorrect if we understand that the positive root has been taken.

Being given a set of n linearly independent vectors,

$$(1) \qquad \overset{r}{\xi}_{\alpha|} \qquad (\alpha = 1,2,3, \cdots, n),$$

in a Riemannian manifold of n dimensions, our object is to define from these vectors a set of n orthogonal unit vectors,

$$(2) \qquad \overset{r}{\lambda}_{\alpha|}, \qquad (\alpha = 1,2,3, \cdots, n).$$

Building up the ennuple in the usual manner,* we arrive at the formulas

$$(3) \qquad \overset{r}{\lambda}_{p|} = \epsilon_p(|\, b_p/b_{p-1} |)^{1/2} \sum_{\alpha=1}^{p} \overset{r}{\xi}_{\alpha|} B_p^{\alpha}, \quad (p = 1,2, \cdots, n).$$

The quantities ϵ_p determine the orientation of the ennuple in space. Eisenhart takes all the ϵ's equal to $+1$. I propose to choose them so that when (1) is an orthogonal ennuple the set (2) may reduce to it. We easily find that we must put $\epsilon_p = e_p$, where e_p has its usual meaning. Our orthogonal ennuple is therefore given by

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* See Kowalewski, *Determinantentheorie*, pp. 423–426.

$$(4)\qquad \overset{r}{\lambda}_{p|} = e_p(\,|\,b_p/b_{p-1}\,|)^{1/2}\sum_{\alpha=1}^{p}\overset{r}{\xi}_{\alpha|}\overset{\alpha}{B}_p, \qquad (p = 1,2, \cdots, n).$$

The Frenet formulas, as deduced from Eisenhart's form of the ennuple, are given by equations (32.16) and (32.18) of the above section. In the formulas (32.18) for the curvatures, Eisenhart errs in absorbing under a square root sign which calls always for the positive value a quantity which may be negative. The equations should read

$$(5)\qquad \frac{1}{\rho_p} = e_p(\,|\,b_{p-1}b_{p+1}/b_p^{2}\,|\,)^{1/2}.$$

In order to obtain the principal normals resulting from the ennuple (4), we need only substitute $e_p\overset{r}{\lambda}_{p|}$ for $\overset{r}{\lambda}_{p|}$ in (32.16) which will then read

$$(6)\quad \frac{dx^m}{ds}\overset{r}{\lambda}_{p|,m} = \frac{e_p}{\rho_p}\overset{r}{\lambda}_{p+1|} - \frac{e_p}{\rho_{p-1}}\overset{r}{\lambda}_{p-1|}, \quad (p = 2, \cdots, n-1).$$

It may be preferable sometimes to have all the curvatures positive, as they obviously are in the above equations for a positive definite line-element. Thus, putting $1/\rho_p = e_p k_p$, $k_p > 0$, we have the Frenet formulas in the final form

$$(7)\qquad \frac{dx^m}{ds}\overset{r}{\lambda}_{p|,m} = k_p\overset{r}{\lambda}_{p+1|} - e_p e_{p-1}k_{p-1}\overset{r}{\lambda}_{p-1|}, \qquad (p = 2, \cdots, n-1).$$

These are the normals and curvatures defined by Synge.* Of course, in the case of a positive definite line-element, the two sets of normals coincide.

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* Synge, *Normals and curvatures of a curve in a Riemannian manifold*, Proceedings of the International Mathematical Congress, Toronto, 1924.

ON THE LOCI OF THE LINES INCIDENT WITH k $(r-2)$-SPACES IN S_r

BY B. C. WONG

The problem of the determination of the number N_r of lines that meet $2r-2$ given $(r-2)$-spaces in S_r has been solved.[*] Schubert's symbolic or enumerative method is powerful for the solution of problems of this kind and has indeed been the one used, but it does not offer any insight into the nature of the geometry involved. It is the purpose of this paper to re-determine the number N_r and also to obtain the loci of the ∞^{2r-2-k} lines incident with k given $(r-2)$-spaces in S_r, where $r<k\leqq 2r-2$.

For our purpose we make use of the known theorem:[†] *The locus of the ∞^{r-2} lines incident with r general $(r-2)$-spaces in S_r is a hypersurface V_{r-1}^{r-1}.*

Now consider r of the given $(r-2)$-spaces, say $S_{r-2}^{(i)}$ $[i=1, 2, \cdots, r]$. They yield a V_{r-1}^{r-1} whose generators are incident with them. Any of the remaining $k-r$ given $(r-2)$-spaces, say $S_{r-2}^{(r+1)}$, meets V_{r-1}^{r-1} in a V_{r-3}^{r-1}. Any hyperplane S_{r-1}' through $S_{r-2}^{(r+1)}$ meets $S_{r-2}^{(i)}$ in r $(r-3)$-spaces. The ∞^{r-4} lines that meet these r $(r-3)$-spaces are in S_{r-1}' and hence meet $S_{r-2}^{(r+1)}$, and they form a $V_{r-3}^{M_r'}$ whose order M_r' is to be determined later. Hence the ∞^{r-3} lines incident with $r+1$ $(r-2)$-spaces in S_r form a $V_{r-2}^{M_r+M_r'}$ (writing M_r for $r-1$), for it is met by S_{r-1}' in a $V_{r-3}^{M_r+M_r'}\equiv V_{r-3}^{M_r}+V_{r-3}^{M_r'}$.

Now the $(r+2)$th $(r-2)$-space, $S_{r-2}^{(r+2)}$, meets $V_{r-3}^{M_r+M_r'}$ in a $V_{r-4}^{M_r+M_r'}$. A hyperplane S_{r-1}'' through $S_{r-2}^{(r+2)}$ intersects the other $r+1$ $(r-2)$-spaces in $r+1$ $(r-3)$-spaces and the ∞^{r-5} lines incident with the latter form a $V_{r-4}^{M_r''}$. Hence the

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[*] See C. Segre, *Mehrdimensionale Raume*, Encyklopadie der Mathematischen Wissenschaften, vol. III: 2, pp. 813, 814 where full references are given.

[†] This Bulletin, vol. 32, No. 5, pp. 553–554.

∞^{r-4} lines incident with $r+2$ $(r-2)$-spaces in S_r form a $V_{r-3}^{M_r+M_r'+M_r''}$; for it is met by S_{r-1}'' in a $V_{r-4}^{M_r+M_r'+M_r''} \equiv V_{r-4}^{M_r+M_r'} + V_{r-4}^{M_r''}$.

Continuing this process of reasoning we soon come to $S_{r-2}^{(k)}$, the kth or the last of the given $(r-2)$-spaces, and find that the ∞^{2r-k-2} lines meeting k $(r-2)$-spaces in S_r form a $V_{2r-k-1}^{M_r+M_r'+\cdots+M_r^{(k-r)}}$, for it is intersected by a hyperplane S_{r-1} through $S_{r-2}^{(k)}$ in a $V_{2r-k-2}^{M_r+M_r'+\cdots+M_r^{(k-r)}}$ composed of a $V_{2r-k-2}^{M_r+M_r'+\cdots+M_r^{(k-r-1)}}$ and a $V_{2r-k-2}^{M_r^{(k-r)}}$. The former is contained in $S_{r-2}^{(k)}$ and the latter is the locus of the ∞^{2r-k-3} lines incident with the $k-1$ $(r-3)$-spaces in which S_{r-1} meets the other $k-1$ $(r-2)$-spaces supposed to have already been considered.

If $k=2r-3$, the ∞^1 lines incident with $2r-3$ $(r-2)$-spaces form a $V_2^{N_r}$ where

$$(1) \qquad N_r = \sum_{j=0}^{r-3} M_r^{(j)}.$$

Since a general $(r-2)$-space can have only N_r points in common with $V_2^{N_r}$, the required number of lines meeting $2r-2$ given $(r-2)$-spaces is N_r given by (1).

Now to determine the values of $M_r^{(j)}$. Replacing r by $r+1$ in (1), we have

$$N_{r+1} = \sum_{j=0}^{r-2} M_{r+1}^{(j)},$$

where

$$(2) \qquad M_{r+1}^{(j)} = \sum_{h=0}^{j} M_r^{(h)}.$$

Then, from (1) and (2),

$$(3) \qquad N_r = M_{r+1}^{(r-2)} = M_{r+1}^{(r-3)}.$$

Decreasing r by unity in (2) and writing out the resulting equalities for all values of j from zero to $r-3$, we obtain, without difficulty, remembering $M_r = r-1$,

$$(4)\qquad M_r^{(j)} = \sum_{j=0}^{r-3} M_{j+3}^{(j-1)} = \frac{(r+j-1)!(r-j-2)}{(r-1)!(j+1)!}$$

and also

$$(5)\qquad M_{r+1}^{(j)} = \frac{(r+j)!(r-j-1)}{r!(j+1)!}.$$

Putting $j=r-2$ or $r-3$, we have, from (3),

$$N_r = \frac{(2r-2)!}{r!(r-1)!},$$

the required formula for N_r in terms of r.

The locus of the ∞^{2r-k-2} lines that meet k given $(r-2)$-spaces has been obtained and its order is $\sum_{h=0}^{k-r} M_r^{(h)}$ and, on account of (2) and (5), this is equal to

$$M_{r+1}^{(k-r)} = \frac{k!(2r-k-1)}{r!(k-r-1)!}.$$

Attention is called to the fact that a general $(r-2)$-space meets this locus in a V_{2r-k-3} of the same order but there are exactly k, that is, the given, $(r-2)$-spaces each of which meets it in a V_{2r-k-2} of order

$$\sum_{h=0}^{k-r-1} M_r^{(h)} = M_{r+1}^{(k-r-1)} = \frac{(k-1)!(2r-k)}{r!(k-r)!}.$$

A hyperplane through one of these k $(r-2)$-spaces meets the locus again in a V_{2r-k-2} of order

$$M_r^{(k-r)} = \frac{(k-1)!(2r-k-2)}{(r-1)!(k-r+1)}.$$

It is to be noted that these formulas do not hold for $k=r$; for, if $k=r$, $M_r=r-1$, $M_{r+1}=r$.

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NOTE ON THE PLÜCKER EQUATIONS FOR PLANE ALGEBRAIC CURVES IN THE GALOIS FIELDS

BY A. D. CAMPBELL

In this note we shall consider the Plücker* equations for plane algebraic curves and find out in how far they are valid in the Galois fields of order p^n. We designate these fields by $GF(p^n)$. Let us consider the n-ic

$$\begin{aligned} f(x,y,z) &\equiv az^n + (b_0x + b_1y)z^{n-1} \\ &+ (c_0x^2 + c_1xy + c_2y^2)z^{n-2} \\ &+ (d_0x^3 + d_1x^2y + d_2xy^2 + d_3y^3)z^{n-3} + \cdots \\ &+ (l_0x^n + \cdots + l_ny^n) = 0, \end{aligned} \tag{1}$$

the first polar† of (0, 0, 1) with respect to (1),

$$\begin{aligned} naz^{n-1} &+ (n-1)(b_0x + b_1y)z^{n-2} \\ &+ (n-2)(c_0x^2 + c_1xy + c_2y^2)z^{n-3} + \cdots = 0, \end{aligned} \tag{2}$$

and the polar conic of (0, 0, 1) with respect to (1),

$$\frac{n(n-1)}{2} az^2 + (n-1)(b_0x + b_1y)z + (c_0x^2 + c_1xy + c_2y^2) = 0. \tag{3}$$

In the derivation of the Plücker equations given by Hilton‡, use is made of the number of intersections of the n-ic and its first polar and of the n-ic and its Hessian, also the polar reciprocal of the n-ic with respect to a given conic is utilized. If $n = \alpha p^r + 1$ the polar conic (3) passes through (0, 0, 1) and is degenerate, but (0, 0, 1) can be any point P, hence for these $GF(p^n)$ we cannot use the Hessian in the derivation of any of Plücker's equations. Also in the $GF(2^n)$ the polars of all the points in the plane with respect to any given conic are concurrent (as is easy to prove), hence in these fields we cannot use the polar reciprocal of the n-ic with respect to a conic.

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* See Hilton, *Plane Algebraic Curves*, p. 112.

† See A. D. Campbell, *The polar curves of plane algebraic curves in the Galois fields*, this Bulletin, vol. 34 (1928), pp. 361–363.

‡ Loc. cit., pp. 97, 100, 63, 34, 66.

We get around these difficulties in the following manner. Hilton shows* that if we can establish the three following equations, the others follow from these three:

$$m = n(n-1) - 2\delta - 3\kappa, \tag{4}$$

$$n = m(m-1) - 2\tau - 3\iota, \tag{5}$$

$$\tfrac{1}{2}n(n+3) - \delta - 2\kappa = \tfrac{1}{2}m(m+3) - \tau - 2\iota, \tag{6}$$

where n is the order of the curve under consideration, m is its class, δ the number of nodes on the curve, κ the number of cusps, τ the number of bitangents, ι the number of inflections. Just as in Hilton pp. 96 and 97† so here we can readily see that (2) cuts (1) once at the point of contact of each tangent from (0, 0, 1) to the given curve, twice at each node and thrice at each cusp of the given curve. Also the curves (1) and (2) ordinarily intersect in $n(n-1)$ points, hence equation (4) is in general valid in the $GF(p^n)$. In some cases (1) and (2) do not intersect in $n(n-1)$ points, namely when (2) is composite with a repeated factor. Such a case occurs for $p=2$, $n=3$, and (1) lacking an xyz term.

The equations (5) and (6) are established in Hilton by taking the polar reciprocal of the n-ic with respect to a conic and by the use of Sylvester's dialytic method.‡ The fact is used (p. 63) that from a point on the tangent t at a cusp P only one tangent to the n-ic coincides with t, while at P three tangents coincide with t. This fact is necessary in order to show that the polar reciprocal of a cusp is a point of inflection. The proof of this fact that is given in Hilton (pp. 84, 86) is not applicable to the $GF(p^n)$, so we substitute the following proof. If we take P as (0, 0, 1) with $y=0$ as tangent, then in (1) $a=b_0=b_1=c_0=c_1=0$ and $c_2\neq 0$, hence the curve* and the first polar of P have in common only $n(n-1)-2\cdot 2-2$ points beside P, whereas for P' an ordinary point the curve and the first polar of P' have in common $n(n-1)-3$ points beside the cusp P. From this follows the above fact. In the $GF(p^n)$ for $p>2$ we can take the polar reciprocal of an n-ic with respect to a conic. Also we do not need to use the Hessian because (5) and (6) are proved in Hilton by means of the polar reciprocal of the n-ic;† and from (5) and (6) we can obtain what Hilton obtains from the intersections of the n-ic and its Hessian, namely the equation‡

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* Loc. cit., p. 112.

† Using the eliminant as in Hilton, pp. 10, 11.

‡ Loc. cit., pp. 97, 63, 66.

$$\iota = 3n(n-2) - 6\delta - 8\kappa. \tag{7}$$

Therefore the Plücker equations are valid in the $GF(p^n)$ for $p>2$. In the $GF(2^n)$ we cannot use a polar reciprocal of the n-ic; we can, however, use plane duality§ in these $GF(2^n)$ to replace the use of the polar reciprocal in proving (5) and (6). Therefore also in the $GF(2^n)$ the Plücker equations are still valid.

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* Hilton, loc. cit., p. 11.

† Loc. cit., pp. 33, 66.

‡ Loc. cit., pp. 100, 101.

§ See Veblen and Young, *Projective Geometry*, vol. 1, p. 201.

REMARK ON A THEOREM OF OSGOOD CONCERNING CONVERGENT SERIES OF ANALYTIC FUNCTIONS

BY C. CARATHÉODORY
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1. *Introduction.* One of the most fundamental properties of analytic functions, that a sequence of such functions which converges in a region R must converge uniformly in at least one subregion R_1 of R, was discovered by Osgood* as early as 1901. It seems, however, that it has not yet been remarked that the theorem of Osgood holds also for meromorphic functions if, as is usual now, one extends the definition of uniform convergence in the neighborhood of a point so as to include functions having poles.

2. *Regular Convergence.* Although the principal object of the present note is to prove the theorem we have just stated, it is perhaps worth while to point out that all the theorems used in connection with the uniform convergence of analytic functions can be proved more easily if one replaces the concept of uniform convergence by another differing only slightly from it. We shall say that a sequence of functions $f_n(z)$, meromorphic on a closed region A, is *regularly convergent* at a point z_0 of A, if for every sequence of points $z_1, z_2, \cdots$, belonging to A, and converging towards z_0, the (finite or infinite) limit

$$\lim_{n\to\infty} f_n(z_n)$$

exists†. We note that if this limit (which is obviously independent of the chosen sequence) is finite at every point of a closed and bounded subset B of A, and if the functions $f_n(z)$ converge regularly at every point of B, they must also converge uniformly on this set.

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* W. F. Osgood, *Note on the functions defined by infinite series whose terms are analytic functions*, Annals of Mathematics, (2), vol. 3 (1901), pp. 25–34.

† Although this very concept has been already used by Hans Hahn for the general theory of real functions under the very pregnant name of "*Stetige Konvergenz*" [H. Hahn, *Theorie der reellen Funktionen*, Berlin,

Moreover, it is readily seen that all the extensions of the concept of uniform convergence that have been made are included in the above definition.*

3. *Extension of Osgood's Theorem.* We consider now a sequence of meromorphic functions $f_n(z)$ which converges towards a function $f(z)$ at every (interior) point of the region R. It is well known that if there exists a subregion S of R and two positive numbers M and N such that for $n>N$, one has $|f_n(z)|<M$ at every point of S, the sequence $f_n(z)$ converges *regularly* in S.

Accordingly, it follows that the theorem of Osgood holds also for meromorphic functions, provided that the limiting function $f(z)$ is *finite* at every point of R. For if this were not the case one could find regions $S_1, S_2, \cdots$, with the property that R contains S_1 and S_k contains S_{k+1}, and an increasing sequence of indices $n_1<n_2<\cdots$, such that $|f_{n_k}(z)|>k$ for all z in S_k. The regions S_k contain at least one point z_0 common to all of them. At this point the limit function cannot be finite, and we thus read a contradiction to our hypothesis.

4. *Singularities Everywhere Dense.* We conclude from this that if the theorem of Osgood is not true, the points at which the limit function $f(z)$ is infinite must form a set everywhere dense in R.

Moreover, since we can apply the same reasoning to the sequence of meromorphic functions $1/\{f_n(z)-a\}$, which

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J. Springer, 1921, p. 238], I take the liberty to speak here of "regular", instead of "continuous", convergence, because in doing this the points which for a long time have been called "irregular" coincide with those at which the convergence is not regular in the above sense.

* See my paper, *Stetige Konvergenz und normale Familien*, which is to appear soon in the Mathematische Annalen.

converges in R towards the function

$$\frac{1}{f(z) - a},$$

we see that for every finite number a the roots of the equation $f(z) - a = 0$ must form a set that is everywhere dense in R.

5. *Reductio ad Absurdum.* We consider now the continuous and bounded set of real functions $\phi_n(x,y)$, where $z = x + iy$, and

$$\phi_n(x,y) = \begin{cases} 1 \text{ at the points of } R & \\ \quad \text{for which} & R[f_n(z)] > 1, \\ -1 \text{ at the points of } R & \\ \quad \text{for which} & R[f_n(z)] < 1, \\ R[f_n(z)] \text{ at the points of } R & \\ \quad \text{for which} & -1 \leqq R[f_n(z)] \leqq 1, \end{cases}$$

$R[f_n(z)]$ designating the real part of $f_n(z)$.

If the theorem of Osgood does not hold for our functions $f_n(z)$, the functions $\phi_n(x, y)$ have the following properties:

(a) the limit,

$$\lim_{n \to \infty} \phi_n(x,y) = \phi(x,y)$$

exists everywhere in R;

(b) for every real number α such that $|\alpha| < 1$, the roots of the equation $\phi(x, y) - \alpha = 0$ are everywhere dense in R.

As these properties are in contradiction with the well known theorem of Baire,* that the limit of a sequence of bounded continuous functions cannot be totally discontinuous, the result we have stated must be true.

6. *Questions of Connectivity.* If a sequence of functions $f_n(z)$, meromorphic in R, converges everywhere in that region, the points E of R at which the sequence converges

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* See Carathéodory, *Vorlesungen über reelle Funktionen*, Leipzig, Teubner, 1927, §176.

regularly must accordingly be everywhere dense in R. Moreover, E contains a denumerable set of subregions $R_1, R_2, \cdots$, in which the limit function $f(z)$ is meromorphic or equal to the infinite constant, and the interior points of R_k form also a set that is dense in R.

If the functions $f_n(z)$ have no poles in R, the regions R_k must be simply connected and, as F. Hartogs and A. Rosenthal* have proved in a very remarkable paper, must fulfill certain other conditions less easy to state. *Nothing of this kind happens in our case.*

Consider for instance a finite or denumerable set of regions $R_1, R_2, \cdots$, whose connectivity may be infinite. Suppose that no two of these regions overlap, and that every point of the z plane is either an interior point or a boundary point of some R_k or that it is a limiting point of such points. Assume, moreover, for simplicity that the boundary points of all the R_k form with their limit points a bounded and closed set S.

Let $\zeta_1, \zeta_2, \cdots$ be a denumerable sequence of points on S such that every point of S is a limiting point of this sequence; the ζ_k thus must not only be everywhere dense on S, but an infinite number of them must cover every isolated point of S.

Designate by γ_n the set of points which are at a distance $1/n$ from S. Let z_n be a point of γ_n such that there is no point of γ_n nearer to ζ_n than z_n. Then the distance $|z_n - z_j| > 1/[n(n+1)]$, $j \neq n$, and the distance from z_n to S is likewise $> 1/(n(n+1))$. Furthermore, every point ζ of S is a limiting point of the set $z_1, z_2, \cdots$. It follows that the various circles κ_n

$$\kappa_n: \qquad |z - z_n| < \frac{1}{2n(n+1)}, \qquad (n = 1, 2, \cdots),$$

have no points in common, and contain no points of S.

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* F. Hartogs and A. Rosenthal, *Über Folgen analytischer Funktionen*, Mathematische Annalen, vol. 100 (1928), pp. 212-263.

7. *Approach by Rational Functions to Zero.* We consider now the functions

$$g_n(z) = \frac{1}{n^2}\cdot\frac{1}{4n(n+1)(z-z_n)-1}, \qquad (n = 1,2,\cdots).$$

We note that for all z exterior to the circle k_n,

$$|g_n(z)| < \frac{1}{n^2}\cdot$$

With every number n we can associate an integer p_n such that no point ζ of S is at a distance from the set of points $z_n, z_{n+1}, \cdots, z_{p_n}$ greater than $1/n$.

The sequence of rational functions

$$f_n(z) = \sum_{k=n}^{k=p_n} g_k(z)$$

then converges identically towards zero in the whole plane. This convergence is regular at every interior point of each R_k, but the convergence is irregular at every point ζ of S, because every such point is a limiting point of poles of the $f_n(z)$.

8. *Approach by Rational Functions to Meromorphic Functions.* It is possible to choose the rational functions $f_n(z)$ such that they converge on R_k to an arbitrary function $\phi_k(z)$, meromorphic in R_k, and converge on S to any function $\phi(z)$ which may be represented as the limit of a sequence of rational functions. The proof of this fact can be based upon the proposition that any function which is meromorphic in a region D, bounded by regular curves and of finite connectivity, can be represented by a definite integral taken over the boundary of D, the integral vanishing identically outside of D. By approximating these integrals with finite sums of rational functions, and applying a method analogous to that of §7, we can prove the theorem stated above without great difficulty.

THE UNIVERSITY OF CALIFORNIA

THE NORM OF A SPACE CONFIGURATION

BY L. M. BLUMENTHAL

1. *Introduction.* In a recent paper,* the writer has shown that attached to an ordered set of $2n$ points in a plane there is the lineo-linear invariant

$$2(N + iA) = \sum_{k=1}^{n} \bar{x}_{2k}(x_{2k-1} - x_{2k+1}) = \left(\frac{1}{n}\right)\sum_{i=1}^{n-1}(1-\epsilon^{n-i})v_i\bar{u}_i,$$

where x_k $(k=1, 2, \cdots, 2n)$ are points in the complex plane, $\bar{x}_k$ denotes the conjugate of x_k, ϵ is a primitive nth root of unity, A is the area of the ordered $2n$-gon, and the *norm* N is defined by

$$2N = -\tfrac{1}{2}(\delta_{12}^2 - \delta_{23}^2 + \cdots + \delta_{2n-1,2n}^2 - \delta_{2n,1}^2),$$

with $\delta_{ij} = |x_i - x_j|$. The Lagrange resolvents†

$$v_i = \sum_{k=0}^{n-1}\epsilon^{ik}x_{2k+1}, \quad \bar{u}_i = \sum_{k=0}^{n-1}\epsilon^{i(n-k)}\bar{x}_{2(k+1)},$$

$$(i = 1, 2, \cdots, n-1),$$

are absolute invariants under translations $y_i = x_i + b$, while the combinations $v_i\bar{u}_i$ are likewise invariant under rotations $y = tx_i$, where t is a complex number with unit modulus.

This paper extends the preceding results to S_3, the theorems that are obtained holding, mutatis mutandis, for S_n. Denoting the $2n$ points by $X_i (i=1, 2, \cdots, 2n)$ and calling the two sets X_{2i}, X_{2i-1} $(i=1, 2, \cdots, n)$ the *component* n-points of the whole set, it is shown that the norm of the $2n$-point is expressible in terms of the Lagrange resolvents of its vertices and is *consequently absolutely invariant under a translation of either of its component n-points.*

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* *Lagrange resolvents in euclidean geometry,* American Journal of Mathematics, vol. 49 (1927). pp. 511–522.

† Pascal, E., Repertorium, 2d. ed., vol. 1, p. 307.

The vanishing of the norms of the space 6-point and 8-point leads to two interesting theorems. The theorem for the latter case is seen to be the space analog of the orthologic triangle relation of Steiner.

2. *Norm of Space 2n-Point.* To express the norm of $2n$ ordered points $X_i \equiv (x_i, y_i, z_i)$, $(i=1, 2, \cdots, 2n)$, in terms of Lagrange resolvents, we consider first the case of the 6-point. The resolvents are

$$v_1(x) = x_1 + \omega x_3 + \omega^2 x_5, \qquad u_1(x) = x_2 + \omega x_4 + \omega^2 x_6,$$
$$v_2(x) = x_1 + \omega^2 x_3 + \omega x_5, \qquad u_2(x) = x_2 + \omega^2 x_4 + \omega x_6,$$

with two additional sets in y and z. If we adjoin to these sets

$$v_0(x) = x_1 + x_3 + x_5, \qquad u_0(x) = x_2 + x_4 + x_6,$$

with the accompanying forms in the other letters, we can readily solve for the coordinates in terms of v_i and u_i and hence show that the invariants v_i, u_i, $(i=1, 2)$, form a complete system.*

From the definition of N we have

$$2N = \sum \sum_{k=1}^{3} x_{2k}(x_{2k-1} - x_{2k+1}) = \sum \sum_{k=1}^{3} x_{2k-1}(x_{2k} - x_{2(3+k-1)}),$$

where the summation extends over x, y, z and subscripts are to be reduced mod 6, with $x_0 \equiv x_6$. Substitution in the above expression yields, after some reductions,

$$2N = \frac{1}{1-\omega} \sum [v_1(x)\bar{u}_1(x) - \omega v_2(x)\bar{u}_2(x)]$$

which expresses the norm of the 6-point in terms of the Lagrange resolvents of the vertices. We have, then, the following theorem.

THEOREM 1. *The norm of a space 6-point is absolutely invariant under a translation of either of its component 3-points.*

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* The expressions v_0, u_0 are, of course, not invariants and will drop out of any invariant relation.

The method illustrated above is applied to the $2n$-point; the result being set down without further details as

$$2N = (1/n) \sum \sum_{i=1}^{n-1} (1 - \epsilon^{n-i}) v_i(x) \bar{u}_i(x) .$$

THEOREM 2. *The norm of a space $2n$-point is absolutely invariant under translation of either of its component n-points.*

3. *Vanishing of the Norm.* In the plane it is seen that the complex number $2(N+iA)$ is determined by an ordered set of $2n$ points. Viewed in a somewhat different light, the expression may be regarded as a simultaneous invariant of the two component sets of n points. It is easily shown that for a plane 6-point, the vanishing of the norm of the figure is a sufficient condition that lines from the vertices of one component 3-point perpendicular to corresponding joins of the other 3-point meet in a point.*

For the space 6-point we consider the three planes through the points X_{2i} perpendicular to the joins of $X_{2i-1}X_{2i+1}$ $(i=1, 2, 3)$. The equations of the planes are

$$\sum(x_3 - x_1)(x - x_2) = 0, \quad \sum(x_5 - x_3)(x - x_4) = 0,$$
$$\sum(x_1 - x_5)(x - x_6) = 0.$$

The determinant of the coefficients is evidently zero. Also

$$K_1 = -\begin{vmatrix} \sum x_2(x_3 - x_1), & y_3 - y_1, & z_3 - z_1 \\ \sum x_4(x_5 - x_3), & y_5 - y_3, & z_5 - z_3 \\ \sum x_6(x_1 - x_5), & y_1 - y_5, & z_1 - z_5 \end{vmatrix} = -2N \begin{vmatrix} y_5 - y_3, & z_5 - z_3 \\ y_1 - y_5, & z_1 - z_5 \end{vmatrix},$$

$$K_2 = 2N \begin{vmatrix} x_5 - x_3, & z_5 - z_3 \\ x_1 - x_5, & z_1 - z_5 \end{vmatrix}, \quad K_3 = -2N \begin{vmatrix} x_5 - x_3, & y_5 - y_3 \\ x_1 - x_5, & y_1 - y_5 \end{vmatrix}.$$

Hence the vanishing of the norm will make the expressions

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* Triangles so related were called by Steiner "orthologique." He showed the relation to be reciprocal, but does not seem to have considered criteria for the existence of the relation.

K_i, $(i=1, 2, 3)$, zero, and the three planes will be coaxial.[*] Hence we have the following theorem.

THEOREM 3. *If the norm of an ordered space* 6-*point vanish, planes through the vertices of one component* 3-*point perpendicular to the corresponding joins of the other are coaxial.*

Considering the case of the space 8-point we have the following theorem.

THEOREM 4. *If the norm of an ordered space* 8-*point vanish, the four planes through the points* X_{2i} *perpendicular to the joins of* $X_{2i-1}X_{2i+1}$, $(i=1, 2, 3, 4)$, *meet in a point.*

The four planes have equations

$$\sum(x_3 - x_1)(x - x_2) = 0, \qquad \sum(x_7 - x_5)(x - x_6) = 0,$$
$$\sum(x_5 - x_3)(x - x_4) = 0, \qquad \sum(x_1 - x_7)(x - x_8) = 0.$$

They will meet in a point if and only if the determinant

$$\Delta = \begin{vmatrix} x_3 - x_1, & y_3 - y_1, & z_3 - z_1, & -\sum x_2(x_3 - x_1) \\ x_5 - x_3, & y_5 - y_3, & z_5 - z_3, & -\sum x_4(x_5 - x_3) \\ x_7 - x_5, & y_7 - y_5, & z_7 - z_5, & -\sum x_6(x_7 - x_5) \\ x_1 - x_7, & y_1 - y_7, & z_1 - z_7, & -\sum x_8(x_1 - x_7) \end{vmatrix}$$

vanishes and not all four minors of the elements in the last column are zero. (Evidently these minors do not in general vanish.) Adding the last three rows to the first we obtain a row with three elements zero and the fourth element

$$-\sum \sum_{k=1}^{4} x_{2k}(x_{2k+1} - x_{2k-1}),$$

that is,

$$\Delta = -2N \begin{vmatrix} x_5 - x_3, & y_5 - y_3 & z_5 - z_3 \\ x_7 - x_5, & \cdots, & \cdots \\ x_1 - x_7, & \cdots, & \cdots \end{vmatrix}.$$

which proves the theorem.

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[*] The K_i's will be zero also if

$$\begin{vmatrix} y_5 - y_3, & z_5 - z_3 \\ y_1 - y_5, & z_1 - z_5 \end{vmatrix} = \begin{vmatrix} x_5 - x_3, & y_5 - y_3 \\ x_1 - x_5, & y_1 - y_5 \end{vmatrix} = 0.$$

The points X_1, X_3, X_5 will then be collinear and the planes will be coaxial with the line at infinity as axis.

4. *Rotation.* If a rotation about the Z-axis of angle θ be applied to the component n-point X_{2k-1}, $(k=1, 2, \cdots, n)$, it is seen that

$$2N'(x,y) = \cos\theta \cdot \sum \sum_{k=1}^{n} x_{2k}(x_{2k-1} - x_{2k+1})$$

$$+ \sin\theta \cdot \left[\sum_{k=1}^{n} x_{2k}(y_{2k-1} - y_{2k+1}) - y_{2k}(x_{2k-1} - x_{2k+1}) \right],$$

where $N'(x, y)$ denotes that part of the transformed norm containing only terms in x and y (the terms in z, of course, are unaltered by the rotation). Whence we may write

$$N' = N \cos\theta + 2N(z) \sin^2 \theta/2 + A \sin\theta,$$

where A is the area obtained by projecting the original $2n$ point on the XY-plane.

5. *Points on a Sphere.* If the points X_i, $(i=1, 2, \cdots, 2n)$, be placed upon a sphere, we shall define the norm of the ordered $2n$-point by

$$2N = \prod_{i=2,4,\cdots,2n}^{i=1,3,\cdots,2n-1} \cos X_iX_{i+1},$$

where the product obtained by allowing the index to assume the values $2, 4, 6, \cdots, 2n$ is to be subtracted from the product produced by letting the index take on the values $1, 3, 5, \cdots, 2n-1$, and $\cos X_iX_{i+1}$ denotes the cosine of the arc of the great circle join of X_i, X_{i+1}.

Consider the 6-point X_i, $(i=1, 2, \cdots, 6)$, on the sphere and through X_{2i} draw arcs of great circles perpendicular to the great circle joins of X_{2i-1}, X_{2i+1}. Suppose the arcs meet at O. It is readily shown that the norm of the six-point vanishes.

THE JOHNS HOPKINS UNIVERSITY

ON THE POTENTIAL OF CERTAIN SURFACE DISTRIBUTIONS

BY HILLEL PORITSKY*

The surface distributions that are commonly studied in potential theory are those known as *single* and *double layers*, or as surface distributions of *poles* and *doublets*. A. Wangerin† first investigated potentials of what he called triple layers (dreifach belegte Flächen), that is, potentials of the form

$$v(P) = \int \mu(\Pi) \frac{\partial^2[1/r(P,\Pi)]}{\partial n^2}\, dS. \tag{1}$$

Here $r(P, \Pi)$ is the distance between a point Π of the surface S and a point P in space, dS is the element of area of S at Π, $\mu(\pi)$ is a given function along S (the density function), and $\partial/\partial n$ indicates differentiation along the normal toward a specified side of S that might be called its positive side.‡

He showed that both v and its first derivatives are discontinuous at S. M. Freund§ studied more complex distributions over a spherical surface of the form

$$v(P) = \int \mu(\Pi) \frac{\partial^k[1/r(P,\Pi)]}{\partial n^k}\, dS. \tag{2}$$

It is the object of this paper to point out that by means of certain integrations by parts these integrals may be replaced, in case the surface S is closed, by other surface integrals that represent the potential of certain single and double layers; if, however, the surface S is open, these integrations by parts lead to a replacement of the original distributions by curve distributions over the boundary of S in addition to the above single and double surface layers. Similar conclusions apply to the more general integrals

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* National Research Fellow in Mathematics.

† A. Wangerin, *Über das Potential dreifach belegter Flächen*, Jahresbericht der Vereinigung, vol. 29 (1920), p. 174.

‡ The considerations that lead to the name triple layers are analogous to the ones that suggest the double layer nomenclature: if the derivative is replaced by the limit of a proper quotient, v is seen to be the limit of the potential of three single layers spread over three nearby surfaces.

§ M. Freund, *Über das Potential mehrfach belegter Flächen*, Dissertation, Universität Halle (1922).

$$v(P) \doteq \int \mu(\Pi) \frac{\partial^k[1/r(P,\Pi)]}{\partial h_1 \partial h_2 \cdots \partial h_k} dS, \tag{3}$$

where $\partial/\partial h_i$, $i=1, 2, \cdots, k$, indicate differentiations in k specified directions at each point Π of S, so that for a fixed Π the integrand is a spherical harmonic of degree $-(k+1)$ in the coordinates of P when the origin is at Π. From this it would appear that except for features presented at the boundary, if any, of S, the surface distributions of the above types introduce no new element into potential theory.

This reduction to simpler surface distributions may be carried out as follows. Introduce a triply orthogonal family of surfaces of which S is one by taking for one family the surfaces parallel to S and for the other two the developable surfaces consisting of the normals to S through its lines of curvature. Let u, v be the parameters along the latter, while n is the (directed) distance from S; u, v, n may be used as parameters for the above surfaces or as curvilinear coordinates for points near S. Let the element of distance be given by

$$ds^2 = g_1^2 du^2 + g_2^2 dv^2 + dn^2;$$

g_1, g_2 are completely determined by their values for $n=0$. Indeed,

$$g_i(u,v,n) = g_i(u,v,0)(1 + n/R_i(u,v)), \qquad (i = 1,2,) \tag{4}$$

where R_1, R_2 are the radii of curvature along $v=\text{const.}$, $u=\text{const.}$ respectively, measured positively if the corresponding centers of curvature are on the side of negative n. Using the well known expression for the Laplacian in orthogonal coordinates, we get

$$(5)\quad \nabla^2 = \left\{\frac{1}{g_1g_2}\left[\frac{\partial}{\partial u}\left(\frac{g_2}{g_1}\frac{\partial}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{g_1}{g_2}\frac{\partial}{\partial v}\right)\right]\right\} + \left\{\frac{1}{g_1g_2}\frac{\partial}{\partial n}\left(g_1g_2\frac{\partial}{\partial n}\right)\right\}.$$

If using (5) we form the equation $\nabla^2(1/r)=0$, where r is the distance from a fixed point P not on S, we obtain an equation solvable for $\partial^2(1/r)/\partial n^2$ in terms of first and second order derivatives of $1/r$ among which not more than one n-differentiation occurs. In a similar way, by the use of this equation and of the equations that are obtained from it by differentiating it with respect to n once, twice, $\cdots$, $(k-2)$ times, one could express $\partial^k(1/r)/\partial n^k$ in terms of u-, v-derivatives of $1/r$ and of $\partial(1/r)/\partial n$. Likewise, the spherical harmonic $\partial^k(1/r)/\partial h_1 \cdots \partial h_k$ in the integrand of (3) may be expressed in terms of such differentiations as follows. Introduce rectangular Cartesian coordinates x, y, z and replace $\partial/\partial h_i$ by $\cos(xh_i)\partial/\partial x+\cos(yh_i)\partial/\partial y+\cos(zh_i)\partial/\partial z$ (the parentheses denote angles between the direction h_i and the positive x, y, z axes), and transform to u-, v-, n-derivatives by means of elementary formulas for change of derivatives under a transformation of variables. The n-derivatives of second or higher order may now be eliminated from the result by the use of the equations obtained from $\nabla^2(1/r)=0$ (expressed in the form (5)) by differentiating it the requisite number of times with respect to u, v, n.

Having thus expressed the integrands in (1), (2), (3), we may then integrate the resulting integrals by parts enough times to make all the u-, v-derivatives of $1/r$, $\partial(1/r)/\partial n$ disappear from the remaining surface integrals. The resulting surface integrals can be collected into the form

$$\int a(1/r)\,dS + \int b[\partial(1/r)/\partial n]\,dS,$$

where a, b are derived from μ by operating on it with certain linear differential operators that depend upon the shape of S near Π but are independent of P. These surface integrals, therefore, represent potentials of a single and of a double surface layer. If the surface S is open, the integrations by parts will also introduce certain integrals over the bounding curves that represent potentials of poles, doublets, and more complex singularities (in cases (2), (3)) distributed over these boundaries. Such curve distributions cannot, in general, be reduced to simpler curve distributions.*

For the case of the triple layers of Wangerin the above computations may be simplified by the use of the well known Green-Beltrami integration theorems. Returning to (5) we recognize in the first brace the second differential operator of Beltrami (see W. Blaschke, *Vorlesungen über Differentialgeometrie*, vol. I, p. 116 (121)); we shall denote it by B^2. Expanding the second brace and using (4) we get for the Laplacian

$$\nabla^2 = B^2 + 2H\,\partial/\partial n + \partial^2/\partial n^2,$$

where $H=(1/2)(1/R_1+1/R_2)$ is the mean curvature. The elimination of $\partial^2(1/r)/\partial n^2$ from the integrand of (1) outlined above results in the equation

$$v(P) = -\int B^2(1/r)\mu\, dS - \int 2H\mu\,[\partial\mu(1/r)/\partial n]dS.$$

The first integral may now be transformed by the integral relation of Beltrami

$$\int (fB^2g - gB^2f)dS = \int\left(f\frac{\partial g}{\partial \xi} - g\frac{\partial f}{\partial \xi}\right)dC$$

(see Blaschke, loc. cit. p. 117 (129); we denote the element of arc length by dC and the derivative along the outer normal to C by $\partial/\partial\xi$). We get

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* It may be pointed out in this connection that the potential of a *volume* distribution of singularities of the type (3) can always, for points outside the region of integration, be reduced to a potential of a volume distribution of poles only, in addition to surface distributions of the form (3). A familiar instance of this is furnished by Poisson's theory of magnetism.

$$v(P) = -\int B^2\mu(1/r)\,dS - \int 2H\mu\,[\partial(1/r)/\partial n]\,dS$$
$$- \int \mu[\partial(1/r)/\partial\xi]dC + \int (1/r)(\partial\mu/\partial\xi)dC.$$

Hence, the density of the single layer is $-B^2\mu$ and that of the double layer $-2H\mu$. The results of Wangerin for the discontinuities in v and its derivatives at the surface S readily follow from the known properties of single and double layers.*

Without going through the explicit computation of the densities of the equivalent single and double surface layers in cases (2), (3), we can tell from the mere existence of these layers that for approach of an inner point of S along the normal, these potentials approach limits which depend, in general, upon the side of approach. This even sheds some light on the behavior near S of higher order derivatives (with respect to Cartesian variables) of the familiar single and double layer potentials, since such derivatives may be written in the form (3);† such a form may also be given to derivatives of any order of the potentials (1), (2), (3).

We conclude with a simple example that will serve to illustrate the behavior of these potentials of distributions over *open* surfaces near the edge of the surface. Consider a distribution of the form (3) over an infinite plane strip, say $y=0$, $0<x<a$, with the k directions $h_1, \cdots, h_k$ constant and μ depending only on the distance from the edge. Expressing the directional derivatives in terms of derivatives

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* But the converse is in a sense also true. Thus, if S is closed, v is uniquely determined if it is harmonic except on S, if it and its derivative along the normal approach limits for approach of S from opposite sides and if a certain behavior is ascertained at infinity.

† The explicit values of the discontinuities for second-order derivatives of potentials of single layers have been obtained by Poincaré in his *Théorie du Potentiel Newtonien*, §105, etc.

with respect to x, y, z and integrating with respect to z first, we get for v a sum of terms of the form*

$$\int_0^a \mu(x) \frac{\partial^k \log r}{\partial x^l \partial y^{k-l}} dx$$
$$= \int_0^a \mu(x) R\left(\frac{\partial^k \log (z-h)}{\partial x^l \partial y^{k-l}}\right) dx,$$

where z is the complex quantity $x+iy$, h is the value of z at P and $R(\alpha)$ denotes the real part of α. But by the Cauchy-Riemann equations for an analytic function F of a complex variable

$$\frac{\partial F}{\partial x} = \frac{\partial F}{i\partial y} = F'(z).$$

Hence the integral becomes

$$R \int_0^a i^{k-l} \mu(x) \frac{\partial^k \log (z-h)}{\partial x^k} dx,$$

and, integrating by parts k times, we obtain an integral that represents a bounded function of h for $|h|<$const. and integrated terms that yield singularities at $h=0$, $h=a$. The terms resulting from the lower limit of integration are given by $R(c_0 \log h + c_1/h + c_2/h^2 + \cdots + c_{k-1}/h^{k-1})$ where the c_i are constants; similar terms result from the upper limit.

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* This furnishes an example of a two-dimensional potential of singularities analogous to those of (3) for space. The general treatment of such potentials could of course be given along similar lines or could be deduced from the above by putting $1/R_2=0$.

CANONICAL CONFIGURATIONS ASSOCIATED WITH A SURFACE*

BY E. P. LANE

1. *Introduction.* After summarizing briefly for subsequent use some of the theory of the projective differential geometry of a surface referred to its asymptotic net, we next present some very convenient formulas for differentiating the coordinates of a point referred to the local tetrahedron customarily employed when the differential equations of the surface are in Fubini's canonical form. Their use is first illustrated in connection with certain configurations generated by different parts of the local tetrahedron at a variable point on the surface, and then they are applied in a brief study of the first and second canonical surfaces, the former being enveloped by the canonical plane and the latter being generated by the canonical point. Finally, the general theory of a pencil of conjugate nets on a surface is applied to a new covariant pencil of conjugate nets, and a sequence of such pencils is constructed.

2. *Analytic Foundations.* In ordinary projective space the four homogeneous coordinates $x^{(1)}, \cdots, x^{(4)}$ of a point P on an analytic non-ruled surface S are functions of two independent variables u, v. If the asymptotic net is parametric and if the proportionality factor of the coordinates is suitably chosen, the functions x satisfy two partial differential equations in Fubini's canonical form

$$(1)\qquad \begin{cases} x_{uu} = px + \theta_u x_u + \beta x_v, \\ x_{vv} = qx + \gamma x_u + \theta_v x_v, \end{cases} \qquad [\theta = \log(\beta\gamma)].$$

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* Presented to the Society, September 6, 1928.

The most general transformation of proportionality factor and parameters preserving this canonical form is

$$x = \text{const. } \bar{x}, \qquad \bar{u} = U(u), \qquad \bar{v} = V(v). \tag{2}$$

A point y given by an expression of the form

$$y = x_1 x + x_2 x_u + x_3 x_v + x_4 x_{uv} \tag{3}$$

is said to have local coordinates $x_1, \cdots, x_4$ referred to the local tetrahedron x, x_u, x_v, x_{uv} with suitably chosen unit point. The vertices of this tetrahedron are covariant points; in fact the line xx_{uv} is the projective normal and the line $x_u x_v$ is the reciprocal polar of this line with respect to the quadric of Lie; the vertex x_{uv} is the point where the tangent of the curve $u=$const. on the surface generated by the point x_u meets the projective normal. The effect of (2) on the local coordinates is the change of unit point given by

$$\bar{x}_1 = x_1, \quad \bar{x}_2 = U' x_2, \quad \bar{x}_3 = V' x_3, \quad \bar{x}_4 = U'V' x_4, \tag{4}$$

accents indicating differentiation with respect to the variables of the functions concerned.

The equation of the canonical plane of S at P is

$$\phi x_2 - \psi x_3 = 0, \tag{5}$$

where

$$\phi = \frac{\partial}{\partial u} \log \beta\gamma^2, \qquad \psi = \frac{\partial}{\partial v} \log \beta^2\gamma. \tag{6}$$

And the coordinates of the canonical point are $(0, \psi, -\phi, 0)$.

3. *The Derivatives of the Local Coordinates.* We shall first give some *formulas for the local coordinates of the two first-order derivative points of the point y given by* (3). By easy calculation we find

$$\begin{cases} y_u = y_1 x + y_2 x_u + y_3 x_v + y_4 x_{uv}, \\ y_v = z_1 x + z_2 x_u + z_3 x_v + z_4 x_{uv}, \end{cases} \tag{7}$$

where

$$
(8)\qquad \begin{cases} y_1 = x_{1u} + px_2 + (p_v + \beta q)x_4, \\ y_2 = x_{2u} + x_1 + \theta_u x_2 + (\beta\gamma + \theta_{uv})x_4, \\ y_3 = x_{3u} + \beta x_2 + \pi x_4, \\ y_4 = x_{4u} + x_3 + \theta_u x_4, \\ z_1 = x_{1v} + qx_3 + (q_u + \gamma p)x_4, \\ z_2 = x_{2v} + \gamma x_3 + \chi x_4, \\ z_3 = x_{3v} + x_1 + \theta_3 x_v + (\beta\gamma + \theta_{uv})x_4, \\ z_4 = x_{4v} + x_2 + \theta_v x_4, \end{cases}
$$

the functions π, χ being defined by

$$
(9)\qquad \pi = p + \beta\psi, \quad \chi = q + \gamma\phi.
$$

It occasionally happens that we have the equation in local coordinates of a surface covariantly connected with a point P of S and wish to find the envelope of this surface as P varies on S. The usual method of procedure is to differentiate partially with respect to u and with respect to v in the equation and then to solve the two resulting equations with the original to obtain the points of contact of the surface with its envelope. We shall now obtain the differentiation formulas to be used in such a situation.

The coordinates of a point $\bar{x}$ on S near P_x are given* by the expansions

$$
(10)\qquad \begin{cases} x_1 = 1 + \cdots, & x_2 = \Delta u + \cdots, \\ x_3 = \Delta v + \cdots, & x_4 = 0 + \cdots . \end{cases}
$$

And the coordinates y, z, w of the corresponding points $\bar{x}_u$, $\bar{x}_v$, $\bar{x}_{uv}$ near x_u, x_v, x_{uv} respectively are given in order by the power series

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* Stouffer and Lane, *Recent developments in projective differential geometry*, this Bulletin, vol. 34 (1928), pp. 453–472. See part B, equation 2.

$$(11)\qquad \left\{\begin{aligned} y_1 &= p\Delta u + \cdots, & y_2 &= 1 + \theta_u \Delta u + \cdots, \\ y_3 &= \beta\Delta u + \cdots, & y_4 &= \Delta v + \cdots; \end{aligned}\right.$$

$$(12)\qquad \left\{\begin{aligned} z_1 &= q\Delta v + \cdots, & z_2 &= \gamma\Delta v + \cdots, \\ z_3 &= 1 + \theta_v\Delta v + \cdots, & z_4 &= \Delta u + \cdots; \end{aligned}\right.$$

$$(13)\qquad \left\{\begin{aligned} w_1 &= (p_v + \beta q)\Delta u + (q_u + \gamma p)\Delta v + \cdots, \\ w_2 &= (\beta\gamma + \theta_{uv})\Delta u + \chi\Delta v + \cdots, \\ w_3 &= \pi\Delta u + (\beta\gamma + \theta_{uv})\Delta v + \cdots, \\ w_4 &= 1 + \theta_u\Delta u + \theta_v\Delta v + \cdots. \end{aligned}\right.$$

If a point has coordinates $x_1, \cdots, x_4$ referred to the original tetrahedron x, x_u, x_v, x_{uv} and coordinates $\bar{x}_1, \cdots, \bar{x}_4$ referred to the neighboring tetrahedron $\bar{x}, \bar{x}_u, \bar{x}_v, \bar{x}_{uv}$, the transformation between the tetrahedrons is found to be given by

$$(14)\qquad \left\{\begin{aligned} \bar{x}_1 &= x_1 - x_2 p\Delta u - x_3 q\Delta v - x_4\big[(p_v + \beta q)\Delta u \\ &\qquad + (q_u + \gamma p)\Delta v\big] + \cdots, \\ \bar{x}_2 &= -x_1\Delta u + x_2(1 - \theta_u\Delta u) - x_3\gamma\Delta v \\ &\qquad - x_4\big[(\beta\gamma + \theta_{uv})\Delta u - \chi\Delta v\big] + \cdots, \\ \bar{x}_3 &= -x_1\Delta v - x_2\beta\Delta u + x_3(1 - \theta_v\Delta v) \\ &\qquad - x_4\big[\pi\Delta u + (\beta\gamma + \theta_{uv})\Delta v\big] + \cdots, \\ \bar{x}_4 &= -x_2\Delta v - x_3\Delta u + x_4(1 - \theta_u\Delta u - \theta_v\Delta v) + \cdots. \end{aligned}\right.$$

Holding $v = \text{const.}$ and taking the limit in these formulas as Δu approaches zero (having first transposed certain terms and then divided by Δu), and then interchanging the roles of u and v and repeating the process, we obtain the desired *formulas for the differentiation of the local coordinates*:

$$(15)\qquad \left\{\begin{aligned} x_{1u} &= -px_2 - (p_v + \beta q)x_4, \\ x_{2u} &= -x_1 - \theta_u x_2 - (\beta\gamma + \theta_{uv})x_4, \\ x_{3u} &= -\beta x_2 - \pi x_4, \\ x_{4u} &= -x_3 - \theta_u x_4, \\ x_{1v} &= -qx_3 - (q_u + \gamma p)x_4, \\ x_{2v} &= -\gamma x_3 - \chi x_4, \\ x_{3v} &= -x_1 - \theta_v x_3 - (\beta\gamma + \theta_{uv})x_4, \\ x_{4v} &= -x_2 - \theta_v x_4. \end{aligned}\right.$$

4. *Applications to the Local Tetrahedron.* It follows from statements in §2 that the loci of the points x_u, x_v, x_{uv} are three surfaces covariant to S. Moreover these surfaces and S have their points in correspondence, corresponding points being those that have the same curvilinear coordinates. Without giving details of the calculations we shall simply remark that if we place $x_1=x$, $x_2=x_u$, $x_3=x_v$, $x_4=x_{uv}$, $u^1=u$, $u^2=v$, it is easy to determine a system of linear differential equations* of the form

$$\frac{\partial x_i}{\partial u^p} = \sum_{j=1}^{4} a_{ijp} x_j, \quad (i = 1, \cdots, 4; p = 1,2), \tag{16}$$

satisfied by the coordinates of these four points. Then the results of D. Sun on quadruples of surfaces with points in correspondence can be applied to these four surfaces. Moreover, each of the three pairs of opposite edges of the local tetrahedron generates two congruences with generators in correspondence, to which the results of A. Cook on pairs of congruences with generators in correspondence can be applied.†

The envelopes of the faces of the local tetrahedron are again a quadruple of surfaces with points in correspondence. It is geometrically obvious, and easily demonstrated by (15), that the face $x_4=0$ envelops S itself. *The points of contact of the faces* $x_1=0$, $x_2=0$, $x_3=0$ *with their envelopes are found by* (15) *to have respectively the coordinates*

$$\begin{cases} [0, -q(p_v+\beta q), -p(q_u+\gamma p), pq], \\ [-\gamma(\beta\gamma+\theta_{uv}), 0, -\chi, \gamma], \\ [-\beta(\beta\gamma+\theta_{uv}), -\pi, 0, \beta]. \end{cases} \tag{17}$$

If, for instance, the last two of these points are joined we obtain a generator of a congruence covariant to S, and so a wide variety of possibilities for the construction of covariant configurations is suggested.

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* Lane, *The projective differential geometry of systems of linear homogeneous differential equations of the first order*, Transactions of this Society, vol. 30 (1928), pp. 785-796. See §5.

† Sun, Cook; Chicago Doctoral Dissertations (1928–29).

5. *Applications to the Canonical Surfaces.* If $\psi=0$, the canonical point coincides with the point x_v. Let us suppose $\psi\neq 0$, and place

$$\lambda = \phi/\psi. \tag{18}$$

Then the local coordinates of the canonical point are $(0, 1, -\lambda, 0)$ and the equation of the canonical plane is

$$\lambda x_2 - x_3 = 0. \tag{19}$$

Applying (15) to (19) we find that *the envelope of the canonical plane, as u, v vary, is a surface, which we shall call the first canonical surface S_1, and whose point of contact with the plane is given by the formulas*

$$\tag{20} \left\{ \begin{aligned} x_1 &= -(\beta\gamma+\theta_{uv})x_4+\pi(\lambda_v+\theta_v\lambda-\gamma\lambda^2) \\ &\qquad +\chi\lambda(\lambda_u+\beta-\theta_u\lambda), \\ x_2 &= -\pi+\chi\lambda^2, \quad x_3 = \lambda x_2, \\ x_4 &= \lambda_u+\lambda\lambda_v+\beta-\theta_u\lambda+\theta_v\lambda^2-\gamma\lambda^3. \end{aligned} \right.$$

The locus of the canonical point, as u, v vary, is the second canonical surface S_2, whose tangent plane can be shown by means of (8) *to have coordinates given by*

$$\tag{21} \left\{ \begin{aligned} u_1 &= \lambda_u+\lambda\lambda_v-\beta-\theta_u\lambda+\theta_v\lambda^2+\gamma\lambda^3, \\ u_2 &= \lambda u_3, \quad u_3 = p-q\lambda^2, \\ u_4 &= p(\lambda_v+\theta_v\lambda+\gamma\lambda^2)+q\lambda(\lambda_u-\beta-\theta_u\lambda). \end{aligned} \right.$$

It will be observed that *the two surfaces S_1, S_2 correspond in the dualistic transformation that converts each point P of S into the tangent plane of S at P.* Analytically this kind of duality* is represented in the present notation by the transformation

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* Lane, *The correspondence between the tangent plane of a surface and its point of contact,* American Journal of Mathematics, vol. 48 (1926), p. 204.

$$(22)\quad x_1 = -(\beta\gamma + \theta_{uv})u_1 + u_4,\ x_2 = -u_3,\ x_3 = -u_2,\ x_4 = u_1$$

accompanied by the substitution

$$(23)\qquad \begin{pmatrix} \beta & \gamma & p & q & dv/du \\ -\beta & -\gamma & \pi & \chi & -dv/du \end{pmatrix}.$$

We shall make a few remarks about S_2, understanding that similar remarks, mutatis mutandis, can be made about S_1.

The tangent plane of S_2 passes through P_x in case

$$(24)\qquad \lambda_u + \lambda\lambda_v - \beta - \theta_u\lambda + \theta_v\lambda^2 + \gamma\lambda^3 = 0,$$

and then its line of intersection with the tangent plane of S is the second canonical tangent joining P_x to the canonical point. In this case the canonical curves of the first kind $dv - \lambda du = 0$ are dual union curves of the reciprocal of the projective normal congruence, so that the ray-point of the first canonical curve through P_x lies on the reciprocal $x_4 = x_1 = 0$ of the projective normal, being in fact the canonical point itself.

The tangent plane of S_2 meets the tangent plane of S in the reciprocal of the projective normal in case

$$(25)\qquad p - q\lambda^2 = 0,$$

being in this case the plane $x_1 = 0$. Then the canonical curves

$$(26)\qquad dv^2 - \lambda^2 du^2$$

correspond to the developables of the reciprocal of the projective normal congruence, which is therefore harmonic to the conjugate net of canonical curves.

The locus of the canonical point is a curve in case (24) and (25) are satisfied simultaneously.

6. *The Canonical Pencil of Conjugate Nets.* A pencil of conjugate nets on a surface is the class of ∞^1 conjugate nets every one of which has the property that at every point of the surface its two tangents form with the tangents of a fundamental conjugate net the same cross ratio. *The covariant pencil of conjugate nets determined by the canonical*

conjugate net (26) *will be called the canonical pencil.* The differential equation of a general net of this pencil is

$$dv^2 - \lambda^2 h^2 du^2 = 0 \qquad (h = \text{const.}). \tag{27}$$

The envelope of the osculating planes at P of all curves of the canonical pencil that pass through P is the axis-plane* cone

$$u_1 = u_2 u_3 m - \beta u_3^3 - \gamma u_2^3 = 0 \tag{28}$$

where

$$m = 2u_4 - (\theta_u - \lambda_u/\lambda)u_3 - (\theta_v + \lambda_v/\lambda)u_2. \tag{29}$$

The cusp-axis of this cone joins P_x to the point

$$(0,\ -\theta_v - \lambda_v/\lambda,\ -\theta_u + \lambda_u/\lambda, 2),$$

which lies in the canonical plane in case

$$\lambda(\theta_v + \lambda_v/\lambda) = \theta_u - \lambda_u/\lambda. \tag{30}$$

Let us suppose that (30) is not satisfied. Then *the plane* $\lambda_1 x_2 - x_3 = 0$, *where*

$$\lambda_1 = \frac{\theta_u - \lambda_u/\lambda}{\theta_v + \lambda_v/\lambda} \tag{31}$$

contains the cusp-axis and the projective normal. The curves enveloped by the line of intersection of this plane with the tangent plane of S, together with their conjugates, form a new covariant conjugate net

$$dv^2 - \lambda_1^2 du^2 = 0, \tag{32}$$

and the process can be repeated indefinitely. In this way we obtain a sequence of covariant conjugate nets.

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* Lane, *A general theory of conjugate nets*, Transactions of this Society, vol. 23 (1922), p. 283.

POLYNOMIALS $f[\phi(x)]$ REDUCIBLE IN FIELDS IN WHICH $f(x)$ IS IRREDUCIBLE*

BY LOUIS WEISNER

1. *Introduction.* Professor Ritt recently had occasion to consider the irreducible polynomials which become reducible when each argument is replaced by a power of itself.†

His results suggest the related problem of determining all polynomials $\phi_1(x_1, \cdots, x_m), \cdots, \phi_m(x_1, \cdots, x_m)$, such that $f[\phi_1, \cdots, \phi_m]$ is reducible, $f(x_1, \cdots, x_m)$ being irreducible. There is no such problem for functions of one variable, as every polynomial in a single variable can be factored into linear factors. If, however, we restrict ourselves to a field R, the problem arises: Given a polynomial $f(x)$ with coefficients in R and irreducible in R; to determine all polynomials $\phi(x)$ with coefficients in R such that $f[\phi(x)]$ is reducible in R. The present paper is devoted to a solution of this problem.

2. *Reducibility of* $\phi(x)-x_\iota$ *in* R'. Let

$$f(x) = (x - x_1)(x - x_2) \cdots (x - x_n) \tag{1}$$

be a polynomial with coefficients in R and irreducible in R; and let $\phi(x)$ be an arbitrary polynomial with coefficients in R. An irreducible factor $A(x)$ of

$$f[\phi(x)] = [\phi(x) - x_1] \cdots [\phi(x) - x_n] \tag{2}$$

has a root in common with one of the equations $\phi(x) = x_i$, say $\phi(x) = x_1$. Let $a_1(x)$ be the greatest common divisor of $A(x)$ and $\phi(x) - x_1$, and

$$\begin{cases} \phi(x) - x_1 = a_1(x)b_1(x) \\ \qquad A(x) = a_1(x)c_1(x), \end{cases} \tag{3}$$

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* Presented to the Society, February 25, 1928.

† J. F. Ritt, *A factorization theory for functions* $\sum_{\iota=1}^{\iota=n} a_\iota e^{\alpha_\iota x}$, Transactions of this Society, vol. 29 (1927), pp. 584–596.

where $b_1(x)$, $c_1(x)$ are polynomials (which may be constants) with coefficients in the field R' obtained by adjoining $x_1, \cdots, x_n$ to R.

The coefficients of the powers of x in (3) are rational functions of $x_1, \cdots, x_n$ with coefficients in R and are therefore unaltered by the group G of $f(x)=0$ relative to R. Let us apply to (3) some substitution of G which changes x_1 to x_i obtaining

$$(4)\qquad \begin{cases} \phi(x) - x_i = a_i(x)b_i(x) \\ A(x) = a_i(x)c_i(x), \end{cases}$$

where $a_i(x)$, $b_i(x)$, $c_i(x)$ denote what $a_1(x)$, $b_1(x)$, $c_1(x)$, respectively, become when subjected to this substitution. The coefficients of

$$Q(x) \equiv a_1(x) \cdots a_n(x),$$

being invariant under G, are in R. As $A(x)$ is irreducible in R and has a root in common with $Q(x)$, $A(x)$ is a divisor of $Q(x)$. But $A(x)$ is divisible by each of the polynomials $a_1(x), \cdots, a_n(x)$, which are relatively prime as they are divisors of the relatively prime polynomials $\phi(x)-x_1, \cdots, \phi(x)-x_n$ respectively. Thus, $A(x)$ and $Q(x)$ are equal up to a constant factor, which, without loss of generality, may be taken as unity, so that

$$(5)\qquad A(x) = a_1(x) \cdots a_n(x).$$

The polynomial $a_i(x)$ *is irreducible in* R'. Otherwise, let

$$(6)\qquad a_1(x) = d_1(x)e_1(x),$$

where $d_1(x)$, $e_1(x)$ are (non-constant) polynomials with coefficients in R': these coefficients are rational functions of $x_1, \cdots, x_n$ with coefficients in R, in which form we suppose them to be expressed. Applying to (6) a substitution of G which changes x_1 to x_i, we obtain

$$a_i(x) = d_i(x)e_i(x),$$

where $d_i(x)$, $e_i(x)$ are what $d_1(x)$, $e_1(x)$, respectively, become under this substitution. Thus $d_1(x) \cdots d_n(x)$ is a polynomial with coefficients in R, of degree less than that of

$A(x)$, and divides $A(x)$. As this conflicts with the irreducibility of $A(x)$ in R, $a_i(x)$ is irreducible in R'.

We conclude that *a necessary and sufficient condition that* $f[\phi(x)]$ *be reducible in* R *is that* $\phi(x)-x_1$ *be reducible in* R', *each irreducible factor of* $\phi(x)-x_1$ *in* R' *giving rise to an irreducible factor of* $f[\phi(x)]$ *in* R *by means of an equation similar to* (5).

3. *Divisors of* $f[\phi(x)]$. Because of the relations which exist among $x_1, \cdots, x_n$ the coefficients of a divisor $a_i(x)$ of $\phi(x)-x_i$ may be expressible in different ways as rational functions of $x_1, \cdots, x_n$ with coefficients in R. We proceed to show that *the coefficients of* $a_i(x)$ *equal rational functions of* x_i *alone with coefficients in* R.

The roots of $A(x)=0$ may be separated into systems

$$\begin{array}{c} x_{11}, x_{12}, \cdots, x_{1p} \\ x_{21}, x_{22}, \cdots, x_{2p} \\ \cdots \\ x_{n1}, x_{n2}, \cdots, x_{np}, \end{array}$$

the quantities in the ith system being the roots of $a_i(x)=0$, which form sets of imprimitivity of the group Γ of $A(x)=0$ relative to R if $f(x)$ and $a_i(x)$ are non-linear. Γ permutes these systems in the same way that G permutes $x_1, \cdots, x_n$ respectively. The subgroup G_i of G which leaves x_i fixed corresponds to the subgroup Γ_i of Γ which permutes the symbols of the ith system among themselves. The coefficients of $a_i(x)$ are unaltered by Γ_i when expressed as elementary symmetric functions of $x_{i1}, \cdots, x_{ip}$ and hence are unaltered by G_i when expressed as rational functions of $x_1, \cdots, x_n$ with coefficients in R. As x_i *belongs* to G_i, every function of the roots of $f(x)=0$ which is unaltered by G_i equals a rational function of x_i with coefficients in R; in particular, this is true of the coefficients of $a_i(x)$

We therefore change our notation and write $a(x, x_i)$ in place of $a_i(x)$. We have

$$A(x) = a(x, x_1) \cdots a(x, x_n). \tag{7}$$

The point to be emphasized in connection with (7) is that $a(x, x_1)$ is a divisor of $\phi(x)-x_1$ and that the other factors in the right member are obtained from $a(x, x_1)$ by changing x_1 to $x_2, \cdots, x_n$.

The preceding equation, derived on the assumption that $A(x)$ is irreducible in R, may be extended to any divisor of $f[\phi(x)]$. Let

$$A(x) = A_1(x) \cdots A_r(x),$$

where $A_1(x), \cdots, A_r(x)$ are irreducible in R, and hence are expressible in the forms

$$A_1(x) = a_1(x, x_1) \cdots a_1(x, x_n),$$
$$\cdots$$
$$A_r(x) = a_r(x, x_1) \cdots a_r(x, x_n),$$

$a_1(x, x_1), \cdots, a_r(x, x_n)$ being divisors of $\phi(x)-x_1$ with coefficients in R' and irreducible in R'. Define

$$a(x, x_1) = a_1(x, x_1) \cdots a_r(x, x_1).$$

Then we have (7), in which $A(x)$ now denotes any divisor of $f[\phi(x)]$ and $a(x, x_1)$ is a divisor of $\phi(x)-x_1$. *Every divisor of* $f[\phi(x)]$ *with coefficients in* R *is of the form* (7). This is clearly true of $f[\phi(x)]$ itself.

4. *Construction of* $\phi(x)$. We proceed to prove the converse: if $a(x, y)$ is a polynomial in the independent variables x, y, with coefficients in R, there exists a polynomial $\phi_0(x)$ with coefficients in R such that $f[\phi_0(x)]$ is divisible by

$$A(x) \equiv a(x, x_1) \cdots a(x, x_n) \tag{8}$$

and $\phi_0(x)-x_1$ is divisible by $a(x, x_1)$. Lagrange's interpolation formula suggests taking

$$\phi_0(x) \equiv \sum_{i,j=1}^{i=n,\, j=p} \frac{x_i A(x)}{(x - x_{ij})A'(x_{ij})}, \tag{9}$$

$x_{i1}, \cdots, x_{ip}$ being the roots of $a(x, x_i)=0$. As the coefficients of

$$\frac{x_i A(x)}{(x - x_{i1})A'(x_{i1})} + \cdots + \frac{x_i A(x)}{(x - x_{ip})A'(x_{ip})}$$

are rational functions of the coefficients of $a(x, x_i)=0$, the coefficients of $\phi_0(x)$ are in R. It is evident that $\phi_0(x_{ij})=x_i$, $(j=1, \cdots, p)$. Hence $\phi_0(x)-x_i$ is divisible by $a(x, x_i)$. As

$$f[\phi_0(x_{ij})] = f(x_i) = 0, \quad \begin{pmatrix} i = 1, \cdots, n \\ j = 1, \cdots, p \end{pmatrix},$$

$f[\phi_0(x)]$ is divisible by $A(x)$.

If $\phi(x)$ is another polynomial with coefficients in R such that $\phi(x)-x_1$ is divisible by $a(x, x_1)$ and $f[\phi(x)]$ is divisible by $A(x)$, then $\phi(x)-\phi_0(x)$ is also divisible by $a(x, x_1)$ and hence by $A(x)$; that is,

$$\phi(x) \equiv \phi_0(x), \qquad (\text{mod } A(x)). \tag{10}$$

Conversely, every polynomial $\phi(x)$ with coefficients in R satisfying this congruence has these properties, $\phi_0(x)$ being distinguished from the others by the fact that its degree is less than that of $A(x)$. The degree of every polynomial $\phi(x)$ of the system (10) excepting $\phi_0(x)$ exceeds that of $A(x)$; hence $f[\phi(x)]$ is reducible. But $f[\phi_0(x)]$ may be of the same degree as $A(x)$ and may be irreducible. For the complete solution of our problem it is necessary to determine those polynomials $a(x, y)$ which lead to a $\phi_0(x)$ such that $f[\phi_0(x)]$ is irreducible.

5. *The Polynomials* $P(x)$. We treat first the case in which $A(x)$ is $f(x)$ itself; that is, we consider the polynomials $\phi(x)$ with coefficients in R such that $f[\phi(x)]$ is divisible by $f(x)$.

If $f[\phi(x)]$ is divisible by $f(x)$, $\phi(x_1)$ is a root of $f(x)=0$; and conversely. Suppose $\phi(x_1)=x_2$. The subgroup of the group G of $f(x)=0$ relative to R which leaves x_1 fixed also leaves x_2 fixed. Hence* there exists a substitution t_2 on the symbols $x_1, \cdots, x_n$ (not necessarily in G) which is commutative with every substitution of G. Let H be the group consisting of the substitutions $t_1=1, t_2, \cdots, t_h$ on the symbols $x_1, \cdots, x_n$ which are commutative with all the substitutions of G.* The order of H is equal to the number of symbols fixed by the subgroup of G which leaves one symbol fixed. Let $x_1, x_2, \cdots, x_h$ be the symbols fixed by the subgroup of G which leaves x_1 fixed. Then $x_2, \cdots, x_h$ equal rational functions of x_1 with coefficients in R. These functions are readily constructed with the aid of Lagrange's interpolation formula. By a suitable choice of notation we may suppose that t_i changes x_1 to x_i. The functions in question are

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* See Miller, Blichfeldt and Dickson, *Finite Groups*, p. 37.

$$(11) \qquad P_i(x) \equiv \frac{t_i(x_1)f(x)}{(x - x_1)f'(x_1)} + \cdots + \frac{t_i(x_n)f(x)}{(x - x_n)f'(x_n)},$$

where $t_i(x_j)$ denotes the effect of t_i on x_j. Evidently $P_i(x_j) = t_i(x_j)$.

That the coefficients of $P_i(x)$ are in R may be seen as follows. Apply to the right member of (11) any substitution s of G, obtaining

$$\frac{s[t_i(x_1)]f(x)}{[x - s(x_1)]f'[s(x_1)]} + \cdots + \frac{s[t_i(x_n)]f(x)}{[x - s(x_n)]f'[s(x_n)]}.$$

As s and t_i are commutative, this expression equals

$$\frac{t_i[s(x_1)]f(x)}{[x - s(x_1)]f'[s(x_1)]} + \cdots + \frac{t_i[s(x_n)]f(x)}{[x - s(x_n)]f'[s(x_n)]},$$

which is clearly the same as the right member of (11), except possibly for the order in which the terms are written. Having shown that the coefficients of $P_i(x)$ are unaltered by G, we conclude that they equal numbers in R.

If $\phi(x)$ is a polynomial with coefficients in R such that

$$\phi(x_i) = P(x_i), \qquad (i = 1, \cdots, n),$$

where $P(x)$ is one of the polynomials (11), $\phi(x) - P(x)$ is divisible by $f(x)$; and conversely. Thus *every polynomial with coefficients in R having the property that $f[\phi(x)]$ is divisible by $f(x)$ is congruent modulo $f(x)$ to one of the polynomials $P(x)$.*

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* A method for constructing H when G is known is explained in Burnside's *Theory of Groups*, 2d edition, pp. 224–227.

We note in passing that the polynomials $P(x)$ form a group, the "product" of $P_i(x)$ and $P_j(x)$ being $P_i[P_j(x)]$ reduced modulo $f(x)$. This group is simply isomorphic with the group H previously described.

6. *Determination of all Polynomials $\phi(x)$ such that $f[\phi(x)]$ is Reducible.*

As noted at the end of §4 we must consider the case in which $f[\phi_0(x)]$ and $A(x)$ are of the same degree. We have

$$[\phi_0(x) - x_1] \cdots [\phi_0(x) - x_n] = ca(x,x_1) \cdots a(x,x_n),$$

where c is in R. It was shown in §3 that $a(x, x_1)$ is a divisor of one of the factors in the left member, say $\phi_0(x)-x_k$; that is,

$$a(x,x_1) = q[\phi_0(x) - x_k],$$

where q is a *constant* in R. Hence x_k equals a rational function of x_1 with coefficients in R. By taking the equation $f(x_1)=0$ into account, this rational function may be expressed as a polynomial in x_1 of degree less than n, and then must be one of the polynomials $P(x)$ of §5. Thus

$$a(x,y) = a(x) - qP(y). \tag{12}$$

Conversely, if $a(x, y)$ is of this form, $f[\phi_0(x)]$ and $A(x)$ are of the same degree. If, therefore, we avoid choosing $a(x, y)$ of this form, we may be certain that $f[\phi_0(x)]$ is reducible. However, the polynomials (12) may not be ignored. For, although they lead to polynomials as $\phi_0(x)$ such that $f[\phi_0(x)]$ may be irreducible, other polynomials $\phi(x)$ are determined from (10) with the aid of $\phi_0(x)$ which have the property that $f[\phi(x)]$ is reducible.

7. *Summary.*

Each polynomial $a(x, y)$ with coefficients in R determines ∞^1 polynomials $\phi(x)$ such that $f[\phi(x)]$ is reducible, by means of (9) and (10). The only exception occurs when $a(x, y)$ is of the form (12), in which case $\phi_0(x)$ is the only polynomial of the system (10) for which $f[\phi_0(x)]$ may be irreducible.

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CONCERNING R. L. MOORE'S AXIOMS Σ_1 FOR PLANE ANALYSIS SITUS[*]

BY R. L. WILDER

1. *Introduction.* R. L. Moore has proposed[†] a system, Σ_1, of eight axioms for plane analysis situs. That a space S satisfying this system is in one-to-one continuous correspondence with two-dimensional euclidean space was shown by Moore in a later paper.[‡]

It is the purpose of the present paper to show that the set Σ_1 may be reduced to a set of *seven* Axioms, by the elimination of Axiom 6, which is a consequence of the other Axioms. Doubt as to the independence of Axiom 6 was raised in the mind of the author by noticing that the independence examples given for Axioms 6 and 7 on pp. 162 and 163 of F.A. are not valid, and by the subsequent finding of an independence example for Axiom 7 accompanied by failure to find any independence example for Axiom 6.

2. *Independence of Axiom* 7. The independence of Axiom 7 is established be the following example:

Let the space considered by ordinary euclidean space of two dimensions. Choose a pair of rectangular axes OX and OY. For every positive integer n, let points be defined as follows: $A_n=(0,-1/n)$, $B_n=(1/n,-1/n)$, $C_n=(1/n,\ 1/n)$, $D_n=(0,\ 1/n)$, $E_n=(0,\ 0)$, $F_n=(1/(2n),\ 0)$. Let T_n be the bounded domain whose boundary is the rectangle $A_nB_nC_nD_n$ together with the straight line interval E_nF_n. Then a point set R is a region if and only if R is either the interior of some simple closed curve or identical with some T_n. That the conditions of Axiom 7 are violated at E_n for every T_n is easily evident.

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[*] Presented to the Society, December 28, 1927.

[†] *On the foundations of plane analysis situs*, Transactions of this Society, vol. 17 (1916), pp. 131–164. Referred to hereafter as F. A.

[‡] *Concerning a set of postulates for plane analysis situs*, Transactions of this Society, vol. 20 (1919), pp. 169–178.

3. *Proof of Axiom* 6 *on the Basis of Axioms* 1–5, 7, 8. In the proof of Theorems 1–27, inclusive, of F.A., no use is made of Axiom 6. The proof given of Theorem 28 is based on both Axioms 6 and 7, but can be made independent of Axiom 6 as follows:

Proof of Theorem 28 *without use of Axiom* 6: Let the proof for the case where M denotes the exterior of J be given as in F.A. Denote the interior of J by R.

Let $\overline{R}$ be any region about O, and let K be a region about O which lies wholly interior to $\overline{R}$ and fails to contain some point, D, of J. Then by the preceding part of the argument there exists an arc AXB such that (1) A and B are on J; (2) AXB, except for its end points, is common to M and K, (3) of the two arcs into which A and B divide J, that one which contains O lies in K.

Clearly that arc of J from A to B which contains D has only A and B in common with AOB. By Theorem 27 of F.A., either D is without $AXBOA$ or O is without $AXBDA$. But since $AXBOA$ is in K, the latter case would imply that D is a point of K (Theorem 21), so that the former case must hold. Then by the second part of Theorem 27, the interior of $AXBDA = AOB$ (except for its end points) + the interior of $AOBDA$ + the interior of $AXBOA$.

Let T be a region about O and lying wholly interior to $AXBDA$ and K. Then by the preceding part of the proof, there exist two points A' and B' on $AXBOA$ and an arc $A'X'B'$ such that $A'X'B'$ except for its end points is interior to T and exterior to $AXBOA$, and that arc of $AXBOA$ from A' to B' which contains O lies in T. As the only points of $AXBOA$ that lie in T are on AOB, it is clear that the arc $A'OB'$ is a subset of AOB and hence of J. That $A'X'B'$, except for its end points, is a subset of R is evident at once since it lies interior to $AXBDA$ but contains no point of AOB or the interior of $AXBOA$.

Since T is a subset of K, which in turn is a subset of $\overline{R}$, we see that $A'X'B'$ is an arc such that (1) A' and B' are on J, (2) $A'X'B'$ is common to $\bar{R}$ and R, (3) of the two arcs into which A' and B' divide J that one which contains O lies in $\bar{R}$.

In the proof of the remainder of the theorems of F.A., that is, Nos. 29–52, inclusive, no use is now made of Axiom 6.

I shall now prove the following theorem without use of Axiom 6:*

THEOREM A. *If R is a region and J is the boundary of R, then J is a simple closed curve.*

Two proofs will be given. The first of these is based entirely on F.A. and may be understood by a reader who has no further familiarity with the existing literature of analysis situs. The second proof has the advantage of being quite brief, and to one who is familiar with the various results that have been obtained in later papers dealing with plane analysis situs—especially with the theory of continuous curves—it is by far preferable.

FIRST PROOF. This proof is based on Theorem 48 of F.A. That J satisfies all conditions† of this theorem except (3) is readily apparent if we let $S_1 \equiv R$, and $S_2 \equiv S - R'$. I shall proceed to show that condition (3) is satisfied.

Let x be any point of J, and let P be a point of R. Let $J_1, J_2, J_3, \cdots$ be a sequence of simple closed curves every one of which encloses x and such that (1) for every positive integer i, J_{i+1} lies interior to J_i, and (2) x is the only point common to the regions bounded by the simple closed curves of this sequence. (Theorems 5 and 36 of F.A.)

As x is a limit point of R, there exists in R a sequence of distinct points, $P_1, P_2, P_3, \cdots$, having x as a sequential limit point and such that for every positive integer i, P_i is interior to J_i. By Theorem 16, P and P_1 are the extremities of an arc a_1, and for every positive integer n, P_n and P_{n+1} are the extremities of an arc t_{n+1}, such that the arcs a_1, t_2, t_3, t_4, $\cdots$ all lie in R. Let x_2 be the last point of a_1 on t_2 in the order from P_1 to P_2. That portion of t_2 from x_2 to P_2 is an arc a_2. Let x_3 be the last point of a_1+a_2 on t_3 in the order from P_2 to P_3. That portion of t_3 from x_3 to P_3 is an arc a_3. Continuing in this way indefinitely, there is obtained a sequence of arcs a_1, a_2, a_3, $\cdots$, such that for every positive integer n, a_n has only one point, that is an end point x_n, in common with the set $a_1+a_2+\cdots+a_{n-1}$.

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* See Theorem A, p. 163 of F. A.

† That the first of these conditions is superfluous has been shown by P. M. Swingle. See this Bulletin, vol. 34 (1928), pp. 607–618. This theorem (No. 48 of F. A.) was first stated by Schoenfliess.

(1) If for every value of n there exists a positive integer k, such that a_k is the last arc of the sequence of arcs $\{a_i\}$ having points on J_n, then it is easily shown that the set $\sum_1^\infty a_n$ together with the point x contains an arc from P to x which lies, except for x, wholly in R.

(2) Suppose there exists a positive integer n such that infinitely many of the arcs of the sequence $\{a_i\}$ have points in common with J_n. Then infinitely many arcs a_i of the sequence have points exterior to J_{n+1} as well as P_i interior to J_{n+2}. For every such arc, a_i, let $\overline{A}_i$ be the first point of J_{n+1} on a_i in the order from P_i to x_i (letting x_1 denote P). Then let $\overline{B}_i$ be the first point of J_{n+2} on that portion of a_i from $\overline{A}_i$ to P_i, in the order from $\overline{A}_i$ to P_i. From $\overline{A}_i$ to B_i there exists an arc $\overline{A}_i\overline{B}_i$ which is a subset of a_i, and such that if I_{n+1} is the set of all points between J_{n+1} and J_{n+2} (see footnote, p. 157 of F.A.), $\overline{A}_i\overline{B}_i$, except for its end points, is a subset of I_{n+1}. Call the set of all such arcs $\{a_i^*\}$. It is easily shown that no two arcs of the set $\{a_i^*\}$ have any points in common.[†] Since J_{n+1} is bounded and closed, the infinite set of points of type $\overline{A}_i$ has at least one limit point A on J_{n+1} (Theorem 13). If z is any other point on J_{n+1} then at least one of the arcs into which A and z divide J_{n+1} must contain an infinite set of points of type $\overline{A}_i$ having A as a limit point; call this arc Az. Then

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[†] See pp. 344–345 of my paper *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), pp. 341–377.

from the points of type $\overline{A}_i$ can be selected an infinite sequence $A_1, A_2, A_3, \cdots$, in the order from z to A on Az, and having A as a sequential limit point. The set of points $B_1, B_2, B_3, \cdots$, where B_n is the other end point of the arc of $\{a_i^*\}$ to which A_n belongs has a sequential limit point B. There is obtained thus a sequence A_1B_1, A_2B_2, $A_3B_3, \cdots$, of arcs of the set $\{a_i^*\}$ arranged in a definite order. Call this sequence the set $\{A_iB_i\}$.

The set $\{A_iB_i\}$ has a limiting set, M_1, which[†] evidently contains A and B. Let w be a point of M_1 in I_{n+1} (that such a point exists is easily shown; for instance, any simple closed curve which encloses J_{n+2} and lies within J_{n+1} will contain such a point), and let C_1 be a simple closed curve enclosing w and lying wholly in I_{n+1} but not enclosing J_{n+2}. Let A_kB_k be an arc of the set $\{A_iB_i\}$ which has points interior to C_1, and let y be one such point. Let s be an arc from y to w lying wholly interior to C_1, and let m be the first point of M_1 on s in the order from y to w. Denote that portion of s from y to m by t. From the order of the arcs $\{A_iB_i\}$ it follows that there exists a positive integer j such that every arc A_iB_i for which $i>j$ has a point y_i on t, and such that m is the sequential limit point of the sequence $y_j, y_{j+1}, y_{j+2}, \cdots$.

(a) If m is a point of R, there exists a region R_1 which contains m and lies wholly in R. Then all but a finite number of the arcs of the set $\{A_iB_i\}$ are joined to one another by a sub-arc of t which lies wholly in R.

(b) If m is not a point of R, then it must be a point of J. By Axiom 7 there exist in C_1 two regions K and $\overline{K}$ such that $\overline{K}$ contains m, K lies in $S-R'$, and all those points of J that lie in $\overline{K}$ are points also of the boundary of K.

For each i, A_iB_i and $A_{i+1}B_{i+1}$, together with those arcs A_iA_{i+1} and B_iB_{i+1} of J_{n+1} and J_{n+2}, respectively, that contain no end points of the set $\{A_iB_i\}$, form a simple closed curve K_i which cannot enclose m. For every positive integer

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[†] A point x is contained in M_1 if it is a sequential limit point of a sequence of points $x_1, x_2, x_3, \cdots$, where for every n, x_n is a point of A_nB_n

$r \geqq k$, there is an arc b_r, subset of t, which has one end point on each of the arcs A_rB_r and $A_{r+1}B_{r+2}$, and which lies, except for these end points, entirely within K_r. For each r there is at most a finite number of such arcs b_r.

If only a finite number of arcs of type b_r contain points of $S-R$, then all but a finite number of the arcs $\{A_iB_i\}$ are joined by arcs of R which lie wholly interior to C_1. That this is indeed the case will follow if it can be shown that there cannot be infinitely many of the arcs of type b_r that contain points of $S-R$. This is done as follows: If an arc b_r contains a point of $S-R$, then, since its end points are in R, it must contain at least one point, P_r, of J. Let one such point be selected for each simple closed curve K_r $(r \geqq k)$. The point m is the sequential limit point of the set of points $\{P_r\}$, provided infinitely many such points exist. Then, since $\overline{K}$ contains m, $\overline{K}$ will contain at least two distinct points P_c and P_d, of type P_r, and by the conditions of Axiom 7 these points are on the boundary of K. If I_c and I_d denote, respectively, the interiors of K_c and K_d, then I_c and I_d have no point in common since no two points of type P_r lie interior to the same curve of type K_r. But then K must contain points in both I_c and I_d, and being a connected set must also contain a point, Q, on K_c. That Q is not on J_{n+1} or J_{n+2} is evident, since K is a subset of C_1. Hence Q is a point of some arc of the set $\{A_iB_i\}$. But this is impossible, since K lies in $S-R'$, and the arcs $\{A_iB_i\}$ all lie in R. Thus there cannot exist infinitely many points of type P_r, and the conclusion stated in the first sentence of this paragraph holds in any case.

There must, then, exist a number N_1, such that for $u > N_1$ and $v > N_1$, A_uB_u and A_vB_v are joined by an arc of R which lies wholly interior to C_1. A fortiori, those arcs of the set $\{a_i\}$ of which the arcs of the set $\{A_iB_i\}$ of subscript $> N_1$ are subsets, are joined by arcs of R in the same way. Call the set of such arcs S_1.

For each arc a_i of S_1 that has points interior to J_{n+3} let $\overline{E}_i$ be the last point of J_{n+2} on a_i in the order from x_i to P_i, and let $\overline{F}_i$ be the first point of J_{n+3} on that portion of a_i from $\overline{E}_i$ to P_i, in the order from $\overline{E}_i$ to P_i. Each such arc a_i contains an arc $\overline{E}_i\overline{F}_i$, such that $\overline{E}_i\overline{F}_i$, except for its end points, lies wholly in I_{n+2} (where I_{n+2} is the set of all points between J_{n+2} and J_{n+3}). It can be shown, by the methods employed above, that from the arcs of the latter collection can be selected an infinite sequence, E_1F_1, E_2F_2, E_3F_3, $\cdots$, having the property that there exists a number N_2 such that for $u>N_2$ and $v>N_2$, E_uF_u is joined to E_vF_v by an arc of R which lies wholly within I_{n+2}. The set of all those arcs of S_1 which contain arcs of this sequence of subscript $>N_2$ denote by S_2.

Continuing in this way it is shown that there exists an infinite sequence of sets S_1, S_2, S_3, $\cdots$, of arcs of $\{a_i\}$, such that for any positive integer j (1) S_{j+1} is a subset of S_j, (2) if a_u and a_v are any two arcs of $\{a_i\}$ which belong to S_j, there is an arc ab of R such that (i) one end point of this arc, a, is a point of a_u, and the other end point, b, is a point of a_v, (ii) the arc ab lies wholly interior to I_{n+j} (the set of all points between J_{n+j} and J_{n+j+1}), and (iii) the arcs aP_u and bP_v, subsets of a_u and a_v, respectively, lie wholly interior to J_{n+j}.

The first arc of the sequence $\{a_i\}$ which belongs to S_1 denote by f_1. The first arc of $\{a_i\}$ after f_1 which belongs to S_2 denote by f_2. Then there exists, in I_{n+1}, an arc of R whose end points are a point x' of f_1 and a point y' of f_2, and such that the arc $y'P_e$, a subset of the arc $f_2\equiv a_e$, lies wholly interior to J_{n+1}. The first arc of $\{a_i\}$ after f_2 which belongs to S_3 denote by f_3. Then there exists, interior to I_{n+2}, an arc of R having as end points a point x'' of $y'P_e$ and a point y'' of f_3, and such that the arc $y''P_h$, subset of $f_3\equiv a_h$, lies wholly interior to J_{n+2}. Continue this process indefinitely. Then, if $f_1\equiv a_q$, the set $a_1+a_2+\cdots+a_q+x'y'+y'P_e+x''y''+y''P_h+\cdots+x$ can be shown to contain an arc from P to x which lies, except for x, wholly in R.

To show that if P is exterior to R, then there exists an arc from P to x which lies, except for x, wholly in $S-R'$, we can proceed as above, with only slight modification of the proof.

It follows that condition (3) of Theorem 48 of F.A. is satisfied, and therefore J is a simple closed curve.

SECOND PROOF.* By virtue of the results obtained in R. L. Moore's paper *Concerning a set of postulates for plane analysis situs*,† space S is equivalent to a euclidean plane. We may, then, make free use of theorems established for the euclidean plane. By the Phragmen-Brouwer theorem, since J is the common boundary of the domains R (bounded) and $S-R'$ it is a bounded continuum. Suppose it is not a continuous curve. Then there exist two concentric circles k_1 and k_2 and a countable infinity of mutually exclusive continua $\overline{M}, M_1, M_2, \cdots$, satisfying all the conditions of the theorem of section 3 of R. L. Moore's *Report on continuous curves from the viewpoint of analysis situs*.‡ Let P denote a point of $\overline{M}$ lying in the point set $\overline{H}$ composed of the set of all points between k_1 and k_2 and let $\overline{R}$ denote a region containing P and lying, together with its boundary, in $\overline{H}$. By Axiom 7 there exist, in $\overline{R}$, two regions K and $\overline{K}$ such that $\overline{K}$ contains P, K lies in $S-R'$ and all points of the boundary of R that lie in $\overline{K}$ belong to the boundary of K. There exist four continua of the sequence $M_1, M_2, \cdots$, that contain points of $\overline{K}$. They may be so labelled x, y, z, w that if a connected subset of $\overline{H}$ contains a point of x and a point of z then it contains a point of $y+w$. The boundary of K contains a point A of x and a point B of z. The point set $K+A+B$ is a connected subset of $\overline{H}$. Hence K contains a point of $y+w$. But $y+w$ and K are subsets of J and of $S-J$ respectively. Thus the supposition that J is not a continuous curve leads to a contradiction. Furthermore, by Theorem 20 of F.A., J is the outer boundary of R. Hence J is§ a simple closed curve.

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* This proof was suggested to the author by R. L. Moore, to whom the original manuscript of this paper was sent for criticism prior to its being offered for publication.

† Loc. cit.

‡ This Bulletin, vol. 29 (1923), pp. 296–297.

§ R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), p. 258, Theorem 4.

THEOREM B. (Axiom 6 of F.A.) *If R and $\overline{R}$ are regions and P is a point in $\overline{R}$ and on the boundary of R, then there exist in $\overline{R}$ two regions K and $\overline{K}$ such that $\overline{K}$ contains P, K lies in R and all those points of the boundary of R that lie in $\overline{K}$ are points also of the boundary of K.*

PROOF. By Theorem A the boundary of R is a simple closed curve J. By Theorem 28 of F.A. there exists a simple continuous arc AXB such that (1) A and B are on J, (2) AXB, except for its end points, is common to R and $\overline{R}$, (3) of the two arcs into which A and B divide J, that one, APB, which contains P lies in $\overline{R}$. Let the interior of the simple closed curve formed by AXB and APB be denoted by K, and let $\overline{K}$ be any region which contains P, lies wholly in $\overline{R}$, and contains no point of AXB nor of that arc of J which does not contain P. The regions K and $\overline{K}$ defined in this manner satisfy the conditions of the theorem.

In conclusion, it is perhaps interesting to note that in the above proof of Theorem A we have also proved the following theorem.

THEOREM C. *If D is a bounded domain (as defined in F.A., p. 136) such that (1) $S-D'$ is connected, and (2) if P is a boundary point of D and R is a region containing P then there exist in R two regions K and $\overline{K}$ such that $\overline{K}$ contains P, K lies in $S-D'$ and all those points of the boundary of D that lie in $\overline{K}$ are points also of the boundary of K, then the boundary of D is a simple closed curve.*

Furthermore, by virtue of a theorem due to J. R. Kline[*] we can state the following theorem.

THEOREM D. *If D is an unbounded domain satisfying conditions (1) and (2) of Theorem C, then the boundary of D is an open curve.*

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[*] *The converse of the theorem concerning the division of a plane by an open curve*, Transactions of this Society, vol. 18 (1917), pp. 177–184. See in this connection the article of P. M. Swingle referred to in a preceding footnote.

THE MONTESANO QUINTIC SURFACE*

BY A. R. WILLIAMS

In a previous paper in this Bulletin (vol. 34 (1928), pp. 631–638), I have mentioned the fact that Montesano† has shown how to obtain a quintic surface with two consecutive skew double lines by quadratic transformation of a rational sextic. The latter has a quadruple point Q, at which are concurrent three coplanar double lines d_1, d_2, d_3, and a triple line k, which of course does not lie in the plane of the double lines. The plane system for the rational sextic and the derived quintic are given by Montesano. That of the sextic consists of the web of curves of order nine having in common 8 triple points $A_1, A_2, \cdots, A_8$ and 3 simple points B_1, B_2, B_3. The image of the triple line is the sextic with double points at $A_1, A_2, \cdots, A_8$ and passing through the points B. The image of a double line d_i is the cubic determined by the 8 points A and B_i. To B_i corresponds the residual line of the surface in the plane of the triple line and d_i. If we now apply a quadratic transformation whose fundamental system is an isolated point O on one of the double lines, say d_3, and the degenerate conic kd_1, the resulting surface will be of degree five. For it loses the plane kd_3 three times by reason of the triple line, the plane d_1d_3 twice by reason of the double line d_1, and the plane kd_1 twice on account of the double point at O. The transform of any ray through Q is another ray through Q. Therefore the quintic has a triple point at Q. A generic plane of the pencil k becomes a plane of the pencil d_1, and the relation between these planes is a collineation. Since a section of the sextic by a plane through k is a cubic which meets the plane d_1d_3 three times at Q, the section of the new surface by the corresponding plane through d_1 is a cubic which has d_1 for inflectional tangent at Q. Moreover, the plane kd_2 intersects the sextic in a residual line. The transform of this plane, which we will call π, meets the quintic, therefore, in d_1 and one other line; that is, d_1 counts for two coincident double lines on the quintic. Two sheets of the surface are tangent along d_1, the tangent plane being π. The trace of the quintic in the plane kd_1, which we will call σ, is the double line d_1, the line k, which is a simple line on the quintic corresponding to the residual intersection of the plane kd_3 with the sextic, and two lines passing through Q which correspond to the tangent planes to the sextic at O. They are in fact simply the intersections of those tangent planes with the plane kd_1. The tangent cone to the quintic at Q is the plane π and the plane σ taken twice. The equation of the surface is of course easily obtained and will be used later.

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* Presented to the Society, San Francisco Section, April 7, 1928.

† Montesano, Rendiconti di Napoli, vol. 46 (1907), p. 66.

When the quadratic transformation is applied to the sextic the intersections of the homoloidal quadrics with the sextic become the plane sections of the quintic. The image of the intersection of a general quadric with the sextic is a curve of degree 18 passing 6 times through each of the 8 points A and twice through each point B. Since the homoloidal quadrics contain the triple line and the double line d_1 and have in common the point O on d_3, the image of the intersection of such a quadric with the sextic is of degree 9, having triple points at the 8 points A, passing through B_2 and B_3, and through the two points on the cubic determined by B_3 which correspond to O on d_3. To directions at these points correspond directions at O in the two tangent planes to the sextic, which we have seen give two lines through the node on the quintic. Therefore the plane system for the quintic consists of the web of curves of order nine having in common 8 triple points $A_1, A_2, \cdots, A_8$ and four simple points, say B, and E_1, E_2, E_3, where the three latter are the residual intersections of a cubic of the pencil with any proper nonic which has triple points at the 8 points A and passes through B. Let ψ be the cubic of the pencil determined by B, and ϕ the cubic that contains the points E. Then it is easily seen that ϕ corresponds to the node (triple point). To the nodal sections correspond the sextics of the net B; that is, the sextics that have double points at $A_1, A_2, \cdots, A_8$ and pass through B and B', where B' corresponds to B in I_{17}, the plane involution determined by the sextics that have double points at the 8 points A. Therefore B' lies on ψ and the complete image of the node is ϕ and the point B'. To directions at B', which corresponded on the sextic to directions at the node in the plane of the triple line and d_2, correspond on the quintic directions at the node in the plane π, the transform of that plane. To B there corresponds the line b, or residual intersection of π with the quintic. To E_1, E_2, E_3 correspond e_1, e_2, e_3, the simple lines of the quintic passing through the node in the plane σ. The cubic ψ is the image of the double line $\pi\sigma$. The points of ϕ are paired by I_{17}. A pair correspond to a direction at the node in σ. Through such a pair pass a pencil of sextics of the net B corresponding to the pencil of plane sections whose axis is the associated direction in σ. To a pencil of nodal sections having its axis in π correspond a pencil of sextics having a common tangent at B'. To a pencil of such sextics having a common tangent at B correspond a pencil of nodal sections whose axis joins the node to a point on b. No proper sextic of the net B, and no proper nonic of the fundamental system can pass through A_0, the ninth base point of the pencil of cubics. To A_0 corresponds at the node the direction of the double line. The images of the sections by planes through the double line are the cubics of the pencil. Twelve of these have a node; that is, the corresponding planes are tangent to the surface.

The number of fundamental curves of a pencil that have an additional node, that is, the class of the surface, is 30. For we have in the first place $3(9-1)^2$ less 20 for each of the eight common triple points, that is, 32. But a general pencil of planes contains one nodal section, and the corresponding pencil of fundamental curves has one degenerate curve made up of ϕ and a sextic of the net B. The two points where the latter meets ϕ are to be deducted, and the class of the surface is thus 30. Further, it is obvious geometrically that the Jacobian of a general net, which is the image of the curve of contact of the corresponding tangent cone, must contain ϕ as a factor. For through any point of space go a pencil of nodal sections; that is, in the corresponding net of fundamental curves there is a pencil of degenerate curves one of which has a double point at any given point of ϕ. The image of the curve of contact of a general tangent cone is therefore of degree 21, and has a seven-fold point at each of the 8 points A, a double point at B and simple points at E_1, E_2, E_3. Of the intersections of two such curves, 399 take place at the fundamental points. The class of the surface accounts for 30 more, leaving 12 to be found. We must, in fact, account for 5 more intersections of such a curve with ψ (the image of the double line), 4 more with ϕ, and 7 more with a general cubic of the pencil, or image of a plane section through the double line. Such a cubic meets ψ only at the fundamental points and A_0. This shows, as is otherwise obvious, that the curve of contact of a tangent cone can meet the double line only at the node and at pinch points, and therefore its image (of degree 21) meets ψ only at the fundamental points and at points which correspond to pinch points or to the node. Since in any net there is a pencil of degenerate curves corresponding to nodal sections, the net may be determined by three curves of the form $U\phi$, $V\phi$, and N, where U and V are sextics of the net B, and N is a proper fundamental nonic. We now find easily that $zJ=9\phi[\phi R+NS]$, where R is of degree 19 and $S=U(\phi_x V_y-\phi_y V_x)+V(\phi_y U_x-\phi_x U_y)$. Now remembering that B is a point of ψ, any sextic of the net B can be written $\alpha(\phi^2 f'-\phi'^2 f)+\beta\psi^2 f'+\gamma\phi\psi f'=0$, where f is any proper sextic having double points at $A_1, A_2, \cdots, A_8$ and f' and ϕ' are the values of f and ϕ at B*. Expressing U and V in this way we find that S vanishes for all points common to ϕ and ψ, that is, the curve of order 21 passes through A_0, a point corresponding to the node. It is immediately seen by use of R that it does not pass through the other point on ψ corresponding to the node, B', where U and V are 0. Hence there are just four pinch points on the double line. The remaining three intersections of the curve of order 21 with ϕ are of interest. The locus of self-corresponding points in the involution I_{17} is a curve of order 9, having a triple point at each of the 8 points A and meeting a cubic of the pencil at 3 other points. At such a point on ϕ the pencil of sextics of the net B that pass through it are tangent to ϕ, one of them to be sure having a double point. Now in a general net of fundamental curves there is one degenerate curve composed of a sextic of this pencil and ϕ. Since any three linearly independent curves of the net determine it we may let U be this sextic. Then at the point in question we have $U=\phi=0$ and $\phi_y U_x-\phi_x U_y=0$, since U and ϕ are tangent there. Hence for such a point S vanishes, and it is a point on the curve of order 21. Since the points on ϕ correspond to directions at the node in the plane σ, the curve of contact of a general tangent cone passes four times through the node, three times in fixed directions in σ and once tangent to the double line. Therefore of the 12 intersections of two curves of degree 21 not accounted for by the fundamental points, either 8 take place at the images of the four pinch points on ψ, with simple intersections at A_0 and the other 3 points on ϕ just considered, or vice versa. It can be shown definitely by use of the above equation that the two curves are not tangent at A_0, so they must be tangent at the images of the four pinch points. That, of course, might be assumed. For at a pinch point on an ordinary double line the curves of contact of any two tangent cones have two points in common, while (if the surface is rational) the corresponding Jacobians have a simple intersection at the image of the pinch point. In the case of two coincident double lines the former number is 4. Hence we should expect the two Jacobians to be tangent. Moreover, one of our curves of degree 21 meets any cubic of the pencil six times, aside from the 8 points $A_1, A_2, \cdots, A_8$ and A_0.

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* Snyder, American Journal of Mathematics, vol. 33, p. 328.

That is, on any plane section through the double line are six points at which the tangent plane passes through a given point. We have seen that the image of the curve of contact of a general tangent cone is of degree 21. Its genus is also 21, and therefore the tangent cone (of order 16) has 42 cuspidal edges and 42 double edges. The quadruple edge at the node counts for 6 of the latter. In fact it may be shown by use of the equation of the surface that the tangent cone has a quadruple edge, 42 cuspidal edges, and 36 double edges with distinct points of contact.*

The sextics having double points at the 8 points A, but which do not pass through B, give on the surface skew sextics of genus 2. One of these plane sextics meets ϕ in two associated points, that is, two points corresponding to the same direction at the node in the plane σ. Hence the skew sextic has a tacnode at the node of the surface and meets the double curve in two other generally distinct points. The plane sextic may be composed of the line joining two of the 8 points A and the quintic having double points at the other 6 and passing through the first two. The line and the quintic meet ϕ in two associated points, and meet each other in 3 points on the locus of self-corresponding points in I_{17}. Thus we get on the surface two skew cubics, intersecting at 3 points, tangent to each other at the node, and each meeting the double line at one other point. There are obviously 28 such pairs. Likewise the plane sextic may be composed of the conic through 5 points A and the quartic having double points at the other 3 and containing the first 5. This gives a similar pair of space cubics. There are 56 of these pairs. The cubics corresponding to the 8 fundamental points A_i are of this type. With the cubic corresponding to one of them, say A_1, is paired the cubic whose image is the sextic corresponding to A_i in I_{17}, that is, the sextic having a triple point at A_i and double points at the other 7 points A. There are thus 92 pairs, or 184 such skew cubics on the surface. To the locus of self corresponding points

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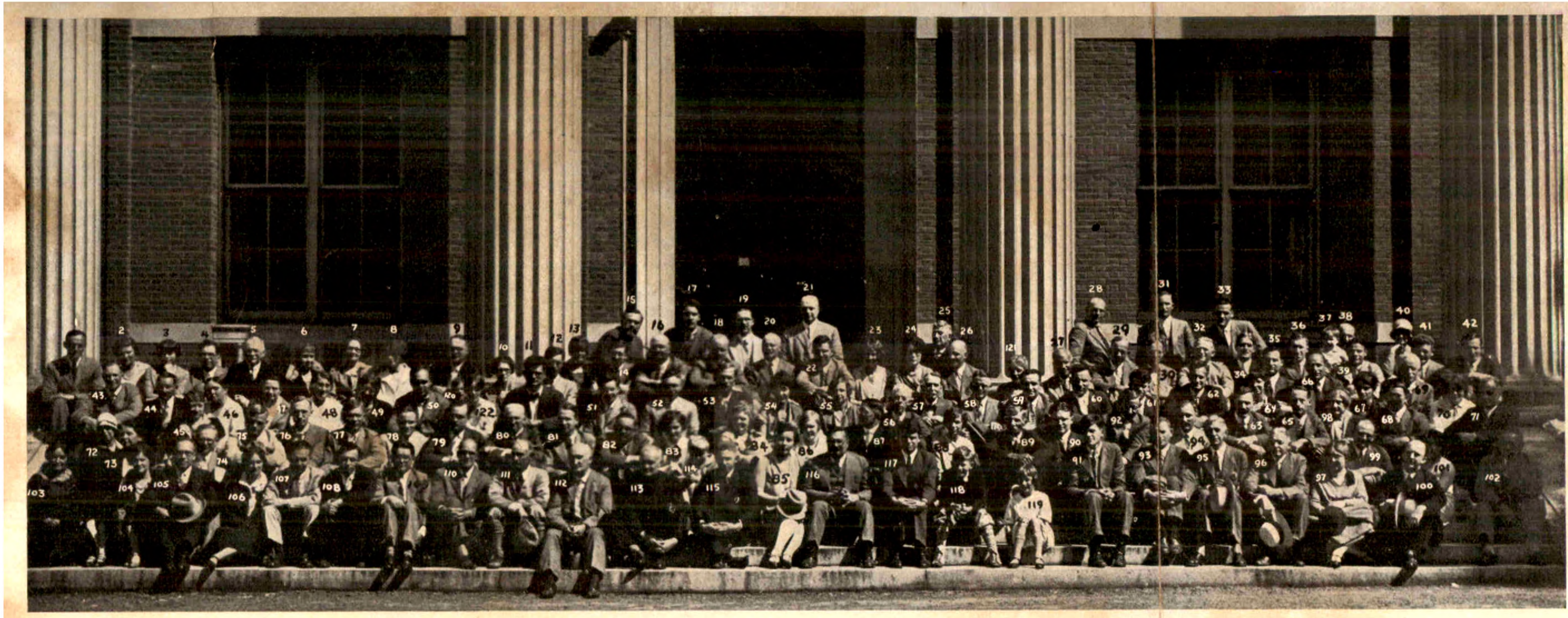
* Salmon, *Solid Geometry*, vol. II, Chap. XVIIa.



Group Photograph Taken at the Amherst Meeting, 1928

1. Jackson, Dunham, 2. Wilson, Elizabeth, 3. MacColl, Mrs. L. A., 4. MacColl, L. A., 5. Olds, G. D., 6. Olds,Mrs. G. D., 7. Perkins, Mrs. Fred W., 8. Perkins, Gwendolyn, 9. Perkins, Fred W., 10. Johnson, Mrs. Roger A., 11, Johnson, Roger A., 12. Struik, D. J., 13. MacDuffee, Mrs. C. C., 14. MacDuffee, C. C., 15. Dresden, Arnold, 16.Richardson, R. G. D., 17. Cairns, Bob, 18. Ingraham, M. H., 19. Langer, R. E., 20. Coolidge, J. L., 21. Tamarkin, J. D., 22. Adams, C. R., 23. Adams, Mrs. C. R., 24. Bennett, Mrs. A. A., 25. Campbell, G. A., 26. Bennett, A. A., 27.Smith, W. M., 28. Cohen, A., 29. Coble, A. B., 30. Bliss, G. A., 31. Wilson, C. R., 32. Rietz, H. L., 33. Thurston, H. S.. 34. Hedrick, E. R., 35. Fry, Thornton C., 36. Mode, E. B., 37. Copeland, Arthur H., 38. Copeland, Arthur H., Jr.,39. Karnow, Herman, 40. Copeland, Mrs. A. H., 41. Brinckman, H. W., 42. Hickson, A. O., 43. Milne, W. E., 44. Schelkunoff, S. A., 45. Coble, Mrs. A. B., 46. Bacon, Clara L., 47. Martin, Emilie N., 48. Simons, Lao, 49. Kryder, Lylah,50. Stetson, J. B., 51. Allison, Eli, 52. Grove, Vernon G., 53. Fort, T., 54. Cairns, Mary, 55. Carlen, Mildred E., 56. Michie, J. N., 57. Morris, Richard, 58. Taylor, Eugene, 59. DoBell, H. A., 60. Jeffery, R. L., 61. Moore, T. W.,62. Tracy, J. I., 63. Kempner, Aubrey J., 64. Fry, Mrs. T. C., 65. Rider, P. R., 66. Douglas, J., 67. Seeley, Caroline, 68. Langman, Harry, 69. Manning, H. P., 70. Thurston, Mrs. H. S., 71. Brown, H. S., 72. Cohen, Mrs. A., 73. Smith,Clara, E., 74. Crathorne, A. R., 75. Gillespie, D. C., 76. Dederick, L. S., 77. Ingold, L., 78. Bennett, Theodore, 79. Johnston, F. E., 80. Roe, E. D., Jr., 81. Porter, Scott, 82. Weaver, Warren, 83. Weaver, Mrs. Warren, 84. Moore, Mrs.C. L. E., 85. Wiener, Mrs. Norbert, 86. Moore, Hazel A., 87. Moore, C. L. E., 88. Torrey, Marian M., 89. Weaver, George, 90. Gehman, H. M., 91. Weaver, James, Jr., 92. Hurwitz, W. A., 93. Weaver, J. H., 94. Blumberg, Henry, 95.Cairns, W. D., 96. Esty, T. C., 97. Yeaton, Mrs. C. H., 98. Johnson, Myra I., 99. Bradshaw, J. W., 100. Yeaton, C. H., 101. Langman, Rebecca J., 102. Slobin, H. L., 103. Camp, Mrs. C. C., 104. Copeland, Lennie P., 105. Camp, C. C.,106. Schoonmaker, Hazel E., 107. Shaub, Howard C., 108. Richmond, D. E., 109. Huber, C. M., 110. Currier, C. H., 111. Dines, L. L., 112. MacMillan, W. D., 113. Feinler, F. J., 114. Weaver, Warren, Jr., 115. Davis, H. T., 116. Wiener Norbert, 117. Berry, Andrew C., 118. Weaver, Paul, 119. Coble, Elinor, 120. Johnson, Duncan, 121. Smith, Mrs. W. M., 122. Stetson, Mrs. J. B.

in I_{17} corresponds on the surface a curve of degree 9 and genus 4. which passes through the 3 points in which any one of these cubics meets the cubic with which it is paired.

The genus of a curve of degree n on the surface does not exceed the greatest integer in $(n^2+n+14)/10$.

A few slightly specialized types will now be mentioned. In setting up the plane system we may choose the 8 points A, which determine A_0, and a point B, which determines B' and the cubic ψ. We take any other cubic of the pencil for ϕ and then any proper curve of degree 9 having triple points at the 8 points A and passing through B will meet ϕ in 3 other points E_1, E_2, E_3. Two of these may fall together. In fact we may make the ninth degree curve tangent to ϕ at any point. Then two of the lines through the triple point will coincide; the class will be reduced by two, and the surface will acquire correspondingly a conic node on the two coincident lines, but in general distinct from the triple point. The Jacobian will have a triple point where the ninth degree curve is tangent to ϕ; but one of the 3 branches is tangent to ϕ and the removal of ϕ leaves the curve of degree 21 with a double point. Two such curves have thus four intersections at the coincident points instead of two, corresponding to the fall in the class of two; and everything else takes place as before. The images of the plane sections through the new node are the ∞^2 proper fundamental curves that have a double point at the two coincident base points on ϕ. The only points on ψ, the image of the double line, that correspond to the node are A_0 and B'; and a proper fundamental nonic can pass through neither. Therefore a general plane section through the new node does not contain the triple point, and the two singular points are distinct. The one pencil of plane sections that contain both nodes have for images the pencil of sextics of the net B that pass through the point of tangency of the ninth degree curve and ϕ. Similarly it might happen that E_1, E_2, E_3 fall together. Then the 3 simple lines through the triple point coincide; the class of the surface will be reduced by 3; and the new node on the coincident lines will be a binode, B_3. The plane of the coincident lines and the double line osculates the surface along the former. The equation of the surface, to be given later, shows that the new node, whether conic node or binode, may be thrown to the origin by suppression of a single coefficient. Apparently that can be accomplished only by allowing B to approach the coincident points E on ϕ. Also if ϕ is fixed we can not select 3 coincident points E arbitrarily. It appears that such a point is a possible ninth triple point for the nonics having triple points at the 8 points A.

If B is taken on the line joining two of the 8 points A, this line with the residual quintic and ϕ constitute the image of a plane section. To the line corresponds a conic on the surface cutting b and passing through the triple point. The quintic must pass through B', and the corresponding curve on the surface is a plane cubic with a double point at the node. If the join of two points A contains B and one of the points E, the corresponding conic consists of the line e and one other line. Finally, if three of the 8 points A lie on a line the surface has a new conic node. A_0 then lies on the conic determined by the other five; and to this conic corresponds a plane rational cubic that lies in a plane through the double line, passes through the original node (triple point) of the surface, and has its own double point at the new node. The images of the plane sections through the new node are of course octic curves which with the line in question constitute fundamental curves. One pencil of these octics consists of ϕ and the quintics which have double points at the remaining 5 points A, simple points at the 3 that are collinear, and which contain B. These are the images of the sections that pass through both nodes. Similarly if 6 of the 8 points A lie on a conic we get a new conic node. To the line joining the other two, and containing of course A_0, corresponds a plane rational cubic like that just mentioned.

It seems clear that the rational sextic described by Montesano is the most general sextic that has those singularities. If we assume this, to obtain the equation of the quintic we have merely to write the general equation of a sextic having xy for triple line and yz; $y+mx$, z; and xz for double lines, and apply the quadratic transformation

$$x':y':z':w' = xy:xz:yz:yw \text{ or } x:y:z:w = x'y':x'z':y'z':y'w'.$$

This gives the quintic

$$\begin{aligned} a_0y^2(z + my)w^2 + wy[a_1xy(z + my) + a_2xz(z + my) \\ + a_3y^3 + a_4y^2z + a_5yz^2 + a_6z^3] + b_1x^3(z + my)^2 \\ + x^2(z + my)L + xM + yzN = 0, \end{aligned}$$

where L, M, and N are binary forms of degree 2, 4, and 3 respectively in y, z. Here the planes π and σ are $z+my$ and y, and yz is the double line. The trace of the surface in $z+my$ is the double line and a residual line. Two sheets of the surface are tangent to each other along yz. It is easily shown without the aid of the plane representation that there are four points where the contact is higher. They are the pinch points that separate the portions of the double line where the surface is real from those where it is imaginary.

We saw above that a new node resulted when two of the three coplanar simple lines through the triple point became coincident. Setting $y=0$ in the equation just obtained we find $x(b_1x^2+b_4xz+b_9z^2)=0$ for the three lines. The terms in L and M that do not involve y are b_4z^2 and b_9z^4. Now the four partial derivatives vanish identically at all points on xy, except the single term b_9z^4 in $\partial/\partial x$ and the terms $a_6z^3w+c_4z^4$ in $\partial/\partial y$.

The term in N that does not involve y is c_4z^3. Hence when we bring two of the above three lines into coincidence by putting $b_4=0$ we get a new conic node at $(0:0:c_4:+a_6)$. We have seen that if we also put $b_4=0$ this node is a binode B_3. But in either case it can be brought to the origin by setting $a_6=0$. This does not change the nature of the surface, apparently; but to accomplish it by the plane representation we must let the point B approach, in an arbitrary direction, the three points E coincident on ϕ.

It is known that a quintic surface having for double curve a proper conic or a pair of intersecting lines, but possessing no other singularities, is not rational. If we write the equation of a quintic having two intersecting double lines and then let the parameter which expresses the angle between them be zero, we obtain a surface of class 36 which is in general not rational, but which has four pinch points and three generally distinct triple points on the coincident double lines. It may be shown that just five relations must obtain among the coefficients of such an equation if it is to represent a rational surface of the type just discussed.

THE UNIVERSITY OF CALIFORNIA

AMOUNTS OF INVESTMENTS AT ANY NUMBER OF RATES OF INTEREST*

BY C. H. FORSYTH

The writer developed certain generalizations of the most common formulas of the mathematical theory of finance in the June-July, 1921, issue of this Bulletin. These generalizations were based upon the use of two and three rates of interest, instead of one.

It is the purpose of this short paper to extend some of these generalizations to the use of any number of rates of interest. In developing such generalizations it will be found that solutions including reasonably complete treatments of the various possible relative frequencies of conversions of the various rates of interest, while not unduly difficult, would be so complicated in form and be so monopolized by terms due only to those treatments that it would be very difficult to sift out the information which we most desire now, namely, the nature of the function itself when the number of rates is increased markedly. We shall therefore omit these objectionable terms by assuming that the period of conversion is the same for all rates of interest.

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* Presented to the Society, October 29, 1927.

We shall denote the rate of interest realized directly from the principal by i_1. If then these interest payments are accumulated at a second rate i_2 the amount of an investment of one unit of principal by the end of n conversion periods will be $1+i_1 s_{\overline{n|}}$ where $s_{\overline{n|}}$ is to be valued at rate i_2. This, then, is the amount of a principal of one unit at two rates of interest.

If three rates of interest are used, the corresponding amount is the one unit plus the sum of the accumulations of the n payments of i_1, where each of these accumulations is given by $i_1(1+i_2 s_{\overline{n|}})$ and n has the successive values $n-1$, $n-2, \cdots, 2, 1, 0$; $s_{\overline{n|}}$ is to be valued at rate i_3. The amount is easily found and can be expressed as

$$1 + {}_nC_1 i_1 + \frac{i_1 i_2}{i_3}\left(s_{\overline{n|}} - {}_nC_1\right),$$

where ${}_nC_1$ is the usual symbol for "the number of combinations, etc."

The procedure just illustrated is the same for all successive steps; the formula for r rates of interest is used to accumulate the interest payments direct from the principal to give the formula for $r+1$ rates of interest. As the formula for $r+1$ rates is of the same form as that for r rates, the formula is established automatically by induction as it is derived. The formula for the amount of an investment of one unit of principal for n conversion periods, at r rates of interest, may be written:

$$1 + {}_nC_1 i_1 + {}_nC_2 i_1 i_2 + \cdots + {}_nC_{r-2} i_1 i_2 \cdots i_{r-2}$$
$$+ \frac{i_1 i_2 \cdots i_{r-1}}{i_r{}^{r-2}}\left(s_{\overline{n|}} - {}_nC_1 - {}_nC_2 i_r - \cdots - {}_nC_{r-2} i_r{}^{r-3}\right)$$

($s_{\overline{n|}}$ to be valued at rate i_r).

The reader may find it of interest to see how the successive cases "nestle." Thus, if i_r be replaced by i_{r-1} in the formula given above, the reduction of that formula to the one for $r-1$ rates is easily followed.

For those who would care to check the summations leading to the formula given above it may be of help to note that $\sum s_{\overline{x}|}$ from $x=n-1$ to 0 inclusive is $(s_{\overline{n}|}-{}_nC_1)\div i$ and that $\sum {}_xC_r$ from $n-1$ to 0 inclusive is ${}_nC_{r+1}$. Evidently, the subscripts of the rates are all to be shifted one unit for each set of summations.

The formula for the amount of an *annuity* of one unit, at the end of each conversion period, for n conversion periods, at r rates of interest, would be obtained by accumulating each payment by the formula obtained above. The desired formula would be given then by the finite integral or sum of the formula given above, from $n-1$ to 0 inclusive. Evidently, this set of summations involves no shift in the subscripts of the rates of interest. The formula for the amount of such an annuity (at rate i_r) becomes

$$
\begin{aligned}
&{}_nC_1 + {}_nC_2 i_1 + \cdots + {}_nC_{r-1} i_1 i_2 \cdots i_{r-2} \\
&\qquad + \frac{i_1 i_2 \cdots i_{r-1}}{i_r{}^{r-1}}\left(s_{\overline{n}|} - {}_nC_1 - {}_nC_2 i_r - \cdots - {}_nC_{r-1} i_r{}^{r-2}\right).
\end{aligned}
$$

Attention was called, in the paper cited at the beginning, to the fact that the present value of an annuity at several rates of interest is indeterminate or meaningless unless the term "present value" of the annuity is defined more fully than usual and even then any such definition will refer to only one of several possible interpretations.

DARTMOUTH COLLEGE

CURTISS ON COMPLEX VARIABLES

Analytic Functions of a Complex Variable. By D. R. Curtiss. Chicago, The Open Court Publishing Co., 1926. ix+173 pp. Price $2.00.

The present volume is the second one in the series of Carus Monographs, the first being Professor Bliss's *Calculus of Variations.* The reader is assumed to have a preparation equivalent to elementary differential and integral calculus. However, without such preparation one may from the present volume, "obtain some idea of the scope and purpose of the theory of functions."

The purpose of the book seems to be admirably in line with that of Mrs. Mary Hegeler Carus, whose generosity has made these monographs possible. This purpose is appropriately described to be the "diffusion of mathematical and formal thought as contributory to exact knowledge and clear thinking not only for mathematicians and teachers of mathematics but also for other scientists and the public at large."

The author says that inasmuch as the theory of functions of a complex variable has been developed by thousands of workers through the last hundred years, it would be impossible to give an account of that theory in all of its branches within the limits of this book. He attempts to present "fundamental principles with sufficient details of proof and discussion to avoid the style of a mere summary . . . . ."

Three practically equivalent points of view may be adopted in the treatment of the theory of analytic functions of a complex variable. One of these, that of Méray-Weierstrass, bases its developments on the properties of power series. Another is the Cauchy-Riemann point of view in which the functions are required to possess a differential quotient, that is, the definition is here made by a descriptive property rather than by a constructive process. The development may also be made from considerations of the curvilinear integral—the Cauchy-Morera point of view. The volume under review gives a rather interesting account primarily of the Cauchy-Riemann point of view with attention directed occasionally to other methodologies.

A brief survey of the contents of the book is as follows:

The first chapter gives an historical account of the development of the idea of analytic function and calls attention to the usefulness of the theory.

Chapter two deals with complex numbers. In its historical account, in its references to a more general number system, and in its general treatment, it is much more than an elementary exposition. The reviewer would like to have seen an even fuller account of the ideas embodied here, having in mind the fact that in many cases students and the general reader come up to a study of the theory of functions without ever having heard of DeMoivre's theorem and at the same time with very little working equipment in complex numbers. In this particular case it seems wiser to include this elementary material than to omit it.

Chapter three considers real functions. It treats the usual elementary topics in this field including limits, continuity, derivatives and integrals, functions of two variables, double integrals, curvilinear integrals, Green's lemma, integrals independent of the path, and implicit functions.

Chapter four introduces the Cauchy idea of analytic function as one having a derivative. It then discusses necessary and sufficient conditions for the existence of a derivative. The chapter ends with the definition of the exponential, the trigonometric and the logarithmic functions for complex values.

Chapter five, entitled "Applications in Geometry and Physics," gives the usual treatment of conformal representation and shows how the Laplace equation enters in the theory of fluid motion. It also indicates the connection of the mathematical theory with the flow of heat and logarithmic potential.

Chapter six defines complex integrals, deduces Cauchy's integral theorem and formula with the customary corollaries, and ends by deriving Taylor's theorem with remainder.

Chapter seven takes up infinite series, both real and complex. The reviewer would like to have seen here a fuller discussion of power series together with a clearer presentation of the relation between the Cauchy and the Weierstrass points of view.

Chapter eight deals with singularities of single-valued analytic functions, proving some important theorems concerning removable and essential singularities and the nature of the infinity at a pole. It closes with a discussion of definitions of functions in terms of their singularities.

Chapter nine is entitled "Analytic Continuation. Many-Valued Analytic Functions." Here, among other things, we find a discussion of analytic continuation in general and by the method of overlapping circles, monogenic analytic functions, the permanence of functional equations, branches and branch points, the monodromic group, and Riemann surfaces.

At the end of each chapter full and explicit page references to several standard treatises in English and one in German are given for additional reading on the various topics under discussion.

Such typographical errors, few in number, as do occur would perhaps be evident to the reader. For example, the first two subscripts in equation 10, p. 127 should be -2, -1.

The book is attractive in form, binding, and printing, and is an admirable introduction to the subject. Its mastery would most certainly induce the pleasure of further conquests in this beautiful theory.

S. E. RASOR

LEVI-CIVITA ON TENSOR CALCULUS

The Absolute Differential Calculus (Calculus of Tensors). By Tullio Levi-Civita. Edited by Enrico Persico. Authorized Translation by Miss M. Long. London and Glasgow, Blackie & Son.

When Einstein arrived at his General Relativity Theory he found ready for use a mathematical instrument in the form of the absolute differential calculus (Bateman had already pointed out its applicability in the theory of space-time transformations). This calculus had been worked out by Ricci and was not well known outside of Italy. The only comprehensive exposition was contained then in a paper prepared for the Mathematische Annalen on the invitation of Klein by Ricci and the author of the book under review. In the years following the introduction of the general relativity, the tensor calculus—as the absolute calculus is now often called—has been much perfected, the most notable development being the introduction of the idea of parallel displacement due again to Levi-Civita, who thus among the living contributed most to the subject. In the more than twenty years that passed between the publication of the Annalen paper and the Italian edition of the present book (1923) very many text books appeared. But although many of them introduce parallel displacement, it always appears as a kind of afterthought; a distinctive feature of the present book is a certain unity of exposition achieved by making parallel displacement the basis of the whole edifice.

The fundamental idea stated in a schematic form is this. in Levi-Civita's exposition the absolute differential calculus appears as a modified form of ordinary differential calculus, the modification consisting mainly in that for the second partial derivatives of a function of n variables (and also more complicated expressions of that type) certain other differential expressions of the second order are substituted; the reason why such a modification appears desirable is that the laws according to which the second partial derivatives are changed under a general transformation of the independent variables are quite complicated and much is to be gained by using instead of them other expressions whose laws of transformation are simple. Both the idea of using such expressions and the form in which they appear is suggested by the theory of surfaces; the original feature of the book under review is the following way in which this suggestion is carried out. The variables (two, in the first place) are interpreted as parameters specifying the position of a point on a surface; the first derivatives of a function are then components of the gradient vector tangent to the surface; the second order differential expressions—which are to be used instead of the second derivatives—are obtained as quantities characterizing the change of the gradient vector as we move from point to point on the surface; this involves the comparison of the gradient vector at different points of the surface which necessitates the consideration of displacement of vectors on the surface. Everybody knows how to move vectors on a plane; it is easy to extend the notion to surfaces which are applicable on a plane, that is, developable surfaces; in the case of a general surface, when he wants to displace a sector along a curve Levi-Civita displaces it in the developable surface which is tangent to the given surface along the curve; this is *his parallel displacement.* He finds an analytic expression for this displacement and thus arrives at the required expressions with a simple law of transformation; these expressions are called the covariant derivatives. The author then abandons geometry without abandoning geometrical language and generalises to more than two variables. The difference between using geometrical language—and conceptions—in questions of analysis and doing geometry is not quite clear to the reviewer but the author seems to draw a line between these two things.

The introduction of covariant differentiation as outlined above occupies the center of the original portion of the book—the first 286 pages of the English edition; what precedes is largely an introduction. The reader who is supposed to know or to believe that there exists a solution of a system of ordinary differential equations is introduced in a very neat way into the theory of total and linear partial differential equations. Then tensors are introduced as algebraic systems and the fundamental difficulty mentioned above with second partial derivatives is pointed out. A geometric chapter follows in which parallel displacement is explained and this ends the first part—Introductory Theories. Next comes a formal introduction of covariant differentiation and the remainder of the second part is occupied by the development of the absolute calculus and includes several interesting and original features. We may mention the sections on *geodesic deviation* in which the contents of the author's recent paper in the Mathematische Annalen is reproduced—these sections as well as the discussion of the geometry of the curvature tensor in three-space in the same (seventh) chapter did not appear in the original Italian edition but this fact is not mentioned in the preface.

The third part—Physical Applications—is also original in the English edition; it is devoted to the theory of relativity. Certainly the most noticeable feature in this part is the careful avoidance of electromagnetism which is barely mentioned a few times. This in spite of the fact that as the author himself says in the preface "Electromagnetism, in common with every other physical phenomenon, now comes within the orbit of General Relativity." The reviewer cannot see that this omission can be justified and in his opinion this forms a grave defect in this otherwise wonderful book. Of the many reasons which could be adduced to establish this point it will be enough to mention only two. The electromagnetic tensor and the electromagnetic energy tensor furnish the most natural illustrations and application of the tensor calculus which certainly must be of importance in a treatment intended in the author's own words to exhibit the fundamental principles of Einstein's general relativity theory as an application of the absolute calculus. Then again, the four-dimensional way of viewing things grows naturally from the consideration of the fundamental equations of electrodynamics without making it necessary to introduce into them any modifications, whereas the mechanical equations have to be modified, adapted to the new point of view. It is true that just this fact makes perhaps the theory of relativity more important and fruitful in the application to mechanics; but both esthetically and didactically it is so much more satisfactory to deduce the general principles on the example of electricity and then to apply them to gravitation.

Aside from that hardly any general criticism can be directed against Levi-Civita's exposition of relativity; it is original, complete (with the above exception) and elegant; of the many departures from the usual presentation perhaps Palatini's introduction of centro-symmetric curved space deserves to be particularly commended. A surprising fact is that Levi-Civita does not use in physical applications the notion of parallel displacement which plays such a fundamental role in the theoretical part and which could help to unify the exposition further if it were used in the discussion of the motion of material particles and of the propagation of light.

Throughout the book the author draws freely on the rich material of Italian works which is scarcely used (with few important exceptions) in other text-books. The result is an increased elegance of exposition.

The style is that of theoretical physics. The author comes very close, in the opinion of the reviewer, to striking the happy mean between the Scilla of the epsilon-delta symbolism and the Charibdis of loose reasoning. A pure mathematician would probably prefer a less careful avoidance of the Scilla; it sometimes becomes hard to form a clear idea of what is being neglected and why, but in view of the character of the material treated it is seldom possible to blame the author or to suggest a better course.

G. Y. RAINICH

KENT ON FINANCE

Mathematical Principles of Finance. By Frederick Charles Kent. Second Edition. xiii+177 pp. *Compound Interest and Annuity Tables.* By Frederick C. Kent and Maud E. Kent. viii+214 pp. New York, McGraw-Hill Book Company, 1927. Bound in one volume.

The second edition of Professor Kent's text differs from the first published in 1924 only by the insertion of six pages on interpolation, made possible by the abbreviation of the chapter on logarithms, now entitled "Interpolation and Logarithms," together with a few plate corrections and changes in the remainder of the book. The problem lists are unchanged.

The text appeals to the reviewer as an average book, neither better nor worse than many that have appeared in the same field in the last fifteen years. The conventional topics are covered in the conventional way. Rather more attention is given to the work of the Federal Farm Loan Board than is usual, but there the increase in material is due rather to the description of the administrative processes of the Board than to additional mathematical material.

The book is intended for college freshmen with only high school mathematics as a prerequisite. The result is the introduction of an amount of detail that seems at times excessive. For the student with a knowledge of geometric progressions it would not be necessary each time one is encountered to go through the whole process of finding the sum.

The definition of amortization as "the extinction of an interest bearing debt, principal and interest, in a series of equal periodical payments" is scarcely warranted either by the etymology of the word or by current usage; nor is it adhered to by the author himself when he comes to consider the amortization of the premium on a bond. A debt may be amortized by payments that are neither equal nor periodical. The definition of the equation of value as the relation which gives the value of the unknown quantity in a definite problem (expressed in general terms to be sure) is misleading to say the least. An equation of value arises wherever sums are accumulated or discounted to the same date and any definition should be broad enough to cover all cases.

As a piece of writing, the book is not above criticism. It is sometimes difficult, when long reductions are made, or when a number of related formulas are presented, to tie text and formulas into a good paragraph, but it is not impossible. The most flagrant illustrations of failure in this respect are found on pages 53 and 54 where, in presenting the problem of finding the term of an annuity from the equations for the amount and the present value, the author has little regard for the rules of grammar, punctuation, or sentence structure. A certain amount of neglect of the rules for good writing may be permitted in blackboard work that is to be supplemented by oral explanation, but such neglect has no place in the pages of a textbook. One may go further and say that the author who writes for freshmen is under peculiar obligation to present his material in the best possible form.

The real contribution of the book lies in the fine ten-place tables which have been worked out by Professor Kent and his wife since the publication of the first edition. The three principal functions of the mathematics of finance and their reciprocals are tabulated for thirty-six different rates for times to 300 years for rates from $\frac{1}{4}$ to $\frac{3}{4}$ per cent, to 200 years for rates from $\frac{7}{8}$ to $1\frac{1}{4}$ per cent, and to 100 years for rates from $1\frac{3}{8}$ to $10\frac{1}{2}$ per cent. The auxiliary functions $(1+i)^{1/p}$, $j_{(p)}$, and $i/j_{(p)}$ for the same rates are given for $p=2, 4, 6, 12$. The values of $(1+i)^n$ are given to fifteen decimal places for twelve fractional rates for one-year intervals to 10 years, 10-year intervals from 10 to 100 years and for 200 years. A few column sums taken at random indicate that the work is reasonably accurate. In one case the column sum differed from the computed value by 45 tenth place units but in all other cases the agreement was much closer.

These admirably arranged tables will form a useful addition to existing tables, and for their patient work in computing them Professor and Mrs. Kent deserve much credit.

E. B. SKINNER

MACROBERT ON HARMONIC FUNCTIONS

Spherical Harmonics. An elementary treatise on Harmonic Functions. By T. M. MacRobert. New York, Dutton, 1928. xii+302 pp. $4.50.

The subject matter of the book falls naturally into three main divisions. The first three chapters deal with Fourier series and applications, Chapters 4 to 13 are on spherical harmonics and the remainder of the book, Chapters 14 to 16, treats of Bessels Functions with applications. As the author states in the preface, he limits himself to those parts of the theory which can be developed expediently without the use of contour integration.

Chapter 1, on the theory of Fourier series, is a very careful and lucid exposition of the elementary theory. The author derives the expansion in Fourier series of the function $f(x)$ subject to the Dirichlet conditions. Here, as throughout the book, he is careful to keep before the reader the validity of the formulas as actually derived in the text. This is a very welcome quality in the book, a quality which is so often lacking in texts designed to appeal to the applied mathematician.

In Chapter 2 the partial differential equation of heat conduction is derived, and the theory of the previous chapter used to obtain solutions for a number of different bodies and with a variety of initial and boundary conditions.

Chapter 3 is a very interesting presentation of the theory of the vibrations of a string. The author solves the wave equation for the case of the harp, the violin, and the piano and brings out strikingly the variations in the results due to the different initial conditions for each of the three instruments.

Chapter 4 begins the study of Spherical Harmonics. Legendre and Laplace coefficients are introduced and the hypergeometric function is studied to some extent. The author departs momentarily from his usually clear style near the bottom of page 79 where the Beta function suddenly appears without explanation.

In Chapter 5 the Legendre Polynomials are studied in considerable detail and in Chapter 6 the Legendre functions of the first and second kind are introduced.

In Chapter 7 the solutions of Legendre's associated equation are investigated quite fully. The Zonal, Tesseral, and Sectorial Harmonics are presented and the expression of a surface spherical harmonic in terms of them obtained.

Chapter 8 begins the application of the theory of the preceding four chapters to mathematical physics. Chapters 8 and 9 contain the proofs of some of the fundamental theorems of potential theory and expressions for the potentials of various bodies, particularly of spheres, spherical shells and spheroids. The second boundary problem for the sphere is also solved in terms of spherical harmonics.

Chapter 10 begins the application of spherical harmonics to electrostatics. The expressions for the surface distributions and potentials for spheres, spherical shells and spheroids are obtained.

In Chapter 11 elliptic coordinates are used to obtain expressions for the potentials and surface distributions of ellipsoids of revolution; and in Chapter 12 the corresponding problem for eccentric spheres is handled by means of bipolar coordinates.

Chapter 13 is devoted to a brief account of Maxwell's theory of spherical harmonics.

In Chapter 14 the expressions for the Bessel functions of the first and second kind are derived and the fundamental properties and formulas involving these are proved.

Chapter 15 is concerned with the modified Bessel functions and asymptotic expansions. The chapter also contains a very lucid account of the zeroes of Bessel functions. Fourier-Bessel expansions are discussed very briefly.

Chapter 16 concludes the book with the application of Bessel functions to the vibration of a circular membrane and to the flow of heat in a circular cylinder and in a sphere.

Very few misprints were found, all of such an obvious nature that it seems useless to list them. It may be worth mentioning, however, that the first line on page 164 should be the last line.

The theory is well illustrated by worked examples in the text and most of the chapter have at the end a good list of exercises. In a number of instances throughout the book the reviewer would have liked to see the general term of an infinite series derived rather than merely the first three or four terms, but nevertheless the author has handled a subject containing mass of detail in a very clear and skillful manner. The book should appeal strongly to the applied mathematician and to the mathematical physicist. They will find here a scholarly treatment of the type of problems arising in a great many branches of theoretical physics and the tools whereby such problems may be attacked.

The one fault which the reviewer finds with the book is the almost entire lack of references. Except for the mention of four treatises in the preface and a few historical references in the text no explicit reference is made to the literature of which the book treats. The literature is of course so vast that an exhaustive bibliography would be impracticable in a text of this nature. However, the material of the text is capable of a great deal of amplification in so many directions that a few well chosen references would, the reviewer feels, increase its usefulness both to the student of pure mathematics and to the one interested mainly in the applications.

G. E. RAYNOR

HADAMARD'S COURS D'ANALYSE

Cours d'Analyse. By J. Hadamard. Volume I. Paris, J. Hermann, 1925, 1927. xxxii+624 pp.

The title *Cours d'Analyse* calls to mind a long array of prominent names, Jordan, Picard, Goursat, de la Vallée-Poussin, whose books on this subject have long been standard works of reference and sources of inspiration. The name of Hadamard joined to "Cours d'Analyse" would lead one to expect something in the same category, a treatment of some standard subjects, but with some decidedly interesting additional original contributions. If one brings such expectations to the volume under consideration, one is doomed to disappointment. Hadamard is careful to warn the reader in the introduction that he does not contemplate the type of product which his predecessors have made famous, by including topics and developments not considered in lectures, but to limit himself to the material which he has been presenting in the course in analysis at the École Polytechnique. As a consequence, the first volume of this work is much of the character of an advanced calculus with a development of the fundamentals of differential geometry thrown in for good measure.

After an introduction devoted mainly to a brief resumé of the essentials of infinite series and indefinite integration, there are two sections on what might be termed additional material in differentiation and integration. The former includes such topics as differentials of functions of more than one variable, functional determinants, change of variables (with material on contact transformations) and maxima and minima, especially of functions of two variables. The latter includes in addition to the Riemann integral, reference to the Stieltjes integral, elliptic integrals, material on definite integrals with some emphasis on the approximate calculation of the latter by different methods and comparison of these methods as regards accuracy, improper integrals and multiple integrals as well as continuity properties of these.

The application of integrals is limited mainly to a discussion of

$$\int_0^\infty e^{-x^2}dx,$$

and its relatives, and the Euler integrals, together with an elegant introduction to the theory of Fourier series. The applications of differentiation are in the field of differential geometry, beginning with a consideration of orders of contact and obtaining the principal results with respect to space curves and surfaces, including the applicability of surfaces on each other.

Naturally, there is a chapter devoted to the fundamental formulas of mathematical physics, called by Hadamard the formulas of Ostrogradsky, Riemann, and Green, respectively, to which is joined a brief introduction to vector analysis in three space, in which a forward look into higher dimensional vector analysis plays a role. The latter chapter closes with a tabular comparison of the notations in common vogue by different schools of vector analysis. It would be helpful if vector analysts and mathematical physicists could agree on the names of operators and formulas.

The concluding section of the volume is concerned with differential equations, and gives the elementary results in the subject relating to differential equations of the first order, and of special types of higher order, with main emphasis on linear differential equations with constant coefficients and systems of such.

The volume closes with notes on (a) comparison of the Euler-Maclaurin and trapezoid formulas for computing definite integrals and a suggestion of a combination giving in some cases a better approximation than either, (b) summation by means as applied to Fourier series, leading naturally to (c) the Weierstrass Theorem on approximation to continuous functions by polynomials; and (d) a brief introduction to affine vectorial geometry. There is appended an incomplete table of errata.

As outstanding features of the book might be mentioned the introduction and treatment of the Stieltjes integrals on a par with the Riemann integral; further, the derivation of some of the characterizing magnitudes of space curves and surfaces by the use of kinematic considerations, making these seem perhaps more natural to a student preparing for a technical career. He bows to the modern trend, especially where technical students are concerned, in devoting considerable attention to approximation theorems and calculations.

The presentation of the material is lucidly and interestingly done, remarks and subremarks being frequently added in the hope of adding further points of interest to the main material. In matters of rigorous treatment, involving so-called ϵ, δ considerations and existence theorems, he prefers to pass over the finer points and refer to consideration of these in the next volume, so as not to divert the attention too much from the material expounded,—a method in common practice and perhaps justifiable on pedagogical grounds. As a careful and clear exposition of topics in advanced calculus this volume is worthy of a high place, but it is not at all remarkable in presenting material in a novel way, or being revolutionary in its point of view.

T. H. HILDEBRANDT

SHORTER NOTICES

Collected Papers of Srinivasa Ramanujan, edited by G. H. Hardy, P. V. Seshu Aiyar and B. M. Wilson. Cambridge University Press, 1927. xxxvi+355 pp.

Ramanujan's great and extremely individual work is now too well known to all who specialize in the theory of numbers to require detailed comment by any reviewer. The editors of his works have reprinted thirty-seven of his papers, each one of which merits the closest attention by students of algebra, the theory of numbers, and analysis. In addition they have added two extremely helpful appendices, the first consisting of notes on the papers, the second of further extracts from Ramanujan's letters to G. H. Hardy. Ramanujan's brilliant and tragic career is feelingly portrayed in the introduction. Without doubt, he was one of the few great mathematicians who have had a superlative genius for numbers, and it is indeed a tragedy that he, like Eisenstein, had to leave the world so early.

To attempt a detailed review of any of these great papers would be impertinent. We may glance, however, at one or two human problems raised by the general color of Ramanujan's brilliant work. The first of these concerns editors of mathematical journals. To the reviewer it is incredible that certain of these papers would have been accepted for publication in at least one of our American periodicals, and the like holds for more than one European journal. It is easy to deny this after Ramanujan's genius has been accepted and established. Yet on these same doubtful papers the stamp of genius is flamingly apparent to all who are not blighted by the rot of academic rigor. What does it matter, some may say, that many of his most original ideas are not backed by anything that even faintly resembles proof, when these same ideas of Ramanujan have already initiated what our successors may perhaps look back on as the first golden age of the analytic theory of numbers? The critical reader will recognize the justice of these remarks on a careful study of those papers designated by Professor Hardy as Ramanujan's greatest. To these may be added number 20, *On the expression of a number in the form* $ax^2+by^2+cz^2+du^2$. If this be sifted to the bottom, it will be found that little, if anything, is *proved*. The defects, of course, have since been supplied by later writers. Still, if proof means anything in mathematics, it should surely mean something in the theory of numbers above all other branches.

A second observation is this. The one theorem, or formula, selected by Major MacMahon as the most beautiful in all of Ramanujan's work, is indeed a thing of beauty. But, as recognized by the editors, this, and many more of equal elegance, are already implicit in the neglected papers of Professor L. J. Rogers. This assertion is not meant to detract in any way from the brilliancy of Ramanujan's totally independent rediscovery of a beautiful theorem. It is meant to direct the attention of young mathematicians to much formal, unfashionable, and unadvertised work on the algebraic side of the theory of numbers which has been overlooked. Without the great publicity which Ramanujan's work has given to these neglected researches, it is doubtful whether they would have survived oblivion. The conclusion of the whole matter is this: the man who is capable of reading will neglect the official and perfunctory abstracts of work that are ground out yearly in the reviews, and glance through the papers as printed in the journals themselves.

Ramanujan has been likened to Jacobi. To the reviewer he seems also to be akin to Eisenstein, Hermite in his earlier work, and to Galois, for the boldness of his thought. Whatever may be the ultimate estimate, it seems reasonable to predict that Ramanujan will be placed high among creative mathematicians. Above all he was an algorist and an arithmetician of the first rank. The brilliance of his papers of the later period, written under the influence of the English school, which is today transforming the analytic theory of numbers, give us a just measure of our loss in the death of a man at the age of thirty-two who might also have been one of the world's great analysts.

E. T. Bell

Encyklopädie der mathematischen Wissenschaften. Volume III. Part 3. Leipzig, Teubner, 1902–1927.

Of the ten articles which compose this volume of more than 800 pages, mostly devoted to differential geometry, some were first published as early as 1902, some as late as 1927. As a result there is a certain inhomogeneity; it is probable that the earlier parts which cover the classical theory of curves and surfaces and which occupy about one half of the volume already have rendered the greater part of the services as a reference book of which they were capable and a more up to date reference book reflecting the advances made in the eventful last quarter of century seems desirable. Fortunately, the articles by Weitzenboeck and Berwald in which differential geometry of n dimensions is covered—a field that has received a great deal of attention lately, partly under the influence of the theory of relativity—bring the literature up to 1923. Differential geometries under groups other than the metric also are dealt with in these two articles. The references are very complete; if there are omissions they probably occur only in cases of dissertations which did not appear in a periodical (I noticed one such case). These two articles are of inestimable value to one who works in the field.

In addition to the articles mentioned, there is a comparatively recent article on triple orthogonal families (Salkowski) and two articles by Liebmann (dated October 1914) which are but loosely connected with differential geometry proper. One is devoted to contact transformations and the other to the geometrical theory of differential equations; what is meant here is the line of attack on differential equations by means of analysis situs considerations started by Poincaré.

G. Y. Rainich

Differential Geometry of Three Dimensions. By C. E. Weatherburn. Cambridge, University Press, 1927. xii+268 pp.

"The objects of the present volume are to provide an introductory treatise on (metric) differential geometry, and to show how vector methods may be employed to advantage."

The author shows excellent judgement in the way he goes about securing these two ends. He chooses his geometrical material well, and, in adopting Gibbs's notation for his vector treatment, he is undoubtedly in accord with the current preference.

The book begins with a treatment of space curves, and the curves and developable surfaces associated with a space curve. It passes on to the theory of surfaces, taking up the important classes of curves on an arbitrary surface and investigating certain special types of surfaces, and closes with treatments of congruences and triply orthogonal systems of surfaces.

This, at least, was the content of the book as originally planned by the author. After the manuscript went to press, he added two chapters dealing primarily with differential invariants and their applications. In the previous chapters he employs merely vector algebra. Here he introduces a vectorial differential operator ∇ for a surface analogous to the usual ∇ for the plane. By skillful use of it he succeeds in generalizing for a surface the classical differential and integral calculus of vectors and obtains from his generalizations a number of interesting and striking geometrical results.

When a specialist in vector analysis turns his attention to geometry, it is too often the result that the geometry becomes merely a foil for the aggrandizement of the vector analysis. The present writer treats geometry more kindly. He keeps it clearly in mind as the first of the two objects he set out to achieve and does exceedingly well by it.

The author's geometric insight is keen, clever, and instructive. But at times he is found offering, as rigorous proofs, intuitive geometric arguments which, though enlightening and to the point in their proper place, are lacking in substance. We quote, for example, his proof that a developable surface may be applied to a plane: "since *consecutive* generators are *coplanar*, the plane *containing* the *first* and *second* of the family of generators may be turned about the second till it *coincides* with the plane *containing* the *second* and *third*; then this common plane may be turned about the third till it *coincides* with the plane *containing* the *third* and the *fourth*; and so on. In this way the whole surface may be developed into a plane." The author is here led astray by the inaccurate use of the words which have been italicized. He is not again so completely betrayed by the language of little zeros. The reviewer is glad to say that this is an extreme case. Is it not, however, deserving of consideration by those of us who are tempted at times, either in the classroom or in writing for the printed page, to employ for the sake of brevity or convenience inaccurate or approximate terminology?

The author likes unusual terms and seems to enjoy devising new names and changing old ones. "Synclastic and anticlastic" surfaces for surfaces of positive and negative curvature respectively, and "specific curvature" instead of Gaussian curvature, are some examples. A little of this may be all very well, but when the author goes so far as to propose that the Gaussian curvature be subjected to the ignominy of the name "second curvature," surely it is time to call a halt.

An able presentation of the elements of the subject by vector methods, a clearly written text with an abundance of good exercises, this book should prove a welcome addition to the literature in differential geometry.

W. C. GRAUSTEIN

Figures d'Équilibre et Cosmogonie. Mémorial des Sciences Mathématiques, No. XIII. By Alex. Véronnet. Paris, Gauthier-Villars, 1926. 62 pp.

One of the greatest problems of celestial mechanics has been that of determining figures of equilibrium of rotating fluid bodies. Its application to the theory of evolution of planets, stars, and stellar systems, has given it perpetual interest. Its difficulties have long baffled mathematicians. The author gives in the present little book a brief yet fairly comprehensive account of the progress that has been made towards solving those difficulties, from the times of Maclaurin, Clairaut, Laplace and Jacobi, to the more recent successes of Darwin, Poincaré and a number of living mathematicians, amongst whom Véronnet holds a prominent place.

The first chapter gives a resumé of studies of figures of equilibrium for the case of a homogeneous fluid; included are the results concerning the stability of those figures. The second chapter is devoted to heterogeneous bodies and the figures of the planets. Chapter three is concerned with a body having an atmosphere, the Laplacian nebular hypothesis, the figures of comets, and Saturn's rings. The fourth chapter takes up the dynamical equilibrium of stellar systems, and theories of cosmogony. The final chapter considers the thermodynamical equilibrium of the universe and its evolution.

At the end of the monograph there is a six-page bibliography of books and memoirs appertaining to the field covered. While this bibliography is far from exhaustive, it is sufficient to indicate most of the principal sources for a study of the famous problem. One might wish that reference had been made to more of MacMillan's recent papers.

E. J. MOULTON

Geodäsie (Landesmessung und Erdmessung). By Gustav Forster. Sammlung Göschen. Berlin, de Gruyter, 1927. 122 pp.

This small volume gives a popular account of the purpose, scope, mathematical foundation, and technique of surveying and of higher geodesy. Although technical calculations for problems of moderate difficulty are included, the actual mathematical content of the book is small. In the reviewer's judgement the informal but clear definitions, the technical descriptions of procedure, and in particular the explanations of the brief periodic deformations of the earth will be attractive reading particularly for mathematicians whose professional interest in geodesy is slight.

B. H. BROWN

Die gewohnlichen und partiellen Differenzengleichungen der Baustatik. By F. Bleich and E. Melan. Berlin, Springer, 1927. vii+350 pp.

Linear difference equations made their first appearance in the statics of framed structures with Clapeyron's discovery of the relation between the bending moments at three consecutive supports of a continuous beam. More complicated types appear in connection with framed girders, and in recent years, even linear partial difference equations have become of importance for calculating the stresses in the lattice arrangement of columns and beams in buildings. The present book, which contains a number of original investigations by the authors, gives a very clear and readable presentation of the entire subject, profusely illustrated by diagrams, numerical examples and tables.

The further development of the questions treated in the fourth section should provide a number of interesting problems to the mathematician.

The contents of the book is the following: the first section gives the elementary properties of differences and sums, including the Euler summation formula. Section 2 deals with linear difference equations and the boundary and expansion problems which are the exact counterpart of the well known problems in differential equations. Special attention is given to equations with constant and linear coefficients, the latter case being solved by a Laplace transformation. Section 3 contains various applications to stress calculations and stability problems, and Section 4 deals with corresponding questions in partial difference equations.

T. H. GRONWALL

New Methods in Geometrical Optics. By C. S. Hastings. New York, Macmillan, 1927. 103+vi pp.

In the opening chapter of this volume the author develops the general equations in first-order theory of centered optical systems in terms of incident and emergent wave surfaces, surface curvatures, and wave-velocities, in place of the quantities usually used. It is claimed that this method leads to simpler forms for the derivatives of the general equations, in terms of which the aberrations are to be expressed. In the chapters that follow there are examples of the application of these equations to the theory of the telescope and of the microscope, and to certain specific problems in the designing of optical systems.

The discussion of chromatic aberration is interesting in that a number of special types of color adjustment, not usually specifically treated, are discussed. Here one finds terms such as "orthokumatic," "ioskumatic," "iosdynamic," each with a well defined meaning. Spherical aberration and the errors of oblique pencils are discussed rather briefly, and the book closes with an appendix in which is found some interesting material on mirages. Most of the illustrative material has apparently been drawn from the author's experience as technical advisor to makers of astronomical instruments, and the volume contains some well taken criticisms of some past practices in this particular field of instrument making.

G. W. MOFFITT

Questions d'Arithmétique, by B. Niewenglowski. Paris, Librairie Vuibert, 1927. viii+225 pp.

This is an entirely elementary book on the rudiments of the theory of numbers, in which induction and proof are agreeably mixed to bring out the spirit of the arithmetical approach. It is difficult, however, to see for what class of readers the work is intended. From an American point of view it is too elementary and insufficiently scientific to be of interest to even first-year college students. It is of about the grade that would stimulate an intelligent high school senior with a taste for abstract mathematics.

It is unnecessary to go into detail on the contents of the book. Chapter I is devoted to curiosities; Chapter VIII to the theorems of Fermat and Wilson, including the elements of the theory of quadratic residues and the law of reciprocity; Chapter IX, the last in the book, gives numerous instructive numerical examples on the Pellian equation. The treatment throughout is more like what one would expect to find in a book of mathematical curiosities than that appropriate to a serious treatise on the theory of numbers. Nevertheless the book is highly interesting, with many fresh touches, and one that is likely to stimulate young readers. Misprints are frequent, but not serious to fairly sophisticated readers. Some of them might bother the beginners for whom the book seems to be written.

E. T. BELL

Vorlesungen uber Hohere Geometrie. By Felix Klein. Third Edition. (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Band XXII.) Berlin, Julius Springer, 1926. viii+405 pp.

Among the many courses of lectures by Klein that began to appear over forty years ago in "autographic" editions none has been more stimulating than that on Higher Geometry which first appeared in 1893. It was based on the ideas developed in his famous Erlanger Program of 1872 entitled *Vergleichende Betrachtungen uber neuere geometrische Forschungen*, in which he established the group of transformations as the fundamental principle of classification in geometry. The course of lectures on Higher Geometry exhibited the whole of geometry from the group-theoretic point of view.

The original edition consisted of two volumes. The book now under review contains what is virtually a reprint of the first of these volumes, Professor W. Blaschke acting as editor. The mathematical public will hail this new edition, not only because in printed form it is much easier to read, but also because the former editions are difficult to obtain. It is one of the classics of our mathematical literature and should be permanently available.

The second volume of the original edition was devoted to Lie's theory of groups of transformations. In view of more recent developments a new edition of this second volume would have required extensive revision and rewriting. Professor Blaschke has, therefore, omitted it entirely from the new edition. In place of it, he has added one hundred pages of entirely new material, giving in five chapters some of the more recent developments in higher geometry. These last five chapters are especially interesting and stimulating. The first of these, by Professor Blaschke, gives an account of Study's line geometry. Professor Radon contributes the second of these chapters which is devoted to a discussion of the author's mechanical derivation of Levi-Civita's parallel displacement. The third chapter, apparently a joint product of Professors Blaschke and Artin, is devoted to analysis situs. It is of special interest to American readers in that it reproduces Alexander's elegant proof of the deformation theorem of Tietze. Professor Radon contributes also the fourth of these chapters which is devoted to a variety of geometric interpretations in the theory of partial differential equations and the calculus of variations. The last of these chapters furnishes a geometric treatment of the theory of elementary divisors. Each of these chapters is enriched by an adequate bibliography of the subjects under discussion. It is to be hoped that some of our younger devotees will receive inspiration from this work, with a view to bringing geometry back to the front of the stage, from which it has been temporarily crowded by the recent advances in analysis.

J. W. YOUNG

The Evolution of Scientific Thought from Newton to Einstein, by A. d'Abro. New York, Boni and Liveright, 1927. 544 pp.

Explaining scientific theories to people with small knowledge of the technique used by the original developers is the cause of much difficulty in modern thinking. This book is an attempt to present the development of space-time in a manner understandable to a person unfamiliar with the mathematical tools used by the physicists in the new theories.

The author uses the only possible method—that of first acquainting the reader with the foundations of the mathematical and physical problems to be discussed. We find in Chapters I–V and VII an excellent sketch of the non-euclidean geometries particularly from Riemann's point of view. In Chapters VI and VIII–XII the physical questions of time, classical relativity and electromagnetics are dealt with.

We are surprised at the amount of insight it is possible to give without the use of mathematical manipulation. Having wondered how much a student of first year calculus (which is, after all, about as much mathematics as the average layman has in his background, if not more) would make out of such a treatment, we tried Chapter VII on one of our students. This chapter introduces the idea of the curvature of space. To our surprise, the idea was grasped rather well from the author's explanation.

The next section (Chapters XIII–XXII) consists of a most accurate and altogether excellent account of the restricted theory of relativity. One of the most interesting chapters is the one devoted to paradoxes (XXII).

Part three deals with the general theory. As the task is much more difficult here, in view of the extremely complicated mathematics, it is much more to the author's credit that he accomplishes such a fine presentation. The difficulties of discussing tensor equations when your reader has no knowledge of how they are arrived at, seem, at first, insuperable, but the author succeeds in showing the meaning of the law of gravitation and in discussing the questions of the finiteness of the universe. As the author admits, action and conservation are almost impossible subjects to treat rigorously without mathematics, but even here he does well.

The fourth part of the book is more critical, discussing the methodology of science and the general significance of the relativity theory. Many philosophers will disagree with the indictment of lay philosophers (pp. 374–5) but, being on the scientific side, we feel, as d'Abro does, that the lack of deep insight which comes only from a thorough understanding of the workings of a theory will prevent such people from making many valuable contributions to the underlying principles of it. In other words, the study of this book is not sufficient preparation for a man who wishes to get up a theory of space-time. The author says "The sole rôle that a semi-popular book can hope to perform is to serve as a general introduction." This the book does and does it excellently. It is to be hoped that other branches of modern science may find as capable and accurate popularizers as the relativity theory has found in d'Abro.

In matters of printing the book is excellent and the only error we have detected is in the next to last and last line on page 93, where $1/R_2^2$ should be put in place of $1/R_2$, and $1/R^2$ in place of $1/R$.

T. C. Benton

Formules Stokiennes. By A. Buhl. (Mémorial des Sciences Mathématiques, Fasc. xvi.) Paris, Gauthier-Villars, 1926. 60 pp.

This tract is concerned with establishing formal connections between the integrands in the generalized Stokes' formulas and the expressions which occur in various differential equations in geometry and physics, notably the electromagnetic and gravitational field equations. In the opinion of the reviewer, these formal connections are too vague and arbitrary to be of great advantage in dealing with the differential equations in question.

T. H. Gronwall

NOTES

In addition to those on the editorial staff of the Bulletin, the following persons have assisted the editors, either by refereeing papers or by advising concerning papers offered for publication in the present volume: Professors E. T. Bell, B. A. Bernstein, Daniel Buchanan, R. D. Carmichael, E. W. Chittenden, L. E. Dickson, Arnold Dresden, Arnold Emch, H. J. Ettlinger, W. C. Graustein, T. R. Hollcroft, Louis Ingold, Dunham Jackson, J. R. Kline, E. P. Lane, C. H. Langford, Mayme I. Logsdon, C. N. Moore, R. L. Moore, F. D. Murnaghan, G. Y. Rainich, Arthur Ranum, Virgil Snyder, E. B. Stouffer, J. M. Thomas, R. L. Wilder, W. A. Wilson, R. M. Winger. The editors desire publicly to recognize this service.

The Annual Meeting of the Society will be held in New York City, in conjunction with meetings of the Mathematical Association of America and the American Association for the Advancement of Science, on December 27–31, 1928. On Friday morning, December 28, the Bôcher Prize will be awarded, and Professor W. F. Osgood will speak on the life and work of Bôcher. On Friday afternoon, at the usual joint session with other organizations, Professor Dunham Jackson will deliver his retiring address as vice-president of Section A of the A. A. A. S; and Professor J. L. Coolidge, at the joint invitation of the A. M. S. and the M. A. A., will present a paper entitled *The heroic age in geometry.* Later on Friday afternoon, the sixth Josiah Willard Gibbs Lecture will be given by Professor G. H. Hardy of the University of Oxford, on the topic *An introduction to the theory of numbers.* The meetings on Monday, December 31, will consist of a symposium on *Wave mechanics*, in joint session with the American Physical Society.

A regular meeting of the Society will be held at the University of Cincinnati on Friday and Saturday, November 30–December 1, 1928. This is the first fall meeting under the new program of three Western meetings a year. The arrangements are in charge of a special committee consisting of Professor C. N. Moore (chairman), Henry Blumberg, P. P. Boyd, Louis Brand, H. T. Davis, and M. H. Ingraham. At the invitation of this Committee, Professor G. D. Birkhoff will speak on *The mathematical bases of art*; Professor E. R. Hedrick will speak on *Recent developments in the theory of non-analytic functions of a complex variable*; and Professor Archibald Henderson will deliver an address. In the interval between this meeting and the Annual Meeting in New York, Professor Hedrick will be on leave of absence, and will visit and speak at several institutions.

The July, 1928, number of the Annals of Mathematics (series 2, volume 29, No. 3) contains the following papers *On Rayleigh's principle in the theory of differential equations of mathematical physics and on Euler's method in calculus of variations*, by N. Bogoliouboff and N. Kryloff; *On various* *conceptions of correlation*, by F. M. Weida; *Kernels of positive type*, by C. Seely; *The Tschirnhaus transformation*, by R. Garver; *On the factoring of Fredholm minors*, by H. T. Davis; *A set of necessary and sufficient conditions for the Cesàro summability of double series*, by G. M. Merriman; *Certain configurations on cubics*, by F. C. Ogg; *On four mutually orthogonal circles*, by N. Altshiller-Court; *A general theorem on the expression of a determinant in terms of its sub-determinants*, by W. M. Flexner; *Quadratic fields in which cyclotomic polynomials are reducible*, by L. Weisner; *On irreducible cuts of the plane between two points*, by W. A. Wilson; *Collineations of projectively related affine connections*, by M. S. Knebelman; *The discriminant of the m-ary quadratic in the Galois fields of order 2^n*, by A. D. Campbell.

The International Mathematical Congress which met at Bologna during the first ten days of September, was concluded by a meeting held at the Palazzo Vecchio in Florence. This was made the occasion for a celebration and pageant, participated in by trumpeters, soldiers, and heralds in ancient costumes. Greetings were exchanged between the meeting of the Congress and the meeting of this Society at Amherst. A report of the Congress will appear in a later issue of this Bulletin.

The Deutsche Mathematiker-Vereinigung held its annual meeting in Hamburg on September 16–23, 1928.

At the third general assembly of the International Astronomical Union, held at Leiden, July 5–13, 1928, under the presidency of Professor W. de Sitter, the following officers were elected: Sir Frank Dyson, president; Dr. Frank Schlesinger, Professor H. Andoyer, Dr. G. Abetti, Professor N. E. Nörlund, and Professor F. Nušl, vice-presidents. The next congress will be held in the United States.

The New Zealand Institute has awarded its Hector medal and prize for 1928 to Professor D. M. Y. Sommerville, of Victoria University College, Wellington, for his mathematical work as a whole and for his investigations in non-euclidean geometry.

The Royal Academy of Belgium has awarded its De Potter prize in mathematics for the period 1925–27 to Maurice Lecat, for his work entitled *Coup d'œil sur la théorie des déterminants dans son état actuel.*

Dr. Max Planck, professor of physics at the University of Berlin, celebrated his seventieth birthday April 23, 1928. To commemorate this occasion his friends and colleagues have founded a gold medal to be awarded for distinguished work in theoretical physics.

Professor T. von Kármán, of the Aerodynamical Institute of the Aachen Technical School, has been elected a foreign member of the Turin Academy of Sciences.

The University of Edinburgh has conferred the honorary degree of doctor of laws on Professor Niels Bohr, of Copenhagen.

Professor G. H. Hardy, of Oxford, has been elected a foreign member of the Göttingen Society of Sciences.

Dr. E. A. Miller, professor of applied mathematics at the University of Manchester, has been elected the first Rouse Ball professor of mathematics at the University of Oxford.

Mr. H. P. Mullholland, of Queen's College, Cambridge, has been appointed lecturer in mathematics at the University of Newcastle-on-Tyne.

Professor H. S. Carslaw, of Sydney, Australia, has been elected to a Supernumerary Fellowship at Emmanuel College, Cambridge, of which he was formerly a Fellow, and he proposes to spend the whole of next year in residence there, while on leave of absence from the University of Sydney.

At its commencement exercises in June, Dartmouth College conferred the honorary degree of doctor of science on Professor Charles N. Haskins of the department of mathematics. This was an expression of appreciation for his work in connection with the building of the new Baker Memorial Library dedicated at that time.

Dr. H. A. Bender, of the University of Illinois, has been appointed assistant professor of mathematics at the Municipal University of Akron.

Associated Professor C. C. Bramble has been promoted to a professorship of mathematics in the postgraduate school of the United States Naval Academy.

Associate Professor G. R. Clements, of the United States Naval Academy, has been promoted to a professorship of mathematics.

Dr. N. B. Conkwright, of the University of Iowa, has been promoted to the rank of associate in mathematics.

Dr. A. H. Copeland has been appointed assistant professor of mathematics at the University of Buffalo.

Associate Professor Alexander Dillingham has been promoted to a professorship at the United States Naval Academy.

Professor J. C. Fitterer has been appointed associate professor of mathematics at the Colorado School of Mines.

Dr. Orrin Frink has been appointed assistant professor of mathematics at Pennsylvania State College.

Dr. H. C. Hicks has been appointed assistant professor of mathematics at the University of Oregon.

Associate Professor B. V. Hill, of Phillips University, Enid, Oklahoma, has been promoted to a professorship of mathematics.

Assistant Professor R. A. Johnson of Hunter College of the City of New York, has been promoted to an associate professorship of mathematics.

The Reverend E. C. Phillips, director of the Astronomical Observatory of Georgetown University, has been appointed to the office of Provincial of the Jesuit province of Maryland-New York.

Assistant Professor J. B. Rosenbach, of the Carnegie Institute of Technology, has been promoted to an associate professorship of mathematics.

Dr. S. A. Schelkunoff, of the State College of Washington, has been promoted to an assistant professorship of mathematics.

Dr. L. S. Shively, of Mount Morris College, has been appointed associate professor of mathematics at Ball Teachers College, Muncie, Indiana.

Dr. H. W. Stager has been appointed head of the department of mathematics at Salinas Junior College.

Dr. J. M. Stetson has been appointed professor of mathematics at the College of William and Mary.

Professor Evan Thomas, of the University of Vermont, has retired.

Dr. J. H. Van Vleck has been appointed professor of theoretical physics at the University of Wisconsin.

Assistant Professor W. J. Webber, of the University of Toronto, has been promoted to an associate professorship of mathematics.

Associate Professor Mary E. Wells, of Vassar College, has been promoted to a professorship of mathematics.

Associate Professor C. H. Yeaton, of Oberlin College, has been promoted to a full professorship of mathematics.

The following appointments to instructorships are announced:
Brown University, Mr. H. S. Thurston;
Hunter College, Dr. Marguerite D. Darkow, Dr. Rosa L. Jackson;
Milton College, Miss Fannie Hopkins;

United States Naval Academy, Messrs. F. J. Baier, J. R. Bland, W. F. Kern, G. A. Lyle, H. C. Stotz, H. J. Winslow;
Wittenberg College, Mr. G. S. Harris.

Professor F. S. Carey, retired, of the University of Liverpool, died July 26, 1928, at the age of sixty-eight.

Mr. Frank Castle, lecturer in practical mathematics at the Municipal Technical Institute, Eastbourne, author of text books on practical mathematics, died August 4, 1928, at the age of seventy-one.

Professor W. J. King, of the United States Naval Academy, died January 15, 1928.

NEW PUBLICATIONS

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BATCHELDER (P. M.). An introduction to linear difference equations. Published with the cooperation of the National Research Council. Cambridge, Mass , Harvard University Press, 1927. 8+209 pp.

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BIEBERBACH (L.). Differential- und Integralrechnung. 3te Auflage. 2ter Band. Leipzig, Teubner, 1928. 150 pp.

BURNETT (J. C.). Hyper and ornate magic squares, 6th and 12th orders, with non-consecutive numbers. Barkston, the Author, 1928. 64 pp.

CAJORI (F.). The early mathematical sciences in North and South America. Boston, Gorham Press, 1928. 156 pp.

CAMPBELL (N. R.). An account of the principles of measurement and calculation. London, Longmans, 1928. 10+293 pp.

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HAUSSNER (R.). Analytische Geometrie der Ebene. (Sammlung Göschen.) Berlin, de Gruyter, 1928. 164 pp.

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MUIKA (J.). Das Fermatsche Theorem. Bukarest, F. Gobl, 1927. 2 pp.

NATIONAL RESEARCH COUNCIL. Bulletin No. 63: Selected topics in algebraic geometry. Report of the Committee on Rational Transformations, V. Snyder, A. B. Coble, A. Emch, S. Lefschetz, F. R. Sharpe, C. H. Sisam. Washington, National Research Council, 1928. 395 pp.

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